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4-DIMENSIONAL ANTI-KÄHLER MANIFOLDS AND WEYL CURVATURE

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Abstract. On a 4-dimensional anti-Kähler manifold, its zero scalar curvature implies that its Weyl curvature vanishes and vice versa. In particular any 4-dimensional anti-Kähler manifold with zero scalar curvature is flat.

Keywords: 4-dimensional anti-Kähler manifold, zero scalar curvature, Weyl curvature, flat

MSC 2000: 32J27, 53B30, 53C25, 53C56, 53C80

1. Introduction

An anti-Kähler manifold means a triple (M^{2m}, J, g) which consists of a smooth manifold M^{2m} of dimension $2m$, an almost complex structure J and an anti-Hermitian metric g such that $\bigtriangledown J = 0$ where $\bigtriangledown$ is the Levi-Civita connection of g. The metric g is called anti-Hermitian if it satisfies $g(JX, JY) = -g(X, Y)$ for all vector fields X and Y on M^{2m}. Then the metric g has necessarily a neutral signature (m, m) and M^{2m} is a complex manifold and there exists a holomorphic metric on M^{2m} [2]. This fact gives us some topological obstructions to an anti-Kähler manifold, for instance, all its odd Chern numbers vanish because its holomorphic metric gives us a complex isomorphism between the complex tangent bundle and its dual; and a compact simply connected Kähler manifold cannot be anti-Kähler because it does not admit a holomorphic metric. We extend J, g and the Levi-Civita connection $\bigtriangledown$ by C-linearity to the complexified tangent bundle $TM^C = TM \otimes C$. From now on we will use the same notations J, g and $\bigtriangledown$ for the complex extended almost complex structure of J, the complex extended metric of g and the complex

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extended Levi-Civita connection of $\bigtriangledown$, respectively. We will also use the same term anti-Kähler manifold for the anti-Kähler manifold considered as a complex manifold with the complex extended g and J. The purpose of this note is to prove the following result.

Theorem 1. *On a 4-dimensional anti-Kähler manifold, its zero scalar curvature implies that its Weyl curvature vanishes and vice versa. In particular, any 4-dimensional anti-Kähler manifold with zero scalar curvature is flat.*

In case of a 4-dimensional Kähler manifold, the above fact does not in general happen. For instance a K3 surface with the Calabi-Yau metric is not flat while it is a 4-dimensional Kähler manifold with zero scalar curvature. The crucial ingredient in the proof of this Theorem is the following: Every 4-dimensional anti-Kähler manifold is Einstein.

2. Properties of anti-Kähler manifolds

Let (M^{2m}, J, g) be a $2m$-dimensional anti-Kähler manifold. We extend J, g and the Levi-Civita connection of g by C-linearity to the complexification of the tangent bundle $TM^C = TM \otimes C$. Fix a (real) basis $\{X_1, \dots, X_m, JX_1, \dots, JX_m\}$ in each tangent space T_xM, then the set $\{Z_a, Z_{\bar{a}}\}$ where $Z_a = X_a - \mathrm{i}JX_a, Z_{\bar{a}} = X_a + \mathrm{i}JX_a$ form a basis for each complexified tangent space $T_xM \otimes C$. Unless otherwise stated, $a, b, c, \dots$ run from 1 to m while $A, B, C, \dots$ run through $1, \dots, m, \bar{1}, \dots, \bar{m}$. Then $JZ_a = \mathrm{i}Z_a$ and $JZ_{\bar{a}} = -\mathrm{i}Z_{\bar{a}}$. We set $g_{AB} = g(Z_A, Z_B) = g_{BA}$. Then the complex extended metric g satisfies $g_{a\bar{b}} = g_{\bar{b}a} = 0$ and $g_{\bar{A}\overline{B}} = \bar{g}_{AB}$. Conversely if the complex extended metric g_{AB} satisfies the above properties then the initial metric must be anti-Hermitian. It will be customary to write a metric satisfying the above properties as

$$(*) \qquad \mathrm{d}s^2 = g_{ab}\,\mathrm{d}z^a\,\mathrm{d}z^b + g_{\bar{a}\bar{b}}\,\mathrm{d}z^{\bar{a}}\,\mathrm{d}z^{\bar{b}}.$$

