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SOME FULL DESCRIPTIVE CHARACTERIZATIONS OF THE HENSTOCK-KURZWEIL INTEGRAL IN THE EUCLIDEAN SPACE

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Abstract. Using generalized absolute continuity, we characterize additive interval functions which are indefinite Henstock-Kurzweil integrals in the Euclidean space.

Keywords: generalized absolute continuity, Henstock-Kurzweil integral

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1. INTRODUCTION

It is well known that the one-dimensional Henstock-Kurzweil integral is equivalent to the Denjoy-Perron integral. See, for example, [4, Theorems 11.3–11.4]. However, the proof of this result is real-line dependent, so an analogous characterization of the higher dimensional Henstock-Kurzweil integral remained open for many years (see, for example, [6, p. 527] or [3, Problem 5.6 (4)]). In this paper we will use certain versions of the generalized absolute continuity condition to obtain some descriptive characterizations of the higher dimensional Henstock-Kurzweil integral, answering a question of Claude-Alain Faure [3, Problem 5.6 (4)]. Moreover, one of our results also sharpens a result of Kurzweil and Jarník [7, Theorem 4.2], whose original proof does not seem to include the case for the Henstock-Kurzweil integral because of its dependence on the regularity assumption [7, Condition 3.2]. Furthermore, we also give an affirmative answer to a question of Lu and Lee [16]. In the last section of this paper, we sharpen the above results by showing that if f is Henstock-Kurzweil integrable on an m-dimensional compact interval $E \subset \mathbb{R}^m$, then there exists an increasing sequence $\{X_n\}_{n=1}^{\infty}$ of closed subsets of E such that $f \in \mathscr{L}(X_n)$ for all positive integers n and $E \setminus \bigcup\limits_{n=1}^{\infty} X_n$ has m-dimensional Lebesgue measure zero. Moreover, the gauge function in the definition of generalized absolute continuity of the indefinite Henstock-Kurzweil integral of f on each X_n can be chosen to be a positive constant for each $\varepsilon > 0$. See Theorem 5.3 and Remark 5.4 for details.

2. Preliminaries

Unless stated otherwise, the following conventions and notation will be used. The set of all real numbers is denoted by $\mathbb{R}$, and the ambient space of this paper is $\mathbb{R}^m$, where m is a fixed positive integer. The norm in $\mathbb{R}^m$ is the maximum norm. For $x \in \mathbb{R}^m$ and $r > 0$, the open ball $B(x, r)$ is the open cube centered at x with sides equal to $2r$. Let $E = \prod_{i=1}^{m} [a_i, b_i]$ be a fixed non-degenerate interval in $\mathbb{R}^m$. For a set $A \subset E$ we denote by χ_A, $\operatorname{diam}(A)$ and $\mu_m^*(A)$ the characteristic function, diameter and m-dimensional Lebesgue outer measure of A, respectively. If $Z \subseteq E$, we denote its interior, boundary and closure with respect to the subspace topology of E by $\operatorname{int}(Z)$, ∂Z and $\overline{Z}$, respectively. The distance between $x \in E$ and $Z \subseteq E$ will be denoted by $\operatorname{dist}(x, Z)$. The expressions "measure", "measurable", "almost all", "almost everywhere" refer to the m-dimensional Lebesgue measure μ_m. A set $Z \subset E$ is called *negligible* whenever $\mu_m(Z) = 0$. Given two subsets X, Y of E, the symmetric difference of X and Y is denoted by $X \Delta Y$. We say that X and Y are non-overlapping if their intersection is negligible. A function is always real-valued. When no confusion is possible, we do not distinguish between a function defined on a set Z and its restriction to a set $W \subset Z$. If Z is a measurable subset of E, $\mathscr{L}(Z)$ will denote the space of Lebesgue integrable functions on Z, and $\mathscr{M}$ denotes the σ-algebra of all measurable subsets of E. If $f \in \mathscr{L}(Z)$, the Lebesgue integral of f over Z will be denoted by (L) $\int_Z f$.

