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A NOTE ON RIESZ SPACES WITH PROPERTY-b

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Abstract. We study an order boundedness property in Riesz spaces and investigate Riesz spaces and Banach lattices enjoying this property.

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MSC 2000: 46B42, 46B28

1. Introduction and preliminaries

All Riesz spaces considered in this note have separating order duals. Therefore we will not distinguish between a Riesz space E and its image in the order bidual $E^{\sim\sim}$.

In all undefined terminology concerning Riesz spaces we will adhere to [3].

The notions of a Riesz space with property-b and b-order boundedness of operators between Riesz spaces were introduced in [1].

Definition. Let E be a Riesz space. A set $A \subset E$ is called b-order bounded in E if it is order bounded in $E^{\sim\sim}$. A Riesz space E is said to have property-b if each subset $A \subset E$ which is order bounded in $E^{\sim\sim}$ remains order bounded in E.

A Riesz space E has property-b if and only if each net $\{x_\alpha\}$ in E satisfying $0 \leqslant x_\alpha\!\uparrow \leqslant x$ for some x in $E^{\sim\sim}$ is order bounded in E.

Order bounded subsets of E are b-order bounded. If $\{x_\alpha\}$ is a net in a Banach lattice with $0 \leqslant x_\alpha\!\uparrow$ and is norm bounded, then $\{x_\alpha\}$ is a b-bounded subset of E.

Examples. Every perfect Riesz space and therefore every order dual has property-b. Every reflexive Banach lattice has property-b. Every KB space has property-b and if $E^{\sim\sim}$ is retractable on E then E has property-b. For an arbitrary compact Hausdorff space K, $C(K)$ has property-b [1]. On the other hand, by considering $\{e_n\}$ in c_0, we see that c_0 does not have property-b.

The weak Fatou property for directed sets of normed Riesz spaces implies the property-b. The Levi property and Zaanen's B-property also imply the property-b in Frechet lattices [1]. However, $C[0,1]$, with the supremum norm, has neither the Levi nor the b-property [1].

Example. Consider the Riesz space of all bounded real valued functions, $E = C_b[0,1]$ on $[0,1]$. Equipped with pointwise order and sup norm, E is an AM-space with unit. Let F be the Riesz subspace of all functions with countable support. That is, $f \in F$ if and only if $\{t \in [0,1] \colon f(t) \neq 0\}$ is countable. By considering the point evaluations, it is easy to see that F neither has property-b nor is order closed. F is a norm closed ideal of E. Let now $\{f_n\}$ be a sequence in F with $0 \leqslant f_n\uparrow \leqslant f$ for some $f \in E''$. Then $\sup f_n$ exists in F.

Guided by this example, a Riesz space E is said to have the countable b-property if each sequence (x_n) in E satisfying $0 \leqslant x_n\uparrow \leqslant x$ for some $x \in E^{\sim\sim}$ is order bounded in E. Although we will not pursue this line of investigation, let us remark that some of the results in this note have their analogues for the countable b-property.

Definition. Let E, F be Riesz spaces. An operator $T\colon E \to F$ is called b-order bounded if it maps b-order bounded subsets of E into b-order bounded subsets of F.

An order bounded operator between two Riesz spaces is b-order bounded. However, the converse need not be true.

Example. Let $T\colon L^1[0,1] \to c_0$ be defined by $T(f) = (\int_0^1 f(x)\sin x\,\mathrm{d}x, \int_0^1 f(x)\sin 2x\,\mathrm{d}x, \ldots)$. Then T is continuous and b-order bounded, but is not an order bounded operator.

2. Main results

Recall that an operator T between Riesz spaces E and F is called pre-regular if $T\colon E \to F^{\sim\sim}$ is order bounded.

Pre-regular operators between Banach lattices were introduced in [4]. The following result shows the relation between pre-regular and b-order bounded operators. The proof is routine and is omitted.

