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Czechoslovak Mathematical Journal, Vol. 56 (2006), No. 1, 9–18

Persistent URL: `http://dml.cz/dmlcz/128050`

Czechoslovak Mathematical Journal, 56 (131) (2006), 9–18

DIAGONAL REDUCTIONS OF MATRICES OVER EXCHANGE IDEALS

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(Received March 5, 2003)

Abstract. In this paper, we introduce related comparability for exchange ideals. Let I be an exchange ideal of a ring R. If I satisfies related comparability, then for any regular matrix $A \in M_n(I)$, there exist left invertible $U_1, U_2 \in M_n(R)$ and right invertible $V_1, V_2 \in M_n(R)$ such that $U_1V_1AU_2V_2 = \operatorname{diag}(e_1, \ldots, e_n)$ for idempotents $e_1, \ldots, e_n \in I$.

Keywords: exchange ring, ideal, related comparability

MSC 2000: 16E50, 16U99

Let R be an associative ring with identity. We say that R is an exchange ring if for every right R-module A and any two decompositions $A = M \oplus N = \bigoplus_{i\in I} A_i$, where $M_R \cong R$ and the index set I is finite, there exist submodules $A_i' \subseteq A_i$ such that $A = M \oplus \Bigl(\bigoplus_{i\in I} A_i'\Bigr)$. We see that regular rings, π-regular rings, strongly π-regular rings, semiperfect rings, left or right continuous rings, clean rings and unit C^*-algebras of real rank zero are all exchange rings. We refer the reader to [1] for a survey on exchange rings. An ideal I of a ring R is said to be an exchange ideal provided that $e\,\mathrm{Re}$ is an exchange ring for any idempotent $e \in I$.

An exchange ring R is said to satisfy related comparability, provided that for any idempotents $e, f \in R$ with $e = 1 - ab$ and $f = 1 - ba$ for some $a \in (1-e)R(1-f)$ and $b \in (1-f)R(1-e)$, there exists $u \in B(R)$ such that $ueR \lesssim^{\oplus} ufR$ and $(1-u)fR \lesssim^{\oplus} (1-u)eR$. Clearly, an exchange ring R satisfies related comparability if and only if $R = A_1 \oplus B_1 = A_2 \oplus B_2$ with $A_1 \cong A_2$ implies that there exists $e \in B(R)$ such that $B_1e \lesssim^{\oplus} B_2e$ and $B_2(1-e) \lesssim^{\oplus} B_1(1-e)$ (see [6]–[7]).

The class of exchange rings satisfying related comparability is very large. It includes all exchange rings satisfying the general comparability, all exchange rings satisfying the comparability axiom, all exchange rings with stable range one, all one-sided unit-regular rings, all right self-injective regular rings, all right or left continuous regular rings, etc.

In this paper, we introduce related comparability for exchange ideals of a ring. Let I be an exchange ideal of a ring R. We say that I satisfies related comparability provided that for any idempotents $e, f \in I$ with $e = 1 - ab$ and $f = 1 - ba$ for some $a \in (1-e)R(1-f)$ and $b \in (1-f)R(1-e)$, there exists $u \in B(R)$ such that $ueR \lesssim^{\oplus} ufR$ and $(1-u)fR \lesssim^{\oplus} (1-u)eR$. On easily checks that an exchange ideal I of a ring R satisfies related comparability if and only if $R = A_1 \oplus B_1 = A_2 \oplus B_2$ with $A_1 \cong A_2$ and $B_1, B_2 \in 9(I)$ implies that there exists $e \in B(R)$ such that $B_1 e \lesssim^{\oplus} B_2 e$ and $B_2(1-e) \lesssim^{\oplus} B_1(1-e)$, where $9(I)$ denotes the set of all finitely generated projective right R-modules P such that $P = PI$. We will investigate diagonal reduction for regular square matrices over exchange ideals satisfying related comparability. Let I be an exchange ideal of a ring R. If I satisfies related comparability, then for any regular matrix $A \in M_n(I)$, there exist left invertible $U_1, U_2 \in M_n(R)$ and right invertible $V_1, V_2 \in M_n(R)$ such that $U_1 V_1 A U_2 V_2 = \operatorname{diag}(e_1, \ldots, e_n)$ for idempotents $e_1, \ldots, e_n \in I$.

