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PTÁK'S CHARACTERIZATION OF REFLEXIVITY IN TENSOR PRODUCTS

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Abstract. We characterize the reflexivity of the completed projective tensor products $X\widetilde{\otimes}_\pi Y$ of Banach spaces in terms of certain approximative biorthogonal systems.

Keywords: reflexive Banach space, biorthogonal system, π-tensor product

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Pták [10] proved among other results that a Banach space E is reflexive iff every bounded biorthogonal system $\{(e_n, f_n)\}_{n=1}^\infty \subset E \times E^*$ has unbounded sequence of partial sums $b_i = \sum\limits_{n=1}^{i} e_n$. Here a bounded biorthogonal system in (E, E^*) is a sequence $\{(e_n, f_n)\}_{n=1}^\infty \subset E \times E^*$ such that $\langle f_j, e_i\rangle = \delta_{ij}$ and $\sup\limits_n \|e_n\| < \infty$, $\sup\limits_n \|f_n\| < \infty$. Other characterizations of reflexivity which stem from Pták's results are for example the results of Singer [13], [14] and Pełczyński [9]. This paper complements the papers [10], [13], [14], [9], [6], in which reflexivity is characterized by reflexivity of subspaces or quotients having Schauder basis or having complete biorthogonal system [10].

Pták constructs in every nonreflexive Banach space a bounded biorthogonal system $\{(e_n, f_n)\}_{n=1}^\infty \subset E \times E^*$ with bounded sequence of partial sums. Here we construct similar systems in the nonreflexive tensor products $X\widetilde{\otimes}_\pi Y$. For later use we will construct such systems in a special form. Namely we observe that the partial sums $b_i = \sum\limits_{n=1}^{i} e_n$ may be chosen in the form $b_i = x_i \otimes y_i$ where the x_i are elements of the unit ball B_X of X and similarly $y_i \in B_Y$. However, we were able to make such a special choice of b_i's only approximately in the sense that the resulting system is biorthogonal only up to arbitrary small perturbations. As in [10] the sequences $\{b_i\}$ and $\{f_j\}$ are constructed first.

The definition below makes this precise and keeps the notation used in [10] except that the y_j's and y are called here f_j and f.

Definition. Let E be a Banach space, let $r \in E^{**}$ and let $\sigma = \{\sigma_i\}$ be a sequence of positive numbers. We will say that the sequences $\{b_i\} \subset E$ and $\{f_j\} \subset E^*$ form a generalized Pták system relative to (σ, r) if there are numbers $\{\sigma_{ij}\}$ with the properties

1° $\|b_i\| \leqslant 1$, $\|f_j\| \leqslant 1$ for all i, j

2° if we put $\beta_j = \langle r, f_j\rangle$ then the matrix $\langle b_i, f_j\rangle$ has the subdiagonal form

$$\begin{array}{ccccc} \beta_1+\sigma_{11} & 0 & 0 & 0 & \dots \\ \beta_1+\sigma_{21} & \beta_2+\sigma_{22} & 0 & 0 & \dots \\ \beta_1+\sigma_{31} & \beta_2+\sigma_{32} & \beta_3+\sigma_{33} & 0 & \dots \\ \beta_1+\sigma_{41} & \beta_2+\sigma_{42} & \beta_3+\sigma_{43} & \beta_4+\sigma_{44} & \dots \end{array} \tag{1}$$

and all the subdiagonal elements are positive. Thus $\langle b_i, f_j\rangle = \beta_j + \sigma_{ij} > 0$ for $j \leqslant i$ and $\langle b_i, f_j\rangle = 0$ for $j > i$

3° $\inf \beta_j > 0$

4° $|\sigma_{ij}| \leqslant \sigma_i$ for all $i \geqslant j$.

A Pták system is a generalized Pták system for which $\sigma_i = \sigma_{ij} = 0$ for all $i \geqslant j$.

Pták [10] shows the following facts:

1. If $\{(e_n, f_n)\}_{n=1}^{\infty} \subset E \times E^*$ is a bounded biorthogonal system with bounded partial sums $\{b_i\} = \left\{\sum\limits_{n=1}^{i} e_n\right\}$ then $\{b_i\}$, $\{f_j\}$ form a Pták system.
2. The Banach space E is not reflexive iff there is a Pták system $\{b_i\} \subset E$, $\{f_j\} \subset E^*$.
3. If $\{b_i\} \subset E$, $\{f_j\} \subset E^*$ is a Pták system then $\{e_i\} = \{b_i - b_{i-1}\}$ and $\{\overline{f}_j\} = \{\beta_j^{-1} f_j\}$ is a bounded biorthogonal system with bounded partial sums $\{b_i\} = \left\{\sum\limits_{n=1}^{i} e_n\right\}$.

Note that if $\{b_i\} \subset E$ and $\{f_j\} \subset E^*$ form a generalized Pták system then $r \in E^{**} \setminus E$ may be recovered from $\{b_i\}$, $\{f_j\}$. Indeed it suffices to put $r = w^*\text{-}\lim b_{n_\alpha}$ for some subnet $\{b_{n_\alpha}\}$ of $\{b_n\}$.