Proposition 1. *Let E and F be Riesz spaces, and let T be an operator from E into F. Then the following statements are equivalent:*

(i) *T is a pre-regular operator.*

(ii) *$T^{\sim}\colon F^{\sim} \to E^{\sim}$ is order bounded.*

(iii) $T^{\sim\sim}\colon E^{\sim\sim} \to F^{\sim\sim}$ *is order bounded.*
(iv) T *is b-order bounded.*

If $T\colon E \to F$ is an operator between Riesz spaces E and F, it follows from the proposition that if an n-th order adjoint of T is order bounded for some n, then every order adjoint of T is also order bounded. As operators mapping order intervals into norm bounded sets are known to be continuous mappings between Banach lattices, it follows that a b-order bounded operator between Banach lattices is continuous.

The proof of the next proposition is similar to the proof of the fact that the order adjoint of an order bounded operator is order continuous; so the proof is omitted. [Cf. 3, Theorem 5.8.]

Proposition 2. *If $T\colon E \to F$ is a b-order bounded operator between two Riesz spaces, then its order adjoint $T^{\sim}$ is order continuous.*

The following proposition gives a characterization of property-b in Banach lattices. The space of order bounded operators between Riesz spaces will be denoted by $\mathcal{L}_b$ (E, F) and it will be equipped with the regular norm.

Proposition 3. *Let E and F be Banach lattices, and let F be Dedekind complete. Then $\mathcal{L}_b(E, F)$ has property-b if and only if F has property-b.*

Proof. Assume $\mathcal{L}_b(E,F)$ has property-b. Take a net $\{y_\alpha\}$ in F with $0 \leqslant y_\alpha\!\uparrow \leqslant y''$ in F''. We have $0 \leqslant \|y_\alpha\|\!\uparrow \leqslant \|y''\|$. Then we choose $x_0 \in E^+$ and $f \in (E')^+$ such that $f(x_0) = 1$. Define T_α by $T_\alpha(x) = f(x)y_\alpha$ for each α. Each T_α is a positive operator and $0 \leqslant T_\alpha\!\uparrow$ holds in $\mathcal{L}_b(E,F)$. We have

$$\|T_\alpha\|_r = \|\,|T_\alpha|\,\| = \|f \otimes y_\alpha\| = \|f\|\,\|y_\alpha\| \leqslant \|f\|\,\|y''\|$$

for all α. We also have $0 \leqslant T_\alpha\!\uparrow \leqslant T^*$ in $[\mathcal{L}_b(E,F)]''$ for some $T^* \in [\mathcal{L}_b(E,F)]''$.

By hypothesis, there exists T in $\mathcal{L}_b(E,F)$ such that $T_\alpha \leqslant T$. In particular,

$$0 \leqslant T_\alpha(x_0) = (f \otimes y_\alpha)(x_0) = f(x_0)y_\alpha = y_\alpha \leqslant T(x_0)$$

for all α. Since $T(x_0) \in F$, F has property-b.

Suppose now F has property-b. Suppose (T_α) is a net in $\mathcal{L}_b(E,F)$ with $0 \leqslant T_\alpha \uparrow T^*$ in $[\mathcal{L}_b(E,F)]''$, then $\|T_\alpha\|_r \leqslant \|T^*\|$ for all α. Therefore, we have

$$\|T_\alpha(x)\| \leqslant \|T_\alpha\|_r\|x\| \leqslant \|T^*\|\,\|x\|$$

for each $x \in E^+$. As $(T_\alpha(x))$ is increasing and norm bounded in F, there exists $\overline{x} \in F''$ such that $0 \leqslant T_\alpha(x) \leqslant \overline{x}$ for each $x \in E^+$. By hypothesis, there exists $y \in F$

such that $0 \leqslant T_\alpha(x) \uparrow y$ for each $x \in E^+$. Since F is Dedekind complete, we conclude that $\sup_\alpha T_\alpha(x) \in F$ and the mapping $T\colon E^+ \to F^+$ defined by $T(x) = \sup_\alpha T_\alpha(x)$ is additive on E^+. We can extend T to a unique positive operator on E. Clearly, $0 \leqslant T_\alpha \leqslant T$ and $\mathcal{L}_b(E,F)$ has property-b. $\square$

Let E, F be Banach lattices and F be Dedekind complete. If $\mathcal{L}_b(E,F) = \mathcal{L}(E,F)$, where $\mathcal{L}(E,F)$ is the space of continuous operators, then F has property-b if and only if $\mathcal{L}(E,F)$ has property-b. As one particular case, let us note that if E is an AL-space and F is a Banach lattice with property-b, then $\mathcal{L}(E,F) = \mathcal{L}_b(E,F)$ [1].