An *interval* is a compact non-degenerate interval of E. $\mathscr{I}$ denotes the family of all non-degenerate subintervals of E. If $I \in \mathscr{I}$, we write $\mu_m(I)$ as $|I|$. For each $J \in \mathscr{I}$, the *regularity* of an m-dimensional interval $J \subseteq E$, denoted by $\operatorname{reg}(J)$, is the ratio of its shortest and longest sides. A function F defined on $\mathscr{I}$ is said to be additive if $F(I \cup J) = F(I) + F(J)$ for each non-overlapping intervals $I, J \in \mathscr{I}$ with $I \cup J \in \mathscr{I}$. In particular, it is shown in [10, Corollary 6.2.4] that if F is an additive interval function on $\mathscr{I}$ with $J \in \mathscr{I}$ and $\{K_1, K_2, \dots, K_r\}$ is a collection of non-overlapping subintervals of J with $\bigcup_{i=1}^{r} K_i = J$, then

$$F(J) = \sum_{i=1}^{r} F(K_i).$$

A *partition* P is a collection $\{(I_i,\xi_i)\}_{i=1}^p$, where $I_1, I_2, \ldots, I_p$ are non-overlapping intervals and $\xi_i \in I_i$ for $i = 1, 2, \ldots, p$. Given $Z \subseteq E$, a positive function δ on Z is called a *gauge* on Z. We say that a partition $\{(I_i,\xi_i)\}_{i=1}^p$ is

(i) a partition *in* Z if $\bigcup_{i=1}^{p} I_i \subset Z$,

(ii) a partition *of* Z if $\bigcup_{i=1}^{p} I_i = Z$,

(iii) *anchored* in Z if $\{\xi_1, \xi_2, \ldots, \xi_p\} \subset Z$,

(iv) δ-*fine* if $I_i \subset B(\xi_i, \delta(\xi_i))$ for each $i = 1, 2, \ldots, p$,

(v) α-*regular* for some $\alpha \in (0,1]$ if $\operatorname{reg}(I_i) \geqslant \alpha$ for each $i = 1, 2, \ldots, p$.

Lemma 2.1 [10, Lemma 6.2.6]**.** *Given a gauge δ on E, δ-fine partitions of E exist.*

Definition 2.2. A function $f\colon E \longrightarrow \mathbb{R}$ is said to be *Henstock-Kurzweil integrable* on E if there exists $A \in \mathbb{R}$ such that for any given $\varepsilon > 0$ there exists a gauge δ on E such that

$$\left|\sum_{i=1}^{p} f(\xi_i)|I_i| - A\right| < \varepsilon \tag{1}$$

for each δ-fine partition $\{(I_i,\xi_i)\}_{i=1}^p$ of E. Here A is called the Henstock-Kurzweil integral of f over E, and we write $A = (\mathrm{HK})\int_E f$.

Remark 2.3.

(a) The linear space of Henstock-Kurzweil integrable functions on E is denoted by $\mathcal{HK}(E)$.

(b) It follows from [10, Theorem 6.4.2] that if $f \in \mathcal{HK}(E)$, then $f \in \mathcal{HK}(J)$ for each subinterval J of E. The interval function $F\colon J \mapsto (\mathrm{HK})\int_J f$ is known as the *indefinite Henstock-Kurzweil integral*, or in short the indefinite $\mathcal{HK}$-integral, of f. By [10, Theorem 6.4.1], F is an additive interval function on $\mathscr{I}$.

(c) By [10, p. 228] and [10, Theorem 3.13.3], we see that $\mathscr{L}(E) \subset \mathcal{HK}(E)$. Furthermore, $(\mathrm{L})\int_E f = (\mathrm{HK})\int_E f$ for each $f \in \mathscr{L}(E)$.

(d) If f is a non-negative, Henstock-Kurzweil integrable function on E, then it follows from [10, p. 228] that $f \in \mathscr{L}(E)$.

By specializing [5, Lemma 1.7] to the case of the Henstock-Kurzweil integral (see [5, Note 1.5]), we have the following important Saks-Henstock Lemma.