Proposition 1. *Suppose that E is a non reflexive Banach space and let $B \subset B_E$. Let $r \in E^{**} \setminus E$ be in the w^* closure of $B \subset E^{**}$ in E^{**}. Then for every sequence of positive numbers $\sigma = \{\sigma_i\}$ there is a generalized Pták system $\{b_i\} \subset B$, $\{f_j\} \subset E^*$ relative to (σ, r) satisfying 1°–4° from the Definition.*

P r o o f. We follow, with the necessary changes, the proof of the Theorem 1 in [10]. There is $f_1 \in E^*$, $\|f_1\| \leqslant 1$ such that $\beta_1 = \langle r, f_1\rangle = 2^{-1}\|r\|$. Having in mind that according to the assumptions $\langle r, f_1\rangle$ is in the closure of $\{\langle b, f_1\rangle;\ b \in B\}$ we see that there exists a vector $b_1 \in B$ such that $\langle b_1, f_1\rangle = \beta_1 + \sigma_{11}$ where $|\sigma_{11}| \leqslant \min\{\sigma_1, 2^{-1}\beta_1\}$. Thus $\langle b_1, f_1\rangle > 0$.

Suppose we have already defined vectors $b_1, \ldots, b_n \in B$ and functionals $f_1, \ldots, f_n \in E^*$ with the following properties:

$$\begin{gathered}
b_i \in B,\ \|f_i\| \leqslant 1 \quad \text{for} \quad 1 \leqslant i \leqslant n \\
|\langle b_i, f_j\rangle - \beta_j| < \min\{\sigma_i, 2^{-1}\beta_j\} \quad \text{for} \quad 1 \leqslant j \leqslant i \leqslant n \\
\langle b_i, f_j\rangle = 0 \quad \text{for} \quad 1 \leqslant i < j \leqslant n \\
\beta_j = \langle r, f_j\rangle \geqslant \frac{1}{2}\varrho_j \quad \text{for} \quad 1 \leqslant j \leqslant n
\end{gathered}$$

where

$$\varrho_j = \sup\{\langle r, f\rangle;\ f \in E_{j-1}^\circ, \|f\| \leqslant 1\}$$

and E_p is the linear span of $\{b_1, \ldots, b_p\}$, $E_0 = \{0\}$. Note that $\varrho_1 = \|r\|$.

We observe that the numbers ϱ_j, $j = 1, 2, \ldots$ are positive. Indeed, otherwise $f \in E_{j-1}^\circ$ would imply $\langle r, f\rangle = 0$ and thus $r \in E_{j-1}^{\circ\circ} = E_{j-1} \subset E$ which is a contradiction. The polars are taken here in the duality $\langle E^{**}, E^*\rangle$ and we consider $E_j \subset E \subset E^{**}$.

Since the number $\varrho_{n+1} = \sup\{\langle r, f\rangle;\ f \in E_n^\circ, \|f\| \leqslant 1\}$ is positive, there exists $f_{n+1} \in E_n^\circ$, $\|f_{n+1}\| \leqslant 1$ such that

$$\beta_{n+1} = \langle r, f_{n+1}\rangle \geqslant \frac{1}{2}\varrho_{n+1} > 0.$$

Having in mind that $r \in E^{**} \setminus E$ is in the w^* closure of the subset B we see that a point $b_{n+1} \in B$ may be found such that $|\langle b_{n+1}, f_j\rangle - \beta_j| < \min\{\sigma_{n+1}, 2^{-1}\beta_j\}$ for $1 \leqslant j \leqslant n + 1$.

The induction is thus complete and we may put $\sigma_{i+1,j} = \langle b_{n+1}, f_j\rangle - \beta_j$ for $j \leqslant n + 1$. Evidently $|\langle b_{n+1}, f_j\rangle - \beta_j| < 2^{-1}\beta_j$ and thus $\langle b_{n+1}, f_j\rangle > 0$.

Clearly $\varrho_1 \geqslant \varrho_2 \geqslant \varrho_3 \geqslant \ldots > 0$. To see 3° we shall show that $\inf \varrho_j > 0$. Suppose not. Then $\lim \varrho_j = 0$. Again we partly follow [10]:

Let ε be an arbitrary positive number. Let n be a natural number such that $\varrho_n < \varepsilon/2$. Let $f \in B_{E^*}$ and let $\tau = \max\limits_{1 \leqslant i \leqslant n} |\langle b_i, f\rangle|$. Let

$$z = \sum_{j=1}^{n} \frac{1}{\beta_j}\langle b_j - b_{j-1}, f\rangle f_j$$

where $b_0 = 0$. Having in mind that $0 < \frac{1}{2}\varrho_j \leqslant \beta_j \leqslant \varrho_j$ we have

$$\|z\| \leqslant \frac{2}{\varrho_n} \cdot 2\tau \cdot n. \tag{2}$$

If $i = 1, \ldots, n$ then

$$\begin{aligned}
\langle b_i, z\rangle &= \sum_{j=1}^{i} \frac{1}{\beta_j} \langle b_j - b_{j-1}, f\rangle(\beta_j + \sigma_{ij}) \\
&= \sum_{j=1}^{i} \frac{1}{\beta_j} \langle b_j - b_{j-1}, f\rangle \beta_j + s_i = \langle b_i, f\rangle + s_i
\end{aligned} \tag{3}$$