In an adapted almost-complex coordinate, $x^\mu = (x^a, y^a \equiv x^{m+a})$, $z^a = x^a + \mathrm{i}y^a$, one has $g_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu = 2\,\mathrm{Re}[g_{ab}\,\mathrm{d}z^a\,\mathrm{d}z^b]$, where μ, $\nu = 1, \dots, 2m$, $a, b = 1, \dots, m$ and Re means real part. We define now the complex Christoffel symbols Γ^C_{AB} by

$$(**) \qquad \bigtriangledown_{Z_A} Z_B = \Gamma^C_{AB} Z_C.$$

Since the complex extended Levi-Civita connection $\bigtriangledown$ has vanishing torsion, the complex Christoffel symbols are symmetric, i.e., $\Gamma^C_{AB} = \Gamma^C_{BA}$. In this case the complex structure J is integrable so that the real manifold M^{2m} inherits the structure of a complex manifold. Let us now recall that there is one to one correspondence between complex manifolds and real manifolds with an integrable complex structure. This means that there exists an atlas of real, adapted (local) coordinates $(x^1, \ldots, x^m, y^1, \ldots, y^m)$ such that $J(\partial/\partial x^a) = \partial/\partial y^a$, $J(\partial/\partial y^a) = -\partial/\partial x^a$. We set $X_a = \partial/\partial x^a$, $Z_a = X_a - \mathrm{i}JX_a = 2\partial_a$, $Z_{\bar{a}} = X_a + \mathrm{i}JX_a = 2\partial_{\bar{a}}$, where $\partial_A = \partial/\partial z^A$, $z^{\bar{a}} = \bar{z^a}$. It follows that (z^a) form an atlas of complex (analytic) coordinate charts on M. By using Christoffel formulas, one gets $\Gamma^C_{AB} = \frac{1}{2}g^{CD}(Z_A g_{BD} + Z_B g_{DA} - Z_D g_{AB}) = g^{CD}(\partial_A g_{BD} + \partial_B g_{DA} - \partial_D g_{AB})$. The curvature tensor R is then given (in complex coordinates) by the classical formula $R^A_{BCD} = \partial_C \Gamma^A_{BD} - \partial_D \Gamma^A_{BC} + \Gamma^A_{EC}\Gamma^E_{BD} - \Gamma^A_{ED}\Gamma^E_{BC}$.

Theorem 2. *Let (M^{2m}, J, g) be a $2m$-dimensional anti-Kähler manifold and $(z^1, \ldots, z^m)$ an inherited (local) complex coordinate system. Then*

(i) *the (complex) Christoffel symbols satisfy $\Gamma^C_{AB} = 0$ except for Γ^c_{ab} and $\Gamma^{\bar{c}}_{\bar{a}\bar{b}} = \bar{\Gamma}^c_{ab}$;*
(ii) *the components of the complex extended metric g_{ab} are holomorphic functions;*
(iii) *the curvature tensors satisfy $R^A_{BCD} = 0$ except for R^a_{bcd} and $R^{\bar{a}}_{\bar{b}\bar{c}\bar{d}} = \overline{R}^a_{bcd}$.*

P r o o f. From (**) we have $\Gamma^{\overline{C}}_{\bar{A}\overline{B}} = \bar{\Gamma}^C_{AB}$. Furthermore, since $\triangledown J = 0$ the connection satisfies the conditions $J(\triangledown_{Z_A} Z_B) = \triangledown_{Z_A}(JZ_B) = \triangledown_{Z_A}(\mathrm{i}Z_B) = \mathrm{i}\triangledown_{Z_A} Z_B$ and $J(\triangledown_{Z_A} Z_{\overline{B}}) = \triangledown_{Z_A}(JZ_{\overline{B}}) = \triangledown_{Z_A}(-\mathrm{i}Z_{\overline{B}}) = -\mathrm{i}\triangledown_{Z_A} Z_{\overline{B}}$. This implies $\Gamma^a_{B\bar{c}} = \Gamma^{\bar{a}}_{Bc} = 0$ for any B. In particular, $\Gamma^a_{b\bar{c}} = g^{aD}(\partial_b g_{\bar{c}D} + \partial_{\bar{c}} g_{Db} - \partial_D g_{b\bar{c}}) = g^{ad}\partial_{\bar{c}} g_{bd} = 0$. So $\partial_{\bar{c}} g_{bd} = 0$. By using the definition of the curvature tensor and the above described properties of Christoffel symbols, the desired properties of the curvature tensor are easily verified. This completes the proof. □

The Ricci tensor R_{AB} and the scalar curvature S are respectively given (in complex coordinate) as follows: by using the Einstein summation convention $R_{AB} = R^C_{ACB}$ and $S = g^{AB}R_{AB} = R^A_A$. Let (M^{2m}, g, J) be an anti-Kähler manifold, i.e., a complex manifold of complex dimension m with a holomorphic metric $g_{ab}(z)$, $a, b = 1, \ldots, m$ and a real metric $g_{\mu\nu}(x)$, μ, $\nu = 1, \ldots, 2m$ defined by (*).