Theorem 2.4 (Saks-Henstock). *If F is the indefinite $\mathcal{HK}$-integral of a function $f \in \mathcal{HK}(E)$, then for $\varepsilon > 0$ there exists a gauge δ on E such that*

$$\sum_{i=1}^{p} \bigl| f(\xi_i)|I_i| - F(I_i) \bigr| < \varepsilon$$

for each δ-fine partition $\{(I_i, \xi_i)\}_{i=1}^{p}$ in E.

Theorem 2.5. *If F is the indefinite $\mathcal{HK}$-integral of a function $f \in \mathcal{HK}(E)$, then F is continuous on $\mathscr{I}$ in the following sense: given $\varepsilon > 0$ there exists $\eta > 0$ such that*

$$|F(E_1) - F(E_2)| < \varepsilon$$

whenever E_1 and E_2 are subintervals of E with $\mu_m(E_1 \Delta E_2) < \eta$.

Proof. This follows from [12, Theorem 3.5] and [12, Lemma 3.6]. □

By using the Vitali covering theorem, it is possible to prove that if F is the indefinite $\mathcal{HK}$-integral of a function belonging to $\mathcal{HK}(E)$, then F is differentiable in the ordinary sense [17, p. 106] at almost all $x \in E$.

Theorem 2.6. *If F is the indefinite $\mathcal{HK}$-integral of some function $f \in \mathcal{HK}(E)$, then the ordinary derivative F' of F exists almost everywhere on E, and $F'(x) = f(x)$ for almost all $x \in E$.*

Proof. The proof is similar to that of [5, Theorem 2.8]. □

Let F be an interval function on $\mathscr{I}$, and let X be an arbitrary subset of E. If δ is a gauge on X, we set

$$V(F, X, \delta) := \sup_{P} \sum_{i=1}^{p} |F(I_i)|,$$

where the supremum is taken over all δ-fine partitions $P = \{(I_i, \xi_i)\}_{i=1}^{p}$ anchored in X.

We put

$$V_{\mathcal{HK}} F(X) := \inf_{\delta} V(F, X, \delta),$$

where the infimum is taken over all gauges δ on X. Then the extended real-valued set function $V_{\mathcal{HK}} F(\cdot)$ is a metric outer measure [3, Proposition 3.3]. Moreover, it follows from [1, Theorem 3.7] that $V_{\mathcal{HK}} F$ is a Borel measure, known as the *Henstock variational measure generated by F*.

For an additive interval function F on $\mathscr{I}$, we say that $V_{\mathcal{HK}}F$ is *absolutely continuous* if the following condition is satisfied:

$$Z \subset E \text{ with } \mu_m(Z) = 0 \Longrightarrow V_{\mathcal{HK}}F(Z) = 0.$$

The next theorem is given in [14, Theorem 3.7].

Theorem 2.7. *Let F be an additive interval function on $\mathscr{I}$. If $V_{\mathcal{HK}}F$ is absolutely continuous, then $V_{\mathcal{HK}}F$ is a measure on $\mathscr{M}$.*

The next theorem is given in [14, Theorem 4.3].

Theorem 2.8. *Let F be an additive interval function on $\mathscr{I}$. Then the following conditions are equivalent:*
(i) *F is the indefinite $\mathcal{HK}$-integral of some function belonging to $\mathcal{HK}(E)$;*
(ii) *the variational measure $V_{\mathcal{HK}}F$ is absolutely continuous.*

3. Henstock variational measure and Lebesgue integrability

Our first lemma is a special case of [11, Lemma 3].

Lemma 3.1. *Let $f \in \mathcal{HK}(E)$. If f is Lebesgue integrable on a non-empty measurable set $X \subseteq E$, then given $\varepsilon > 0$ there exists a gauge δ on X such that*

$$\sum_{i=1}^{p} \left| (\mathrm{L})\int_{I_i \cap X} f - (\mathrm{HK})\int_{I_i} f \right| < \varepsilon$$

for each δ-fine partition $\{(I_i, \xi_i)\}_{i=1}^{p}$ anchored in X.

The next theorem, which was given in [14, Theorem 3.9], can also be deduced from Lemma 3.1.