As we have noted the space $C(K)$, where K is a compact Hausdorff space, has property-b. On the other hand, let $c_0(\Omega)$ be the space of all real-valued functions f on a compact Hausdorff space Ω such that for each $\varepsilon > 0$ there exists a finite subset Φ of Ω satisfying $|f(w)| < \varepsilon$ for all $w \notin \Phi$. Then $c_0(\Omega)$ has property-b if and only if Ω is finite.

In general, property-b does not pass to subspaces. For example, each infinite dimensional closed Riesz subspace of $C(K)$ contains a copy of c_0. Specializing to Riesz subspaces of $C(K)$ we have the following result.

Proposition 4. *If E is an AM-space, then E has property-b if and only if E has an order unit.*

P r o o f. Let E be an AM-space with property-b. Then E'' is a Dedekind complete AM-space with order unit denoted by φ. The set $[0,\varphi] \cap E$ is order bounded in E''. Since E has property-b, there exists an element e in E such that $0 \leqslant f \leqslant e$ for each $f \in [0,\varphi] \cap E$. It is now easy to show that e is an order unit in E. $\square$

Let now Q be an infinite quasi-Stonean space without isolated points. We denote by $CD_0(Q)$ the space $c_0(Q) \oplus C(Q)$. $CD_0(Q)$ is a Riesz space under pointwise ordering, and a normed Riesz space under the supremum norm. Since Q has no isolated points the decomposition of an element of $CD_0(Q)$ into a sum of a function from $C(Q)$ and a function from $c_0(Q)$ is unique. Thus there is a projection from $CD_0(Q)$ onto $c_0(Q)$. Since $CD_0(Q)$ is an AM-space with unit, it therefore has property-b. Although $c_0(Q)$ is the range of a projection from $CD_0(Q)$, it does not have property-b. On the other hand, every projection band in a Riesz space with property-b has the same property.

Proposition 5. *Every order closed Riesz subspace of a Dedekind complete Riesz space E with property-b has property-b.*

P r o o f. Let F be an order closed Riesz subspace of E and $\{x_\alpha\} \subset F$ be a net with $0 \leqslant x_\alpha\uparrow$ such that $0 \leqslant x_\alpha\uparrow \leqslant x''$ in $F^{\sim\sim}$ for some $x'' \in (F^{\sim\sim})^+$. It follows that

$0 \leqslant x_\alpha\uparrow \leqslant x''$ in $E^{\sim\sim}$. Since E has property-b, there exists $z \in E$ with $0 \leqslant x_\alpha \leqslant z$. Let $x = \sup\{x_\alpha\colon\ \alpha \in \Lambda\}$. Then x is in E by Dedekind completeness of E. As the net $\{x_\alpha\}$ is increasing, it is order convergent to x, and so $x \in F$. $\square$

If a Banach lattice E is a band in E'' then E has property-b. On the other hand, property-b and the conditions ensuring that E is an ideal in E'' imply that E is a band of E''. One such case is given below.

Corollary. *Let E be a Banach lattice. E is a band in E'' if and only if E has property-b and every relatively weakly compact subset of E^+ has relatively weakly compact solid hull.*

Let us note that the space c_0 shows that property-b is indispensable in the Corollary.

Let c_{00} be the space of sequences having only finitely many non-zero terms and l_∞ be the space of all bounded sequences. Although c_{00} is not order closed in l_∞, c_{00} has property-b. However, for norm closed ideals the situation is different.