Throughout this paper, rings are associative with identity and modules are right unital modules. Let $B(R)$ denote the Boolean algebra of all central idempotents in R, and $U(R)$ $(U_{\mathrm{r}}(R), U_{\mathrm{l}}(R), U_{\mathrm{l}}(R)U_{\mathrm{r}}(R))$ the set of all units (right units, left units, products of a left unit and a right unit) of R. If A and B are R-modules, the notation $B \lesssim^{\oplus} A$ means that B is isomorphic to a direct summand of A. An element $w \in R$ is called a related unit if there exists some $e \in B(R)$ such that $ew \in U_{\mathrm{r}}(eR)$ and $(1-e)w \in U_{\mathrm{l}}((1-e)R)$. We use $U_w(R)$ to denote the set of all related units of R.

Lemma 1. *Let I be an exchange ideal of a ring R. If I satisfies related comparability, then $aR + bR = R$ with $a \in 1 + I, b \in R$ implies that there exists $y \in R$ such that $a + by \in U_w(R)$.*

P r o o f. Suppose that $ax + b = 1$ with $a \in 1 + I$, $x, b \in R$. Then $a(x+b) + (1-a)b = 1$. Clearly, $1 - a \in I$. Since I is an exchange ideal, there exists an idempotent $e \in R$ such that $e = (1-a)bs$ and $1 - e = (1-(1-a)b)t = a(x+b)t$ for $s, t \in R$. Hence $(1-e)a(x+b)t + e = 1$. Since $a \in 1 + I$, we deduce that $(1-e)a \in 1 + I$ is regular. So we have $c \in R$ such that $(1-e)a = (1-e)ac(1-e)a$ and $c = c(1-e)ac$. Clearly, $1 - c = (1-(1-e)a) - (1-(1-e)a)c(1-e)a - c(1-(1-e)a) \in I$; hence $c \in 1 + I$. Set $f = 1 - (1-e)ac$ and $g = 1 - c(1-e)a$. Then $f, g \in I$; hence, $(1-e)a = (1-e)ac(1-e)a = (1-e)ac(1-e)ac(1-e)a = (1-f)((1-f)(1-e)a)(1-g) \in (1-f)R(1-g)$. Likewise, $c \in (1-g)R(1-e)$. Thus we have $u \in B(R)$ such that $ueR \lesssim^{\oplus} ufR$ and $(1-u)fR \lesssim^{\oplus} (1-u)eR$. Similarly to [6, Theorem 2], there exists $w \in U_w(R)$ such that $(1-e)a = (1-e)aw(1-e)a$. Furthermore, $uw \in U_{\mathrm{r}}(uR)$ and $(1-u)w \in U_{\mathrm{l}}((1-u)R)$. Set $(1-e)aw = l$. Then

$u(1-e)a = ulq$ with $uwq = u$. It follows by $ulq + ue = u(1-e)a(x+b)t + ue = u$ that $u(a+e(y-a)) = u((1-e)a+ey) = u(1+lq(1-l))^{-1}q \in U_l(uR)$. Assume now that $dw = 1$. Then $h(x+b)t + we = w$; hence, $h((x+b)t) + (1-h)we = w$. Set $k = (1-h)wed(1-h)$. Then $hk = kh = 0, h = h^2$ and $k = k^2$. So $h((x+b)t) = hw$ and $(1-h)we = k(1-h)we = kw$. This implies that

$$\begin{aligned} w(a + (1-a)bs(d(1-h)(1+hwed(1-h)) - a))(1-hwed(1-h))w \\ &= w((1-e)a + ed(1-h)(1+hwed(1-h)))(1-hwed(1-h))w \\ &= (h + wed(1-h)(1+hwed(1-h)))(1-hwed(1-h))w \\ &= (h(1-hwed(1-h)) + wed(1-h))w \\ &= (h + (1-h)wed(1-h))w = (h+k)w = w. \end{aligned}$$