Theorem 3.2. *Let $X \subseteq E$ be measurable. If F is the indefinite $\mathcal{HK}$-integral of some $f \in \mathcal{HK}(E)$, then $f \in \mathscr{L}(X)$ if and only if $V_{\mathcal{HK}}F(X)$ is finite. Moreover,*

$$V_{\mathcal{HK}}F(X) = (\mathrm{L})\int_X |f|$$

even if one of the sides is equal to ∞.

P r o o f. We may assume that X is non-empty. Suppose that $f \in \mathscr{L}(X)$ and let $\varepsilon > 0$ be given. An application of Lemma 3.1 shows that there exists a gauge δ on X such that

$$V(F, X, \delta) \leqslant (\mathrm{L})\int_X |f| + \varepsilon,$$

from which the inequality

$$V_{\mathcal{HK}}F(X) \leqslant (\mathrm{L})\int_X |f|$$

follows by the arbitrariness of $\varepsilon > 0$. In particular, $V_{\mathcal{HK}}F(X)$ is finite. By mimicking the proof of [15, Lemma 3.8], we obtain the following equality

$$(\mathrm{L})\int_X |f| = V_{\mathcal{HK}}F(X). \tag{2}$$

Conversely, we assume that $V_{\mathcal{HK}}F(X) < \infty$. We shall prove that (2) holds. For any fixed positive integer n, we set

$$X_n := \{x \in X\colon\ |f(x)| \leqslant n\}.$$

Since f is measurable and bounded on the measurable set X_n, we see that $f \in \mathscr{L}(X_n)$. Consequently, it follows from (2) (with X replaced by X_n) that

$$(\mathrm{L})\int_{X_n} |f| = V_{\mathcal{HK}}F(X_n). \tag{3}$$

By letting $n \to \infty$ in (3), we see that (2) follows by the Monotone Convergence Theorem and Theorem 2.7. The proof is complete. □

Lemma 3.3. *Let F be a continuous additive interval function on $\mathscr{I}$. If $X \subseteq E$ is non-empty and δ_c is a constant gauge on X, then*

$$V(F, \overline{X}, \delta_c) \leqslant 3^m V(F, X, \delta_c).$$

P r o o f. This follows from assertion (ii) of [14, Theorem 5.11]. □

The next theorem, which is a special case of [7, Theorem 2.10], will be proved by means of Theorem 3.2 and Lemma 3.3.

Theorem 3.4. *If $f \in \mathcal{HK}(E)$, then there exists a sequence $\{X_n\}$ of closed sets whose union is E, and $f \in \mathscr{L}(X_n)$ for each positive integer n.*

Proof. Choose a gauge δ_1 on E corresponding to $\varepsilon = 1$ in the Saks-Henstock Lemma. For each positive integer n, put

$$Y_n = \left\{x \in E\colon\ |f(x)| \leqslant n \text{ and } \delta_1(x) \geqslant \frac{1}{n}\right\}$$

and $X_n := \overline{Y_n}$. Clearly, $E = \bigcup\limits_{n=1}^{\infty} X_n$ with

$$V\Big(F, Y_n, \frac{1}{n}\Big) \leqslant n|E| + 1.$$

An application of Theorem 2.5 and Lemma 3 yields

$$V\Big(F, X_n, \frac{1}{n}\Big) \leqslant 3^m(n|E| + 1),$$

which implies that $V_{\mathcal{HK}}F(X_n)$ is finite. In view of Theorem 3.2, $f \in \mathscr{L}(X_n)$. □

As a consequence of Theorem 3.4 and the Baire Category Theorem, we have

Corollary 3.5 [2]. *If f belongs to $\mathcal{HK}(E)$, then f is Lebesgue integrable on a portion of E.*

4. Main results

In this section, we will give some new descriptive characterizations of the higher dimensional Henstock-Kurzweil integral. We begin with

Definition 4.1. Let F be an interval function F on $\mathscr{I}$.

(i) F is $\mathrm{AC}^s(X)$ for some $X \subseteq E$ (the so called "absolutely continuous on X" in the paper [3, Definition 3.6 (b)]) if for $\varepsilon > 0$ there exists $\eta > 0$ such that whenever $A \subseteq X$ with $\mu_m^*(A) < \eta$, we have $V_{\mathcal{HK}}F(A) < \varepsilon$.