Corollary. *Let E be a Dedekind complete Banach lattice with property-b and let F be a norm closed ideal F of E. A necessary and sufficient condition for F to have property-b is the order closedness of F in E.*

The topology $|\sigma|(E', E)$ on E' is the topology of uniform convergence on order intervals of E. In general we have $|\sigma|(E', E) \leqslant |\sigma|(E', E'')$. Equality of these topologies gives rise to property-b.

Proposition 6. *If E is a Banach lattice and $|\sigma|(E', E) = |\sigma|(E', E'')$, then E has property-b.*

Proof. Let (x_α) be a net in E such that $0 \leqslant x_\alpha\uparrow \leqslant u''$ for some $u'' \in E''$. There exists some y'' in $(E'')^+$ satisfying $0 \leqslant x_\alpha \uparrow y''$. We choose z in E with $[-z, z]^\circ \subseteq [0, y'']^\circ \subseteq \{x_\alpha\colon\ a \in \Lambda\}^\circ$ in E'. Taking polars in E, we have $\{x_\alpha\colon\ \alpha \in \lambda\}^{\circ\circ} \subseteq [0, y'']^{\circ\circ} \subseteq [-z, z]^{\circ\circ}$. By the Bipolar theorem, we have $[-z, z]^{\circ\circ} = [-z, z]$. It follows that $\{x_\alpha\colon\ \alpha \in \Lambda\}^{\circ\circ} \subseteq [-z, z]$ which implies $0 \leqslant x_\alpha \leqslant z$ in E, and E has property-b. $\square$

Let us note that $|\sigma|(E', E) = |\sigma|(E', E'')$ does not characterize Banach lattices with property-b. $E = l^1$ has property-b, but on $E' = l^\infty$, $|\sigma|(E', E) < |\sigma|(E', E'')$ as the next proposition shows.

Proposition 7. *Let E be a Banach lattice. Suppose $|\sigma|(E', E)$ and $|\sigma|(E', E'')$, have the same convergent sequences. Then E' is a KB-space.*

Proof. Let $\{f_n\}$ be a sequence in E' with $0 \leqslant f_n\uparrow \leqslant f$ for some $f \in E'$. $g = \sup\limits_n f_n$ exists in E' and $g - f_n \downarrow 0$. Then we have $f_n \to g$ in $|\sigma|(E', E)$ since $|\sigma|(E', E)$ is an order continuous topology. Hence $f_n \to g$ in $|\sigma|(E', E'')$ by the hypothesis. Monotoneness of the sequence $\{f_n\}$ and convergence of it in $|\sigma|(E', E'')$ yield convergence of $\{f_n\}$ in the norm topology by Theorem 11.8 in [3]. Thus E' has order continuous norm. □

One special case of the proposition is when the Banach lattice E is a Grothendieck space. That is, the topologies $\sigma(E', E)$ and $\sigma(E', E'')$ have the same convergent sequences. In this case the topologies $|\sigma|(E', E)$ and $|\sigma|(E', E'')$ also have the same convergent sequences and thus E' is a KB-space.

Let E be a Banach lattice and H be a Banach space. An operator $T\colon\ E \to H$ is called cone absolutely summing if there exists a constant $l \geqslant 0$ with $\sum\limits_i \|Tx_i\| \leqslant l \, \|\sum x_i\|$ for all finite families (x_i) in E^+. The smallest constant l satisfying the preceding inequality is a norm on the cone absolutely summing operators and will be denoted by $\|\cdot\|_l$. $\mathcal{L}_l(E, F)$ will denote the cone absolutely summing operators. Recall that a Banach lattice F is said to have property-P if there exists a positive, contractive projection $P\colon\ F'' \to F$. It is well known if E, F are Banach lattices, F having property-P then $\mathcal{L}_l(E, F)$ is a Banach lattice and order ideal of $\mathcal{L}_b(E, F)$ [5].

We generalize this to Banach lattices with property-b serving as range.

