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LIMIT POINTS OF EIGENVALUES OF (DI)GRAPHS

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Abstract. The study on limit points of eigenvalues of undirected graphs was initiated by A. J. Hoffman in 1972. Now we extend the study to digraphs. We prove:

1. Every real number is a limit point of eigenvalues of graphs. Every complex number is a limit point of eigenvalues of digraphs.
2. For a digraph D, the set of limit points of eigenvalues of iterated subdivision digraphs of D is the unit circle in the complex plane if and only if D has a directed cycle.
3. Every limit point of eigenvalues of a set $\mathcal{D}$ of digraphs (graphs) is a limit point of eigenvalues of a set $\ddot{\mathcal{D}}$ of bipartite digraphs (graphs), where $\ddot{\mathcal{D}}$ consists of the double covers of the members in $\mathcal{D}$.
4. Every limit point of eigenvalues of a set $\mathcal{D}$ of digraphs is a limit point of eigenvalues of line digraphs of the digraphs in $\mathcal{D}$.
5. If M is a limit point of the largest eigenvalues of graphs, then $-M$ is a limit point of the smallest eigenvalues of graphs.

Keywords: limit point, eigenvalue of digraph (graph), double cover, subdivision digraph, line digraph

MSC 2000: 05C50, 15A48

1. Introduction

Since A. J. Hoffman [9] initiated the study of limit points of graph eigenvalues in 1972, many interesting results have been obtained on this topic (see, for example, [2]–[7], [9], [10], [15]). Hoffman [9] studied limit points of the largest eigenvalues of graphs, and he determined all limit points less than $\sqrt{2+\sqrt{5}}$ (the golden mean) and showed that these limit points constitute an increasing sequence (a_n) with $a_1 = 2$ and $\lim\limits_{n\to\infty} a_n = \sqrt{2+\sqrt{5}}$. He also suggested that possibly there exists a real number λ such that every number not less than λ is a limit point of the largest eigenvalues

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of graphs. In fact this is true with $\lambda = \sqrt{2+\sqrt{5}}$, as proved by Shearer [15]. From these works the set of limit points of the largest eigenvalues of graphs has been determined. For $k \geqslant 2$, the limit points of the kth largest eigenvalues of graphs have been studied for a long time, but they have not been completely determined. Dasong Cao and Hong Yuan [2], [3] showed that for each $k \geqslant 2$ there is a gap in the set L_k of limit points of the kth largest eigenvalues of graphs, and that $L_k \subseteq L_{k+1}$ for all k. They conjectured in [3] that $\lim_{k\to\infty} L_k = R$, the set of all real numbers.

In this paper we will first prove that every real number is a limit point of eigenvalues of graphs, and then extend the study to the limit points of eigenvalues of digraphs.

We follow [1] for general graph theoretical terminology. All digraphs in this paper are finite without loops or multiple arcs. Undirected simple graphs are simply called graphs. Let D be a digraph (graph) with adjacency matrix A. The charateristic polynomial of D is $\chi(D, x) = |xI - A|$, where I denotes the identity matrix. The roots (complex numbers) of $\chi(D, x)$ are called the eigenvalues of D. It is well known that all eigenvalues of D must be real numbers when D is a graph.

To state our main results, we need the following

Definition 1. Let $\mathcal{D}$ denote an infinite set of digraphs (graphs). The complex number ζ is said to be a *limit point of igenvalues of* $\mathcal{D}$ if there is an infinite sequence of distinct complex numbers λ_n, each of which is an eigenvalue of a digraph (graph) in $\mathcal{D}$, such that $\zeta = \lim_{n\to\infty} \lambda_n$.

Remarks.

(i) When $\mathcal{D}$ is the set of all digraphs (graphs), we simply say that ζ is a limit point of digraph (graph) eigenvalues.

(ii) Every limit point of graph eigenvalues is a limit point of digraph eigenvalues.

Remark (ii) can be justified as follows. For any graph G there corresponds a unique digraph $D(G)$, called its associated digraph, which is obtained when each edge e of G is replaced by two oppositely oriented arcs with the same ends as e. It is obvious that a graph G and its associated digraph $D(G)$ have the same adjacency matrix, and so they have the same eigenvalues. Hence, when we consider eigenvalues, a graph can be seen as a digraph with a symmetric adjacency matrix.

We say that a digraph is bipartite if its underlying graph is bipartite. Note that a bipartite graph and a bipartite digraph both have an adjacency matrix in the block form $\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}$, where 0 denotes the block with all entries zero. While for bipartite graphs B must equal to the transpose A^T of A, there is no such restriction for bipartite digraphs.