From $dw = 1$, we see that $a + (1-a)bs(d(1-h)(1+hwed(1-h)) - a) \in U_r(R)$. Applying the consideration to $(1-u)R$, we get a $z \in R$ such that $x + (1+z(1-a))b = (x+b) + z(1-a)b \in U_w(R)$ by [5, Proposition 1]. Consequently, we have a $q \in R$ such that $a + bq \in U_w(R)$, as asserted. $\square$

Let I be an exchange ideal of a ring R. If I satisfies related comparability, by Lemma 1 and [3, Lemma 4.1], we show that I is a separative ideal.

Lemma 2. *Let R be a ring. Then $U_w(R) \subseteq U_l(R)U_r(R)$.*

Proof. Suppose that $w \in U_w(R)$. Then there exists $e \in B(R)$ such that $ew \in U_r(eR)$ and $(1-e)w \in U_l((1-e)R)$. Assume that $ewes = e$ and $(1-e)t(1-e)w = 1-e$ for some $s, t \in R$. Set $u = e + (1-e)t$ and $v = es + (1-e)$. Then we check that $uwv = (e + (1-e)t)w(es + (1-e)) = 1$. In addition, we have $(wv)(uw) = (e + (1-e)w)(ew + (1-e)) = w$, hence $w \in U_w(R) \subseteq U_l(R)U_r(R)$, as required. $\square$

The pair (a, b) is called right unimodular if $aR + bR = R$. We say that a right unimodular pair (a, b) is right quasi-reducible if there exists $y \in R$ such that $a + by \in U_l(R)U_r(R)$.

Lemma 3. *Let (a, b) be a right unimodular row in a ring R. Let $u, v \in U(R)$ and $c \in R$. Then $(vau + vbc, vb)$ is also right unimodular. Furthermore, (a, b) is right quasi-reducible if and only if so is $(au + vbc, vb)$.*

Proof. It follows by [4, Lemma 6.3] that $(vau + vbc, vb)$ is right unimodular. Assume that (a, b) is right quasi-reducible. Then there is a $y \in R$ such that $a + by \in U_l(R)U_r(R)$. Set $z = yu - c$. We have $(vau + vbc) + (vb)z = v(a+by)u \in U_l(R)U_r(R)$; whence, $(au + vbc, vb)$ is right quasi-reducible. Conversely, assume that there exists a $z \in R$ such that $vau + vbc + vbz \in U_{\mathrm{l}}(R)U_{\mathrm{r}}(R)$. Then $v(a + b(c + z)u^{-1})u \in U_{\mathrm{l}}(R)U_{\mathrm{r}}(R)$, and $a + b(c + z)u^{-1} \in U_{\mathrm{l}}(R)U_{\mathrm{r}}(R)$. Therefore (a, b) is right quasi-reducible. $\square$

Analogously to [9, Theorem 6], we can derive the following result.

Lemma 4. *Let I be an exchange ideal of a ring R. Suppose that I satisfies related comparability. If $AX + B = I_n$ with $A \in \mathrm{I}_n + M_n(I)$ and $B \in M_n(I)$, then we have $Y \in M_n(R)$ such that $A + BY \in U_{\mathrm{l}}(M_n(R))U_{\mathrm{r}}(M_n(R))$.*