Proposition 8. *Let E, F be Banach lattices with F Dedekind complete and having property-b. Then $\mathcal{L}_l(E, F)$ is a Banach lattice and an order ideal of $\mathcal{L}_b(E, F)$.*

Proof. Let $T \in \mathcal{L}_l(E, F)$. Then there exists an operator S, $0 \leqslant S$ in $\mathcal{L}_l(E, F'')$ such that $\|S\|_l = \|T\|_l$ and $\pm T \leqslant S$. Let $x \in E^+$ and y be such that $|y| \leqslant x$. Then $|Ty| \leqslant S|y| \leqslant S(x)$ in F''. By property-b in F, we conclude that $T[-x, x]$ is order bounded in F. Thus $|T|$ exists in $\mathcal{L}_b(E, F)$. Let (x_n) be a summable sequence in E^+. Then $\sum\limits_n \| \, |T|x_n\| \leqslant \sum\limits_n \|Sx_n\| < \infty$ shows that $|T| \in \mathcal{L}_l(E, F)$. Moreover, $\| \, |T| \, \|_l \leqslant \|S\|_l = \|T\|_l$.

This shows that $\mathcal{L}_l(E, F)$ is a Riesz subspace of $\mathcal{L}_b(E, F)$. As $\|T\|_l \leqslant \| \, |T| \, \|_l$ and the l-norm is monotone on the positive cone of $\mathcal{L}_b(E, F)$, l-norm is a lattice norm. Hence $\mathcal{L}_l(E, F)$ is a Banach lattice and order ideal of $\mathcal{L}_b(E, F)$. □

Let E be a Banach space and F be a Banach lattice. Then an operator $T\colon\ E \to F$ is called majorizing if there exists $y'' \in F''_+$, $\|y''\| \leqslant m$ such that T maps the unit ball of E into an interval $[-y'', y'']$ of F''. The m-norm $\|T\|_m$ is the smallest constant

m satisfying the above condition. $\mathcal{L}_m(E,F)$ will denote the space of majorizing operators.

The proof of the next proposition is similar to the proof of the previous proposition and is omitted.

Proposition 9. *Let E, F be Banach lattices with F Dedekind complete and having property-b. Then $\mathcal{L}_m(E,F)$ is a Banach lattice and order ideal of $\mathcal{L}_b(E,F)$.*

Let $(U(X,Y),\vartheta)$ be an operator Banach space where the norm ϑ satisfies $\|T\| \leqslant c\vartheta(T)$ for each $T \in U(X,Y)$.

Definition. a) An operator Banach space $(U(X,Y),\vartheta)$ is called boundedly closed if whenever $T\colon X \to Y$ is an operator such that there is a bounded sequence (T_n) in $U(X,Y)$ with $\lim_{n\to\infty} T_n x = Tx$ for each $x \in X$ then $T \in U(X,Y)$.

b) An operator Banach space $(U(X,Y),\vartheta)$ is called d-boundedly closed if whenever $T\colon X \to Y$ is an operator such that there exists a bounded sequence (T_n) in $U(X,Y)$ satisfying $T'y' = \lim_{n\to\infty} T_n' y'$, for all $y' \in Y'$, then $T \in U(X,Y)$.

The operator Banach space $\mathcal{L}_m(E)$ of majorizing operators on a Banach lattice E is boundedly closed in the m-norm. As adjoints of cone absolutely summing operators are majorizing [5], it follows that the operator Banach space $\mathcal{L}_l(E)$ on E is d-boundedly closed in the l-norm.

If E and F are Banach spaces, each containing a complemented copy of c_0, then $\mathcal{W}(E,F)$, the space of weakly compact operators, is not complemented in $\mathcal{L}(E,F)$. Among other things, this was extended to other operator spaces in [6]. The next result is an immediate consequence of [6, Theorem 6].

Proposition 10. a) *Let E be a Dedekind complete Banach lattice with property-b. Then if $\mathcal{L}_m(E)$ contains a complemented copy of c_0, then c_0 is contained isomorphically in E.*

b) *If $\mathcal{L}_l(E)$ contains a complemented copy of c_0, then either E' or E contains a copy of c_0.*

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