For an n by n matrix A, let $\ddot{A}$ denote the $2n$ by $2n$ matrix $\begin{bmatrix} 0 & A \\ A & 0 \end{bmatrix}$.

Definition 2. Let D be a digraph (graph) with adjacency matrix A. The digraph (graph) with the adjacency matrix $\ddot{A}$ is called the *double cover of* D, denoted by $\ddot{D}$.

Remarks.

(i) $\ddot{D}$ is a graph when D is a graph, since $\ddot{A}$ is symmetric when A is symmetric.

(ii) $\ddot{D}$ is a bipartite digraph (graph) when D is a digraph (graph).

The concept of line digraphs was introduced in Harary and Norman [8]. For a digraph D with vertex set $V(D)$ and arc set $A(D)$, the *line digraph* $L(D)$ of D has $A(D)$ as its vertex set; (a, b) is an arc of $L(D)$ if and only if there are vertices u, v, w in D with $a = (u, v)$ and $b = (v, w)$, i.e., the head of a coincides with the tail of b. The subdivision digraph $S(D)$ of D is obtained by inserting a new vertex onto every arc of D; that is, each arc (u, v) of D is replaced by two new arcs (u, w) and (w, v) where w is a new vertex. The iterated subdivision digraphs are defined inductively by $S^r(D) = S(S^{r-1}(D))$ with $S^0(D) = D$ and $S^1(D) = S(D)$.

We will prove:

1. Every real number is a limit point of eigenvalues of graphs. Every complex number is a limit point of eigenvalues of digraphs.
2. For a digraph D, the set of limit points of eigenvalues of iterated subdivision digraphs of D is the unit circle in the complex plane if and only if D has a directed cycle.
3. Every limit point of eigenvalues of a set $\mathcal{D}$ of digraphs (graphs) is a limit point of eigenvalues of a set $\ddot{\mathcal{D}}$ of bipartite digraphs (graphs), where $\ddot{\mathcal{D}}$ consists of the double covers of the members in $\mathcal{D}$.
4. Every limit point of eigenvalues of a set $\mathcal{D}$ of digraphs is a limit point of eigenvalues of line digraphs of the digraphs in $\mathcal{D}$.
5. If M is a limit point of the largest eigenvalues of graphs, then $-M$ is a limit point of the smallest eigenvalues of graphs.

2. Lemmas

Lemma 1. *Let D be a digraph or graph. If the eigenvalues of D are λ_i ($i = 1, \ldots, n$), then the eigenvalues of the double cover $\ddot{D}$ of D are $\pm\lambda_i$ ($i = 1, \ldots, n$).*

P r o o f. Let A be the adjacency matrix of D. Then

$$\chi(\ddot{D}, x) = |xI - \ddot{A}| = \begin{vmatrix} xI & -A \\ -A & xI \end{vmatrix}.$$

It is well known in matrix theory (see, for example, [11, p. 45]) that if M is an invertible matrix then $\begin{vmatrix} M & N \\ P & Q \end{vmatrix} = |M| \cdot \left|Q - PM^{-1}N\right|$. So,

$$\begin{aligned}
\chi(\ddot{D}, x) &= x^n \left| xI - (-A)\Big(\frac{1}{x}I\Big)(-A) \right| \\
&= x^n \left| xI - \frac{1}{x}A^2 \right| = |x^2 I - A^2| = |xI - A| \cdot |xI + A| \\
&= (-1)^n |xI - A| \cdot |-xI - A| = (-1)^n \chi(D, x)\chi(D, -x).
\end{aligned}$$

This proves Lemma 1. □

Let D_1 and D_2 be digraphs (graphs) with matrices A and B, respectively. The Kronecker product $D_1 \otimes D_2$ is the digraph (graph) with adjacency matrix $A \otimes B$, the Kronecker product of the matrix A with the matrix B. From matrix theory (see, e.g., [14, p. 24]) we know that the set of the eigenvalues of a Kronecker product $A \otimes B$ of square matrices A and B is the same as the set of all possible products $\lambda_A \lambda_B$ where λ_A and λ_B are eigenvalues of A and B, respectively. Since the Kronecker product is associative, we have the following Lemma 2.