(ii) F is $\mathrm{ACG}^s(E)$ if $E = \bigcup\limits_{k=1}^{\infty} X_k$ for a sequence $\{X_k\}$ of sets such that F is $\mathrm{AC}^s(X_k)$ for each positive integer k.

The next theorem gives an affirmative answer to [3, Open problems 5.6 (4)].

Theorem 4.2. *Let F be an additive interval function on $\mathscr{I}$ and let $f\colon E\to\mathbb{R}$. Then the following conditions are equivalent:*

(i) *$f\in\mathcal{HK}(E)$ and F is the indefinite $\mathcal{HK}$-integral of f;*

(ii) *F is $\mathrm{ACG}^s(E)$ with $F'=f$ almost everywhere on E.*

Proof. (ii) $\Longrightarrow$ (i) If F is $\mathrm{ACG}^s(E)$, then it follows from [3, Lemma 3.9 (2)] that $V_{\mathcal{HK}}F$ is absolutely continuous. An application of Theorem 2.8 shows that $f\in\mathcal{HK}(E)$ with F being the indefinite $\mathcal{HK}$-integral of f.

(i) $\Longrightarrow$ (ii) Suppose that $f\in\mathcal{HK}(E)$ and F denotes the indefinite $\mathcal{HK}$-integral of f. In view of [3, Proposition 4.1] and [3, Lemma 3.9 (1)], F is $\mathrm{ACG}^s(E)$. The conclusion $F'=f$ almost everywhere on E follows from Theorem 2.6. The proof is complete. □

Definition 4.3.

(i) An interval function F on $\mathscr{I}$ is said to be $\mathrm{AC}^*_\delta(X)$ if for $\varepsilon>0$ there exist $\eta>0$ and a gauge δ on X such that

$$\sum_{i=1}^{p}|F(I_i)|<\varepsilon$$

for each δ-fine partition $\{(I_i,\xi_i)\}_{i=1}^p$ anchored in X satisfying $\sum\limits_{i=1}^{p}|I_i|<\eta$.

(ii) F is $\mathrm{ACG}^*_\delta(E)$ if $E=\bigcup\limits_{k=1}^{\infty}X_k$ for a sequence $\{X_k\}$ of sets such that F is $\mathrm{AC}^*_\delta(X_k)$ for each positive integer k.

For the case when $m=1$, it is well known that F is the indefinite $\mathcal{HK}$-integral of some function $f\in\mathcal{HK}(E)$ if and only if $F\in\mathrm{ACG}^*_\delta(E)$. For a proof, see for example, [4, p. 148]. Since this proof depends strongly on derivatives, it does not seem to work well for higher dimensions. The following theorem, which is an extension of [4, p. 148], also improves a result of Lee Peng-Yee and Ng Wee-Leng [9, Theorem 10].

Theorem 4.4. *Let F be an additive interval function on $\mathscr{I}$ and let $f\colon E\to\mathbb{R}$. Then the following conditions are equivalent:*

(i) *$f\in\mathcal{HK}(E)$ and F is the indefinite $\mathcal{HK}$-integral of f;*

(ii) *F is $\mathrm{ACG}^*_\delta(E)$ with $F'=f$ almost everywhere on E.*

Proof. (i) $\Longrightarrow$ (ii) This follows from Theorem 3.4, Lemma 3.1 and the absolute continuity of the indefinite $\mathscr{L}$-integral.

(ii) $\Longrightarrow$ (i) If F is $\mathrm{ACG}^*_\delta(E)$, then $V_{\mathcal{HK}}F$ is absolutely continuous. Hence the result follows from Theorem 2.8. The proof is complete. □

We shall next prove that the Henstock-Kurzweil integral can also be characterized in terms of ACG$^{\nabla}$ functions [7]. In [7], the $\mathscr{L}$-integral, which includes the Henstock-Kurzweil integral and the α-regular integral [8], is introduced so that various kinds of integrals can be studied by means of a unified approach. In particular, a full descriptive definition of the $\mathscr{L}$-integral can be obtained by means of an additive interval function satisfying certain generalized absolute continuity condition. This interesting characterization, however, depends strongly on the assumption of a certain regularity condition [7, Condition 3.2]. See [7, Theorem 4.2] for more details. Since the definition of the multidimensional Henstock-Kurzweil integral does not involve regular intervals, it is unclear whether [7, Theorem 4.2] remains valid for this particular integral. Our next aim is to give an affirmative answer to this problem.