Proof. Suppose that $AX + B = I_n$ with $A \in \mathrm{I}_n + M_n(I), X \in M_n(R)$ and $B \in M_n(I)$. We will show that $A + BY \in U_{\mathrm{l}}(M_n(R))U_{\mathrm{r}}(M_n(R))$ for a $Y \in M_n(R)$. Since I is an exchange ideal, so is $M_n(I)$ for all $n \in \mathbb{N}$. According to Lemma 1 and Lemma 2, the result holds for $n = 1$. Assume inductively that the result holds for n. It suffices to show that the result also holds for $n + 1$. Suppose that

$$(*) \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1(n+1)} \\ a_{21} & a_{22} & \dots & a_{2(n+1)} \\ \vdots & \vdots & \ddots & \vdots \\ a_{(n+1)1} & a_{(n+1)2} & \dots & a_{(n+1)(n+1)} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} & \dots & b_{1(n+1)} \\ b_{21} & b_{22} & \dots & b_{2(n+1)} \\ \vdots & \vdots & \ddots & \vdots \\ b_{(n+1)1} & b_{(n+1)2} & \dots & b_{(n+1)(n+1)} \end{pmatrix}$$

$$+ \begin{pmatrix} c_{11} & c_{12} & \dots & c_{1(n+1)} \\ c_{21} & c_{22} & \dots & c_{2(n+1)} \\ \vdots & \vdots & \ddots & \vdots \\ c_{(n+1)1} & c_{(n+1)2} & \dots & c_{(n+1)(n+1)} \end{pmatrix} = \mathrm{diag}(1, 1, \dots, 1)$$

in $M_{n+1}(R)$, where

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1(n+1)} \\ a_{21} & a_{22} & \dots & a_{2(n+1)} \\ \vdots & \vdots & \ddots & \vdots \\ a_{(n+1)1} & a_{(n+1)2} & \dots & a_{(n+1)(n+1)} \end{pmatrix} \in \mathrm{diag}(1, 1, \dots, 1) + M_{n+1}(I),$$

$$\begin{pmatrix} c_{11} & c_{12} & \dots & c_{1(n+1)} \\ c_{21} & c_{22} & \dots & c_{2(n+1)} \\ \vdots & \vdots & \ddots & \vdots \\ c_{(n+1)1} & c_{(n+1)2} & \dots & c_{(n+1)(n+1)} \end{pmatrix} \in M_{n+1}(I).$$

Obviously, $a_{11}b_{11} + a_{12}b_{21} + \ldots + a_{1(n+1)}b_{(n+1)1} + c_{11} = 1$ with $a_{11} \in 1 + I$. Since I satisfies related comparability, from Lemma 1, there is a $z_1 \in R$ such that $a_{11} + (a_{12}b_{21} + \ldots + a_{1(n+1)}b_{(n+1)1} + c_{11})z_1 \in U_{\mathrm{l}}(R)U_{\mathrm{r}}(R)$. Furthermore, we have $a_{11} +$

$(a_{12}b_{21}+\ldots+a_{1(n+1)}b_{(n+1)1}+c_{11})z_1 \in 1+I$ since $a_{11} \in 1+I, a_{12},\ldots,a_{1(n+1)},c_{11} \in I$. Using Lemma 3, $(*)$ is right quasi-reducible if and only if this is so for the row with elements

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \ldots & a_{1(n+1)} \\ a_{21} & a_{22} & a_{23} & \ldots & a_{2(n+1)} \\ a_{31} & a_{32} & a_{33} & \ldots & a_{3(n+1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{(n+1)1} & a_{(n+1)2} & a_{(n+1)3} & \ldots & a_{(n+1)(n+1)} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & \ldots & 0 \\ b_{21}z_1 & 1 & 0 & \ldots & 0 \\ b_{31}z_1 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_{(n+1)1}z_1 & 0 & 0 & \ldots & 1 \end{pmatrix}$$