Lemma 2. *Let D_i denote digraphs (graphs), $i = 1, 2, \ldots, n$. The set of eigenvalues of $D_1 \otimes D_2 \otimes \ldots \otimes D_n$ is the same as the set of all possible products $\lambda_1 \lambda_2 \ldots \lambda_n$, where λ_i is an eigenvalue of D_i.*

Lemma 3. *If a real number r is a limit point of graph eigenvalues, then*

i) *$-r$ is also a limit point of graph eigenvalues;*

ii) *every point on the circle with radius $|r|$ centered at the origin in the complex plane is a limit point of digraph eigenvalues.*

P r o o f. i) If λ is an eigenvalue of a graph D, then by Lemma 1, $-\lambda$ is an eigenvalue of $\ddot{D}$. This implies i) directly.

ii) From Remark (ii) following Definition 1, r is also a limit point of digraph eigenvalues. Let $r = \lim_{n\to\infty} \lambda_n$, where λ_n are distinct and each λ_n is an eigenvalue of a digraph G_n. It is well known (see [4, p. 53]) that the eigenvalues of a directed cycle D_n with n vertices are $\exp(\mathrm{i}2k\pi/n)$ ($k = 0, 1, \ldots, n-1$; $\mathrm{i} = \sqrt{-1}$), the nth complex roots of unit. Note that these n complex roots are evenly distributed in the unit circle on the complex plane.

Now, let $\mathcal{D}$ denote the set of all digraphs $G_n \otimes D_n$. By Lemma 2, $\lambda_n \exp(\mathrm{i}2k\pi/n)$ ($k = 0, 1, \ldots, n-1$) are eigenvalues of $G_n \otimes D_n$. Note that these n complex numbers are evenly distributed on the circle with radius $|\lambda_n|$ centered at the origin in the complex plane, and that $\lim_{n\to\infty} |\lambda_n| = |r|$. Then it is easily seen that every point on the circle with radius $|r|$ centered at the origin in the complex plane is a limit point of eigenvalues of $\mathcal{D}$, and so it is a limit point of digraph eigenvalues. This completes the proof for Lemma 3. □

Lemma 4. *A digraph D has a nonzero eigenvalue if and only if D has a directed cycle.*

Proof. From the "Coefficients Theorem for Digraphs" (see [4, p. 32]), we know that for a digraph D with n vertices,

$$\chi(D,x) = x^n + a_1x^{n-1} + \ldots + a_n \quad \text{with } a_i = \sum_{L\in\mathcal{L}_i} (-1)^{p(L)} \quad (i = 1, 2, \ldots, n)$$

where $\mathcal{L}_i$ is the set of all such subdigraphs L of D with exactly i vertices that each component of L is a directed cycle; $p(L)$ denotes the number of components of L.

So, the necessity of the lemma follows immediately. To show the sufficiency, since D has a directed cycle, we consider the directed cycles of D with the shortest length ϱ. Then we see that in $\chi(D,x) = x^n + a_1x^{n-1} + \ldots + a_n$, $a_\varrho \neq 0$, and this proves the sufficiency. □

Lemma 5 (Lin and Zhang [12]; also see [13, p. 19])**.** *Let D be a digraph with n vertices and m arcs. Then $\chi(L(D),x) = x^{m-n}\chi(D,x)$.*

Lemma 6 (Zhang, Lin, and Meng [16])**.** *Let D be a digraph with n vertices and m arcs. Then $\chi(S(D),x) = x^{m-n}\chi(D,x^2)$.*

3. Main results

Theorem 1.

i) *Every real number r is a limit point of graph eigenvalues.*

ii) *Every complex number ζ is a limit point of digraph eigenvalues.*

Proof. i) It is well known that a cycle C_n with n vertices has eigenvalues $2\cos(2k\pi/n)$, $k = 1, 2, \ldots, n$. So, it is obvious that $r = 0$ is a limit point of graph eigenvalues. Then, by Lemma 3 (i), we may assume $r > 0$ without loss of generality. Note that C_{4n+1} has a positive eigenvalue $\lambda_n = 2\cos(2n\pi/(4n+1))$. Obviously $\lambda_n \to 0$ as $n \to \infty$. So, for any $0 < \varepsilon < \frac{1}{2}r$, we may take an n such that $\lambda_n < \frac{1}{2}\varepsilon$. Let $N = \lfloor (r - \frac{1}{2}\varepsilon)/\lambda_n \rfloor$. Then $N \leqslant (r - \frac{1}{2}\varepsilon)/\lambda_n < N+1$. It follows that $N\lambda_n \leqslant r - \frac{1}{2}\varepsilon < r$ and that $N\lambda_n > r - \frac{1}{2}\varepsilon - \lambda_n > r - \varepsilon$. Thus we have $r - \varepsilon < N\lambda_n < r$. Now let G_ε denote the product graph of N copies of C_{4n+1}. It is well known (see, e.g., [4]) that the eigenvalues of the product graph $\Gamma_1 \times \Gamma_2$ of two graphs Γ_1 and Γ_2 are the sums of eigenvalues of Γ_1 with those of Γ_2. (Note that $\Gamma_1 \times \Gamma_2$ is denoted differently in [4] as $\Gamma_1 + \Gamma_2$.) Hence, $N\lambda_n$ are eigenvalues of G_ε. Therefore, for any $\varepsilon > 0$ there is a graph G_ε with an eigenvalue in the open interval $(r - \varepsilon, r)$. It clearly implies i).