Definition 4.5 [7, Definition 1.5]**.** Let F be an interval function on $\mathscr{I}$, and $X \subseteq E$. F is said to be AC$^{\nabla}(X)$ if for each $\varepsilon > 0$ there exists a gauge δ on X and $\eta > 0$ such that whenever P_1 and P_2 are δ-fine partitions anchored in X satisfying

$$\mu_m\Bigg(\Bigg(\bigcup_{(I,\xi)\in P_1} I\Bigg) \triangle \Bigg(\bigcup_{(I,\xi)\in P_2} I\Bigg)\Bigg) < \eta,$$

we have

$$\Bigg|\sum_{(I,\xi)\in P_1} F(I) - \sum_{(I,\xi)\in P_2} F(I)\Bigg| < \varepsilon.$$

The next definition is a special case of [7, Definition 3.2].

Definition 4.6. Let F be an additive interval function on $\mathscr{I}$. F is said to be ACG$^{\nabla}(E)$ if E can be expressed as a countable union of measurable sets $\{X_n\}$ and F is AC$^{\nabla}(X_n)$ for each positive integer n.

The next theorem is a special case of [7, Theorem 3.7].

Theorem 4.7. *If F is* ACG$^{\nabla}(E)$*, then $V_{\mathcal{HK}}F$ is absolutely continuous.*

Theorem 4.8. *Let F be an additive interval function on $\mathscr{I}$ and let $f\colon E \to \mathbb{R}$. Then $f \in \mathcal{HK}(E)$ and F is the indefinite $\mathcal{HK}$-integral of f if and only if F is* ACG$^{\nabla}(E)$ *with $F' = f$ almost everywhere on E.*

Proof. ($\Longrightarrow$) Suppose that $f \in \mathcal{HK}(E)$ and F denotes the indefinite $\mathcal{HK}$-integral of f. Then it follows from [7, Theorem 4.1] that F is ACG$^{\nabla}(E)$. By Theorem 2.6 $F' = f$ almost everywhere on E.

($\Longleftarrow$) This follows from Theorems 4.7 and 2.8. $\square$

Definition 4.9. Let F be an additive interval function on $\mathscr{I}$.

(i) F is $\mathrm{AC}^{**}(X)$ for a non-empty set $X \subseteq E$ if for $\varepsilon > 0$ there exist a gauge δ on X and $\eta > 0$ such that whenever $\{(I_i, \xi_i)\}_{i=1}^{p}$ and

$$\{(J_r, x_r)\colon\ J_r \subseteq I_{i(r)} \quad \text{for some } i(r) \in \{1, 2, \dots, p\}\}_{r=1}^{q}$$

are δ-fine partitions anchored in X with

$$\mu_m\biggl(\Bigl(\bigcup_{i=1}^{p} I_i\Bigr) \setminus \bigcup_{r=1}^{q} J_r\biggr) < \eta,$$

we have

$$\sum_{i=1}^{p}\Bigl|F(I_i) - \sum\{F(J_r)\colon\ J_r \subseteq I_i\}\Bigr| < \varepsilon.$$

(ii) F is said to be ACG^{**} on E if E can be written as a countable union of sets $\{X_n\}$ such that F is $\mathrm{AC}^{**}(X_n)$ for each positive integer n.

We can now give an affirmative answer to a question of Lu and Lee [16].

Theorem 4.10. *Let F be an additive interval function on $\mathscr{I}$ and let $f\colon E \to \mathbb{R}$. Then $f \in \mathcal{HK}(E)$ and F is the indefinite $\mathcal{HK}$-integral of f if and only if F is* ACG^{**} *on E with $F' = f$ almost everywhere on E.*

Proof. If $f \in \mathcal{HK}(E)$ and F is the indefinite $\mathcal{HK}$-integral of f, then the conclusion that F is ACG^{**} on E follows from Theorem 3.4, Lemma 3.1 and the absolute continuity of the indefinite $\mathscr{L}$-integral.