$$+ \begin{pmatrix} c_{11} & c_{12} & c_{13} & \ldots & c_{1(n+1)} \\ c_{21} & c_{22} & c_{23} & \ldots & c_{2(n+1)} \\ c_{31} & c_{32} & c_{33} & \ldots & c_{3(n+1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_{(n+1)1} & c_{(n+1)2} & c_{(n+1)3} & \ldots & c_{(n+1)(n+1)} \end{pmatrix} \begin{pmatrix} z_1 & 0 & 0 & \ldots & 0 \\ 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \end{pmatrix}$$

and

$$\begin{pmatrix} c_{11} & c_{12} & c_{13} & \ldots & c_{1(n+1)} \\ c_{21} & c_{22} & c_{23} & \ldots & c_{2(n+1)} \\ c_{31} & c_{32} & c_{33} & \ldots & c_{3(n+1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_{(n+1)1} & c_{(n+1)2} & c_{(n+1)3} & \ldots & c_{(n+1)(n+1)} \end{pmatrix}.$$

Thus we assume that $a_{11} \in (U_{\mathrm{l}}(R)U_{\mathrm{r}}(R)) \cap (1+I)$ in $(*)$. Since $a_{11} \in U_{\mathrm{l}}(R)U_{\mathrm{r}}(R)$, there exist $u, v, s, t \in R$ such that $a_{11} = uv, su = 1$ and $vt = 1$. So $sa_{11}t = 1$. From $a_{11} \in 1+I$, we see that $st \in 1+I$. Clearly, we may assume that

$$\begin{pmatrix} a_{33} & \ldots & a_{3(n+1)} \\ a_{43} & \ldots & a_{4(n+1)} \\ \vdots & \ddots & \vdots \\ a_{(n+1)3} & \ldots & a_{(n+1)(n+1)} \end{pmatrix} \in \mathrm{diag}(1,1,\ldots,1) + M_{n-1}(I),$$

$$\begin{pmatrix} c_{33} & \ldots & c_{3(n+1)} \\ c_{43} & \ldots & c_{4(n+1)} \\ \vdots & \ddots & \vdots \\ c_{(n+1)3} & \ldots & c_{(n+1)(n+1)} \end{pmatrix} \in M_{n-1}(I).$$

It is easy to check that

$$\begin{pmatrix} s & 0 & 0 & \dots & 0 \\ 1-a_{11}ts & a_{11}t & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1(n+1)} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2(n+1)} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3(n+1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{(n+1)1} & a_{(n+1)2} & a_{(n+1)3} & \dots & a_{(n+1)(n+1)} \end{pmatrix}$$

$$\times \begin{pmatrix} t & 1-tsa_{11} & 0 & \dots & 0 \\ 0 & sa_{11} & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix} = \begin{pmatrix} 1 & b_{12} & b_{13} & \dots & b_{1(n+1)} \\ b_{21} & b_{22} & b_{23} & \dots & b_{2(n+1)} \\ b_{31} & b_{32} & * & \dots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_{(n+1)1} & b_{(n+1)2} & * & \dots & * \end{pmatrix},$$

and that

$$\begin{pmatrix} s & 0 & 0 & \dots & 0 \\ 1-a_{11}ts & a_{11}t & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix} = \begin{pmatrix} a_{11}t & 1-a_{11}ts & 0 & \dots & 0 \\ 0 & s & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}^{-1},$$

$$\begin{pmatrix} t & 1-tsa_{11} & 0 & \dots & 0 \\ 0 & sa_{11} & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix} = \begin{pmatrix} sa_{11} & 0 & 0 & \dots & 0 \\ 1-tsa_{11} & t & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}^{-1} \in \mathrm{GL}_{n+1}(R).$$