ii) directly follows from i) and Lemma 3 (ii). □

It should be pointed out that the graphs and digraphs in Theorem 1 can be restricted as regular graphs and regular digraphs, which can be easily seen from the proof of Theorem 1.

Theorem 2. *For a digraph D, the set of limit points of eigenvalues of iterated subdivision digraphs of D is the unit circle in the complex plane if and only if D has a directed cycle.*

Proof. If D has no directed cycles, then all the iterated subdivision digraphs $S^r(D)$ have no directed cycles. Then by Lemma 4, all eigenvalues of $S^r(D)$ are zero so that no limit points exist.

Let D be a digraph with a directed cycle. Then by Lemma 4, D has nonzero eigenvalues. Let α be any one of the nonzero eigenvalues. By Lemma 6, the subdivision digraph $S(D)$ has the two square roots of α among its eigenvalues. Clearly, for any positive integer r, the iterated subdivision digraph $S^r(D)$ has all the 2^rth roots of α among its eigenvalues. Note that in the complex plane the 2^rth roots of α are evenly distributed on the circle centered at the origin with the radius equal to the 2^rth root of the modulus of α. Since $\lim_{r\to\infty} 2^r = \infty$, $\lim_{r\to\infty} \sqrt[2^r]{|\alpha|} = 1$, and every nonzero eigenvalue of $S^r(D)$ is a 2^rth root of a nonzero eigenvalue of D, we see that the set of the nonzero eigenvalues of $S^r(D)$ approaches the unit circle in the complex plane as $r \to \infty$. Hence Theorem 2 follows. □

When we consider the limit points of eigenvalues of a set $\mathcal{D}$ of digraphs (graphs), the next two theorems are helpful.

Theorem 3. *Let $\mathcal{D}$ be an infinite set of digraphs (graphs), and let $\ddot{\mathcal{D}}$ denote the set of the double covers of digraphs (graphs) in $\mathcal{D}$. Every limit point of eigenvalues of $\mathcal{D}$ is a limit point of eigenvalues of $\ddot{\mathcal{D}}$.*

Proof. It follows from Lemma 1. □

From Theorems 1 and 3, we immediately have

Corollary 1. *Every real number is a limit point of eigenvalues of bipartite graphs. Every complex number is a limit point of eigenvalues of bipartite digraphs.*

For an infinite set $\mathcal{D}$ of digraphs, let $\vec{\mathcal{D}}$ denote the set of all line digraphs $L(D)$ with D in $\mathcal{D}$. We have

Theorem 4. *Every limit point of eigenvalues of $\mathcal{D}$ is a limit point of eigenvalues of $\vec{\mathcal{D}}$.*

P r o o f. It follows from Lemma 5. □

From Theorems 1 and 4, we immediately have

Corollary 2. *Every complex number is a limit point of eigenvalues of line digraphs.*

Note that if λ is the largest eigenvalue of a graph G, then by Lemma 1, $-\lambda$ is the smallest eigenvalue of the double cover $\ddot{G}$ of G. This implies

Theorem 5. *If M is a limit point of the largest eigenvalues of graphs, then $-M$ is a limit point of the smallest eigenvalues of graphs.*

By Theorem 5 we can obtain the following result of [6].

Corollary 3. *Every number in the interval $\left(-\infty, -\sqrt{2+\sqrt{5}}\right]$ is a limit point of the smallest eigenvalues of graphs.*

P r o o f. It was initiated by Hoffman [9] and proved by Shearer [15] that every number not less than $\sqrt{2+\sqrt{5}}$ (the golden mean) is a limit point of the largest eigenvalues of graphs. Then Corollary 3 immediately follows from Theorem 5. □

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