Conversely, if F is ACG^{**} on E, then F is $\mathrm{ACG}^{*}_{\delta}$ on E. In view of Theorem 4.4, the proof is complete.

5. On another necessary and sufficient condition for Henstock-Kurzweil integrability

In this section, we will give a slight modification of [14, Theorem 3.10], which will sharpen the results in the previous section. More precisely, if f is Henstock-Kurzweil integrable on E, then there exists an increasing sequence $\{X_n\}_{n=1}^{\infty}$ of closed subsets of E such that $f \in \mathscr{L}(X_n)$ for all positive integers n and $E \setminus \bigcup\limits_{n=1}^{\infty} X_n$ is negligible. Moreover, the gauge function in the definition of the generalized absolute continuity of the indefinite Henstock-Kurzweil integral of f on each X_n can be chosen to be a positive constant for each $\varepsilon > 0$. In what follows, we write $X_n \uparrow X$ if $\{X_n\}$ is an increasing sequence of sets whose union is X.

Theorem 5.1. *Let F be the indefinite $\mathcal{HK}$-integral of some function $f \in \mathcal{HK}(E)$. If f is Lebesgue integrable on a non-empty measurable set $X \subseteq E$, and G denotes the indefinite $\mathscr{L}$-integral of $f\chi_X$, then for $\varepsilon > 0$ there exists a gauge δ on X such that*

(i) $V(F - G, X, \delta) < \varepsilon/3^m$,

(ii) *for each* $p = 1, 2, \dots$ *we have* $V(F - G, \overline{\{x \in X\colon \delta(x) \geqslant 1/p\}}, 1/p) < \varepsilon$,

(iii) $\overline{\{x \in X\colon \delta(x) \geqslant 1/n\}} \uparrow X$, *provided that X is closed.*

Proof. Assertion (i) follows from Lemma 3.1. Assertion (ii) follows from assertion (i) and Lemma 3.3. Assertion (iii) is obvious. $\square$

We shall write $\varepsilon_p \downarrow 0$ if $\{\varepsilon_p\}$ is a decreasing null sequence of positive numbers.

Theorem 5.2. *If $f \in \mathcal{HK}(E)$, then for $\varepsilon_p \downarrow 0$ there exists a sequence $\{Y_p\}$ of closed sets and a negligible set Z satisfying the following conditions:*

(i) $Y_p \uparrow E \setminus Z$;

(ii) $f \in \mathscr{L}(Y_p)$ *for each positive integer* p;

(iii) *for each positive integer p there exists a constant $\delta_p > 0$ such that*

$$\sum_{i=1}^{q} \left| (\mathrm{L})\int_{I_i \cap Y_p} f - (\mathrm{HK})\int_{I_i} f \right| < \varepsilon_p$$

for each δ_p-fine partition $\{(I_i, \xi_i)\}_{i=1}^q$ anchored in Y_p.

Proof. By Theorem 3.4 there exists a sequence $\{X_k\}$ of closed sets with $X_k \uparrow E$, and $f \in \mathscr{L}(X_k)$ for each positive integer k. Moreover, we may assume that $\mu_m(X_k) > 0$ for all positive integers k.

If F_k denotes the indefinite $\mathscr{L}$-integral of $f\chi_{X_k}$, and let F denote the indefinite $\mathcal{HK}$-integral of f. Then it follows from assertions (ii) and (iii) of Theorem 5.2 that there exists a sequence $\{Z_{k,n}\}_{n=1}^{\infty}$ of closed sets such that $Z_{k,n} \uparrow X_k$ with

$$V\Big(F_k - F, Z_{k,n}, \frac{1}{n}\Big) < \frac{\varepsilon_k}{4}.$$

By the absolute continuity of the indefinite $\mathscr{L}$-integral of $f\chi_{X_k}$, there exists $0 < \eta_k < \mu_m(X_k)$ such that

$$(\mathrm{L})\int_Q |f| < \frac{\varepsilon_k}{4}$$

whenever Q is a measurable subset of X_k with $\mu_m(Q) < \eta_k$.