Thus $(*)$ is right quasi-reducible if and only if this is so for the row with elements

$$\begin{pmatrix} 1 & b_{12} & b_{13} & \dots & b_{1(n+1)} \\ b_{21} & b_{22} & b_{23} & \dots & b_{2(n+1)} \\ b_{31} & b_{32} & * & \dots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_{(n+1)1} & b_{3(n+1)} & * & \dots & * \end{pmatrix}, \begin{pmatrix} s & 0 & 0 & \dots & 0 \\ 1-a_{11}ts & at & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}$$

$$\times \begin{pmatrix} c_{11} & c_{12} & c_{13} & \dots & c_{1(n+1)} \\ c_{21} & c_{22} & c_{23} & \dots & c_{2(n+1)} \\ c_{31} & c_{32} & c_{33} & \dots & c_{3(n+1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_{(n+1)1} & c_{(n+1)2} & c_{(n+1)3} & \dots & c_{(n+1)(n+1)} \end{pmatrix}.$$

Clearly, all $b_{ij} \in I$ for $i \neq j$. Furthermore, $b_{22} = ((1 - a_{11}ts)a_{11} + a_{11}ta_{21})(1 - tsa_{11}) + ((1 - a_{11}ts)a_{12} + a_{11}ta_{22})sa_{11}$. Since $a_{11}, a_{22}, st \in 1 + I$ and $a_{12} \in I$, we deduce that $b_{22} \equiv (1-ts)(1-ts)+ts \equiv 1(\operatorname{mod} I)$. Hence $b_{22} \in 1+I$. Using Lemma 3 again, $(*)$ is right quasi-reducible if and only if this is so for the row with elements

$$\begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & * & * & \dots & * \\ 0 & * & * & \dots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & * & * & \dots & * \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ * & 1 & 0 & \dots & 0 \\ * & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ * & 0 & 0 & \dots & 1 \end{pmatrix} \begin{pmatrix} s & 0 & 0 & \dots & 0 \\ 1 - ats & at & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}$$
$$\times \begin{pmatrix} c_{11} & c_{12} & c_{13} & \cdots & c_{1(n+1)} \\ c_{21} & c_{22} & c_{23} & \cdots & c_{2(n+1)} \\ c_{31} & c_{32} & c_{33} & \cdots & c_{3(n+1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_{(n+1)1} & c_{(n+1)2} & c_{(n+1)3} & \cdots & c_{(n+1)(n+1)} \end{pmatrix}.$$

Thus, we assume that $a_{11} = 1, a_{1i} = 0 = a_{i1}$ for $i = 2, \dots, n+1$ in $(*)$. Moreover, we may assume that $(*)$ is in the following form:

$$\begin{pmatrix} 1 & 0_{1\times n} \\ 0_{n\times 1} & D \end{pmatrix} \begin{pmatrix} b_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} + \begin{pmatrix} c_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & I_n \end{pmatrix},$$

where $D \in \operatorname{diag}(1, 1, \dots, 1) + M_n(I)$. Clearly, we have $DB_{22} + C_{22} = I_n$. By the induction hypothesis, we can find a $Z_2 \in M_n(R)$ such that $D + C_{22}Z_2 \in U_{\mathrm{l}}(R)U_{\mathrm{r}}(R)$; hence, we pass to the right unimodular row with elements

$$\begin{pmatrix} 1 & 0_{1\times n} \\ 0_{n\times 1} & D \end{pmatrix} + \begin{pmatrix} c_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} \begin{pmatrix} 0 & 0_{1\times n} \\ 0_{n\times 1} & Z_2 \end{pmatrix}, \begin{pmatrix} c_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}.$$

So it suffices to show that the right unimodular row with elements

$$\begin{pmatrix} 1 & C_{12}Z_2 \\ 0_{n\times 1} & D + C_{22}Z_2 \end{pmatrix} \text{ and } \begin{pmatrix} c_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}$$

is quasi-reducible. Inasmuch as $D + C_{22}Z_2 \in U_{\mathrm{l}}(M_n(R))U_{\mathrm{r}}(M_n(R))$, we see that $\begin{pmatrix} 1 & C_{12}Z_2 \\ 0 & D + C_{22}Z_2 \end{pmatrix} \in U_{\mathrm{l}}(M_n(R))U_{\mathrm{r}}(M_n(R))$. By induction, we conclude that $A + BY \in U_{\mathrm{l}}(M_n(R))U_{\mathrm{r}}(M_n(R))$ for a $Y \in M_n(R)$. □