Theorem 3. *The holomorphic metric $g_{ab}(z)$ is Einstein if and only if the real metric $g_{\mu\nu}(x)$ is a solution of the Einstein equations. In other words we have: $R_{ab} = (S/2m)g_{ab}$ if and only if $R_{\mu\nu} = (S/2m)g_{\mu\nu}$.*

P r o o f. By Theorem 2 we have $R_{a\bar{b}} = 0$. The (complex) Einstein equations $R_{AB}(g) = rg_{AB}$ are thus equivalent to the pair of equations

(***) $$R_{ab}(g_{cd}) = rg_{ab}$$

and

(****) $$R_{\bar{a}\bar{b}}(g_{\bar{c}\bar{d}}) = rg_{\bar{a}\bar{b}}.$$

To get a real solution of Einstein equations from (∗∗∗) and (∗∗∗∗), one uses the real coordinates (x^μ), $\mu = 1,\dots,2m$ on M, i.e., $z^a = x^a + \mathrm{i}x^{m+a}$, $a = 1,\dots,m$ and writes the real Ricci tensor $R_{\mu\nu}$ as $R_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu = R_{ab}\,\mathrm{d}z^a\,\mathrm{d}z^b + R_{\bar a\bar b}\,\mathrm{d}z^{\bar a}\,\mathrm{d}z^{\bar b}$. The result then follows.

3. The main result

In case of a 4-dimensional Kähler manifold, zero scalar curvature does not necessarily imply flatness. For instance a K3 surface with the Calabi-Yau metric is not flat while it is a 4-dimensional Kähler manifold with zero scalar curvature. On the other hand, we prove that on a 4-dimensional anti-Kähler manifold, its zero scalar curvature implies that its Weyl curvature vanishes and vice versa. In particular any 4-dimensional anti-Kähler manifold with zero scalar curvature is flat. The following Lemma 1 is a crucial ingredient to prove our main result.

Lemma 1. *Every 4-dimensional anti-Kähler manifold is Einstein.*

Proof. Let $\{E_1, E_2\}$ be a local unitary basis. Then we obtain $R_{11} = R_{22} = R_{1212}$ and $R_{\bar1\bar1} = R_{\bar2\bar2} = R_{\bar1\bar2\bar1\bar2} = \overline{R}_{1212}$ and $R_{AB} = 0$ for the other cases because we have $R^A_{BCD} = 0$ except for R^a_{bcd} and $R^{\bar a}_{\bar b\bar c\bar d} = \overline{R}^a_{bcd}$. These facts imply that every 4-dimensional anti-Kähler manifold is Einstein. This completes the proof. □

The Weyl curvature tensor W is written (in complex coordinates) as $W_{ABCD} = R_{ABCD} + \frac12(g_{AD}R_{BC} - g_{BD}R_{AC} + R_{AD}g_{BC} - R_{BD}g_{AC}) - \frac16 S(g_{AD}g_{BC} - g_{BD}g_{AC})$. Now we can prove the following main result.

Theorem 4. *On a 4-dimensional anti-Kähler manifold, its zero scalar curvature implies that its Weyl curvature vanishes and vice versa. In particular, any 4-dimensional anti-Kähler manifold with zero scalar curvature is flat.*

Proof. In local complex coordinates (z^1, z^2) on M^4, we have $W_{1212} = R_{1212} + \frac12(g_{12}R_{21} - g_{22}R_{11} + R_{12}g_{21} - R_{22}g_{11}) - \frac16 S(g_{12}g_{21} - g_{22}g_{11}) = -\frac16 S(g_{12}g_{21} - g_{22}g_{11})$. On the other hand, Lemma 1 and $R_{1\bar21\bar2} = 0$ imply that $W_{1\bar21\bar2} = R_{1\bar21\bar2} + \frac12(g_{1\bar2}R_{\bar21} - g_{\bar2\bar2}R_{11} + R_{1\bar2}g_{\bar21} - R_{\bar2\bar2}g_{11}) - \frac16 S(g_{1\bar2}g_{\bar21} - g_{\bar2\bar2}g_{11}) = \frac12(g_{1\bar2}\frac14 Sg_{\bar21} - g_{\bar2\bar2}\frac14 Sg_{11} + \frac14 Sg_{1\bar2}g_{\bar21} - \frac14 Sg_{\bar2\bar2}g_{11}) - \frac16 S(g_{1\bar2}g_{\bar21} - g_{\bar2\bar2}g_{11}) = -\frac1{12}S(g_{11}g_{\bar2\bar2})$. The following results can also be verified in the same manner: $W_{\bar1\bar2\bar1\bar2} = -\frac16 S(g_{\bar1\bar2}g_{\bar2\bar1} - g_{\bar2\bar2}g_{\bar1\bar1})$, $W_{\bar12\bar12} = -\frac1{12}S(g_{22}g_{\bar1\bar1})$, $W_{\bar11\bar11} = W_{1\bar11\bar1} = -\frac1{12}S(g_{11}g_{\bar1\bar1})$, $W_{\bar22\bar22} = W_{2\bar22\bar2} = -\frac1{12}S(g_{22}g_{\bar2\bar2})$ and $W_{ABCD} = 0$ for the other cases.

Hence, zero scalar curvature implies that the Weyl curvature vanishes and vice versa. In particular, zero scalar curvature implies that the curvature tensor must vanish because of the Einstein condition and vanishing Weyl curvature [1]. This completes the proof. □

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