Now, for X_1 we choose a sequence of strictly increasing integers $\{n(1,k)\}_{k=1}^{\infty}$ such that

$$\mu_m\Big(X_1 \setminus \bigcap_{k=1}^{\infty} Z_{k,n(1,k)}\Big) < \eta_1.$$

In view of our choice of η_1, we see that $\bigcap\limits_{k=1}^{\infty} Z_{k,n(1,k)}$ is non-empty. Proceed inductively: if a strictly increasing sequence of integers $\{n(p-1,k)\}_{k=p-1}^{\infty}$ has been chosen for some $p > 1$, we may choose a strictly increasing sequence of integers $\{n(p,k)\}_{k=p}^{\infty}$ such that

$$\mu_m\Bigl(X_p \setminus \bigcap_{k=p}^{\infty} Z_{k,n(p,k)}\Bigr) < \eta_p$$

and $n(p,k) > n(p-1,k)$ for each positive integer $k \geqslant p$.

For each positive integer p, we put $Y_p := \bigcap\limits_{k=p}^{\infty} Z_{k,n(p,k)}$ and $Z := E \setminus \bigcup\limits_{p=1}^{\infty} Y_p$. Since $X_k \uparrow E$ and $Z_{k,n} \uparrow X_k$ for each positive integer k, we see that $Y_p \uparrow E \setminus Z$.

Define $\delta_p\colon Y_p \longrightarrow \mathbb{R}^+$ by $\delta_p(\xi) = 1/n(p,p)$. To this end, selecting any δ_p-fine partition $\{(I_i,\xi_i)\}_{i=1}^{q}$ anchored in Y_p, we have

$$\begin{aligned}\sum_{i=1}^{q}\Bigl|(\mathrm{L})\int_{I_i\cap Y_p} f - (\mathrm{HK})\int_{I_i} f\Bigr| &\leqslant \sum_{i=1}^{q}\Bigl|(\mathrm{L})\int_{I_i\cap X_p} f - (\mathrm{HK})\int_{I_i} f\Bigr| + (\mathrm{L})\int_{X_p\setminus Y_p} |f| \\ &< V\Bigl(F_p - F, Z_{p,n(p,p)}, \frac{1}{n(p,p)}\Bigr) + \frac{\varepsilon_p}{4} < \varepsilon_p.\end{aligned}$$

The proof is complete. $\square$

The next theorem, which follows from Theorem 5.2, is a slight modification of [14, Theorem 3.10].

Theorem 5.3. *Let F be an additive interval function on $\mathscr{I}$ and let $f\colon E \to \mathbb{R}$. Then $f \in \mathcal{HK}(E)$ and F is the indefinite $\mathcal{HK}$-integral of f if and only if the following conditions are satisfied:*

(i) *there exists an increasing sequence $\{X_n\}_{n=1}^{\infty}$ of closed subsets of E such that $X_0 := E \setminus \bigcup\limits_{n=1}^{\infty} X_n$ is negligible;*

(ii) *$f \in \mathscr{L}(X_n)$ for each positive integer n;*

(iii) *for each positive integer n and $\varepsilon > 0$ there exists a constant $\eta_n > 0$ such that*

$$\sum_{i=1}^{p}\Bigl|(\mathrm{L})\int_{I_i\cap X_n} f - F(I_i)\Bigr| < \varepsilon$$

for each η_n-fine partition $\{(I_i,\xi_i)\}_{i=1}^{p}$ anchored in X_n;

(iv) *$V_{\mathcal{HK}}F(X_0) = 0$.*

At this stage, it is still unclear whether we can choose $\{X_n\}_{n=0}^{\infty}$ in Theorem 5.3 so that $\{f\chi_{X_n\cup X_0}\}_{n=1}^{\infty}$ is Henstock-Kurzweil equi-integrable on E. Hence [13, Conjecture 3.5] remains unsolved for $m \geqslant 2$. If $m = 1$, then a version of Theorem 5.3 is given in [13, Theorem 3.5]. However, we have the following remark.

Remark 5.4. By using Theorem 5.3, it is not difficult to sharpen the results from the previous section. We omit the details.

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