Lemma 5. *Let I be an exchange ideal of a ring R. If I satisfies related comparability, then the following hold:*

(1) *For any regular $A, B \in M_n(I)$, $AM_n(R) = BM_n(R)$ implies that there exists $U \in U_{\mathrm{l}}(R)U_{\mathrm{r}}(R)$ such that $A = BU$.*

(2) *For any regular $A, B \in M_n(I)$, $M_n(R)A = M_n(R)B$ implies that there exists $U \in U_{\mathrm{l}}(R)U_{\mathrm{r}}(R)$ such that $A = UB$.*

Proof. (1) Suppose that $AM_n(R) = BM_n(R)$ with regular $A, B \in M_n(I)$. Then $A = BX$ and $B = AY$ for $X, Y \in M_n(R)$. Since A and B are regular, we may assume that $X, Y \in M_n(I)$. Furthermore, we have $B(X + (\mathrm{I}_n - XY)) = BX = A$. Thus we may assume that $X \in \mathrm{I}_n + M_n(I)$. Likewise, we may assume that $Y \in \mathrm{I}_n + M_n(I)$. Since $XY + (\mathrm{I}_n - XY) = \mathrm{I}_n$, by Lemma 4, we have $Z \in M_n(R)$ such that $X + (I_n - XY)Z = U \in U_{\mathrm{l}}(M_n(R))U_{\mathrm{r}}(M_n(R))$. Therefore $A = BX = B(X + (I_n - XY)Z) = BU$, as asserted.

(2) Let $e, f \in R$ be idempotents. Clearly, $eR \cong fR$ if and only if $\mathrm{Re} \cong Rf$ and $eR \lesssim^{\oplus} fR$ if and only if $\mathrm{Re} \lesssim^{\oplus} Rf$. We know that I satisfies related comparability as an ideal of R if and only if I^{op} satisfies related comparability as an ideal of R^{op}. Applying (1) to the opposite ring R^{op} of R, we get the result. $\square$

Lemma 6. *For any regular $a, b \in R$, if $\psi\colon aR \cong bR$, then $Ra = R\psi(a)$ and $\psi(a)R = bR$.*

Proof. Since $\psi\colon aR \cong bR$, we have $\psi(a) \in bR$, and so $\psi(a)R \subseteq bR$. On the other hand, there exists $r \in R$ such that $b = \psi(ar) = \psi(a)r \in \psi(a)R$; hence, $bR \subseteq \psi(a)R$. Thus, $\psi(a)R = bR$. As $b \in R$ is regular, we have an idempotent $e \in R$ such that $bR = eR$; hence, $\psi(a)R = eR$. This implies that $\psi(a) \in R$ is regular. So we have a $c \in R$ such that $\psi(a) = \psi(a)c\psi(a) = \psi(ac\psi(a))$. It follows that $a = ac\psi(a) \in R\psi(a)$, whence $Ra \subseteq R\psi(a)$. One the other hand, we have $a = ada$ for a $d \in R$. This shows that $\psi(a) = \psi(a)da \in Ra$; hence, $R\psi(a) \subseteq Ra$. Therefore we conclude that $Ra = R\psi(a)$, as asserted. $\square$

Theorem 7. *Let I be an exchange ideal of a ring R. If I satisfies related comparability, then for any regular matrix $A \in M_n(I)$, there exist left invertible $U_1, U_2 \in M_n(R)$ and right invertible $V_1, V_2 \in M_n(R)$ such that $U_1V_1AU_2V_2 = \mathrm{diag}(e_1, \ldots, e_n)$ for idempotents $e_1, \ldots, e_n \in I$.*

Proof. Given any regular $A \in M_n(I)$, we have $E = E^2 \in M_n(I)$ such that $AM_n(R) = EM_n(R)$. Clearly, ER^n is a generated projective right R-module. Since I is an exchange ideal, there are idempotents $e_1, \ldots, e_n \in I$ such that $ER^n \cong e_1R \oplus \ldots \oplus e_nR \cong \mathrm{diag}(e_1, \ldots, e_n)R^n$ as right R-modules, so we have $ER^{n\times 1} \cong$

$\mathrm{diag}(e_1,\dots,e_n)R^{n\times 1}$, where $R^{n\times 1} = \{(x_1 \dots x_n)^{\mathsf{T}};\ x_1,\dots,x_n \in R\}$ is a right R-module and a left $M_n(R)$-module. Let $R^{1\times n} = \{(x_1,\dots,x_n);\ x_1,\dots,x_n \in R\}$. Then $R^{1\times n}$ is a left R-module and a right $M_n(R)$-module; hence, $(ER^{n\times 1})\bigotimes_R R^{1\times n} \cong \mathrm{diag}(e_1,\dots,e_n)R^{n\times 1}\bigotimes_R R^{1\times n}$. One easily checks that $R^{n\times 1}\bigotimes R^{1\times n} \cong M_n(R)$ as right $M_n(R)$-modules. Hence $\psi\colon AM_n(R) \cong \mathrm{diag}(e_1,\dots,e_n)M_n(R)$. It is easy to show that all $e_i \in R$. In view of Lemma 6, $M_n(R)A = M_n(R)\psi(A)$ and $\psi(A)M_n(R) = \mathrm{diag}(e_1,\dots,e_n)M_n(R)$. According to Lemma 5, we have left invertible $U_1,U_2 \in M_n(R)$ and right invertible $V_1,V_2 \in M_n(R)$ such that $U_1V_1A = \psi(A)$ and $\psi(A)U_2V_2 = \mathrm{diag}(e_1,\dots,e_n)$. Consequently, $U_1V_1AU_2V_2 = \mathrm{diag}(e_1,\dots,e_n)$, as required. $\square$

Let R be a generalized stable regular ring. For any $A \in M_n(R)$, there exist right invertible $U_1,U_2 \in M_n(R)$ and left invertible $V_1,V_2 \in M_n(R)$ such that $U_1V_1AU_2V_2 = \mathrm{diag}(e_1,\dots,e_n)$ for idempotents $e_1,\dots,e_n \in R$ (see [9, Theorem 16]). As an immediate consequence of Theorem 7, we can derive the following.

Corollary 8. *Let R be an exchange ring satisfying related comparability. Then for any regular $A \in M_n(R)$, there exist left invertible $U_1,U_2 \in M_n(R)$ and right invertible $V_1,V_2 \in M_n(R)$ such that $U_1V_1AU_2V_2 = \mathrm{diag}(e_1,\dots,e_n)$ for idempotents $e_1,\dots,e_n \in R$.*

Let R be a right self-injective, regular ring (see [10]). Then R is an exchange ring satisfying related comparability. By Corollary 8, we show that for any square matrix A over R, there exist left invertible $U_1,U_2 \in M_n(R)$ and right invertible $V_1,V_2 \in M_n(R)$ such that $U_1V_1AU_2V_2 = \mathrm{diag}(e_1,\dots,e_n)$ for idempotents $e_1,\dots,e_n \in R$.

Corollary 9. *Let I be a purely infinite, simple ideal of a ring R. Then for any regular $A \in M_n(I)$, there exist left invertible $U_1,U_2 \in M_n(R)$ and right invertible $V_1,V_2 \in M_n(R)$ such that $U_1V_1AU_2V_2 = \mathrm{diag}(e_1,\dots,e_n)$ for idempotents $e_1,\dots,e_n \in I$.*

Proof. By Ara's result, every purely infinite, simple ideal of a ring is an exchange ideal. Clearly, I satisfies related comparability. Using Theorem 7, we complete the proof. $\square$

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