where $s_i = \sum_{j=1}^{i} \beta_j^{-1} \langle b_j - b_{j-1}, f\rangle \sigma_{ij}$. Evidently

$$|s_i| \leqslant \sum_{j=1}^{i} \frac{1}{\beta_j} 2\tau |\sigma_{ij}| \leqslant 2\tau\sigma_i \sum_{j=1}^{n} \frac{2}{\varrho_j} = \tau h_n$$

where h_n depends on n, $\{\sigma_i\}$ and on $\{\varrho_i\}$ but not on $f \in B_{E^*}$.

Note that the special form of the matrix (1) implies that the vectors $\{b_i\}$ are linearly independent. Since E_n and so (its dual) E_n^* are finite dimensional there is a constant $c = c(b_1, \ldots, b_n)$ independent of $f \in B_E$ such that if $g \in E_n^*$ then

$$\|g\|_{E_n^*} = \sup\{\langle b, g\rangle;\ b \in E_n, \|b\| \leqslant 1\} \leqslant c \max\{|\langle b_i, g\rangle|;\ i = 1, \ldots, n\}.$$

Now we define $g \in E_n^*$ by $\langle b_i, g\rangle = s_i$ for all $i = 1, \ldots, n$. Then

$$\|g\|_{E_n^*} \leqslant c \max\{|s_i|;\ i = 1, \ldots, n\}.$$

Let us choose a Hahn-Banach extension of g to the whole space E and call it g again. Then

$$\|g\| \leqslant c\tau h_n. \tag{4}$$

Using (3) we note that $\langle b_i, z - f - g\rangle = \langle b_i, f\rangle + s_i - \langle b_i, f\rangle - s_i = 0$ so that $z - f - g \in E_n^\circ$. Thus by the definition of ϱ_{n+1}, (2), $\|f\| \leqslant 1$ and by (4)

$$\begin{aligned}
|\langle r, z - f - g\rangle| &\leqslant \|z - f - g\| \cdot \sup\{\langle r, y\rangle;\ y \in E_n^\circ, \|y\| \leqslant 1\} \\
&= \|z - f - g\| \cdot \varrho_{n+1} \\
&\leqslant \Big(\frac{4}{\varrho_n}\tau n + 1 + c\tau h_n\Big)\varrho_n = 4n\tau + \varrho_n + c\tau h_n \varrho_n \\
&\leqslant 4n\tau + \frac{1}{2}\varepsilon + c\tau h_n \varrho_n.
\end{aligned}$$

Further note that $\langle r, z\rangle = \langle b_n, f\rangle$ and that we may have supposed that $\|r\| = 1$. Then

$$\langle r, f\rangle = \langle r, z\rangle - \langle r, z-f\rangle = \langle b_n, f\rangle - \langle r, z-f-g\rangle + \langle r, g\rangle$$
$$\leqslant \tau + \Bigl(4n\tau + \frac{1}{2}\varepsilon + c\tau h_n \varrho_n\Bigr) + c\tau h_n < \varepsilon$$

if $\tau = \max\limits_{1\leqslant i\leqslant n} |\langle b_i, f\rangle|$ is sufficiently small.

The set $\{y \in E^*;\ \max\limits_{1\leqslant i\leqslant n} |\langle b_i, y\rangle| < \tau\}$ being a $w*$ neighbourhood of zero this shows that $r|_{B_{E^*}}$ is w^* continuous at zero on the dual unit ball B_{E^*} and thus it is w^* continuous. This shows that r is w^* continuous on the dual unit ball B_{E^*} and thus it should be w^* continuous. But this is a contradiction because r does not belong to E. We have thus shown that $\inf \beta_j > 0$ and the proof of the Proposition is complete. $\square$

Proposition 1 together with the following observation generalizes the statement 2 before Proposition 1.

Proposition 2. *Let E be a Banach space and let $\{\sigma_i\}$ be any sequence of positive numbers, $\lim \sigma_i = 0$ and let r be any element of E^{**}. Suppose further that $\{b_i\} \subset B_E$ and $\{f_j\} \subset E^*$ is a generalized Pták system in E relative to (σ, r) satisfying $1°$–$4°$ from the Definition. Then E is not reflexive.*

Proof. Let $b \in \overline{\operatorname{span}}\{b_i\}$. Then $\lim\langle f_j, b\rangle = 0$. Indeed, given $\varepsilon > 0$ there are numbers $a_1, \ldots, a_q$ such that $\Bigl\|b - \sum\limits_1^q a_i b_i\Bigr\| < \varepsilon$. If $j > q$, we have $|\langle b, f_j\rangle| = \Bigl|\langle b, f_j\rangle - \Bigl\langle \sum\limits_1^q a_i b_i, f_j\Bigr\rangle\Bigr| \leqslant \Bigl|\Bigl\langle b - \sum\limits_1^q a_i b_i,\ f_j\Bigr\rangle\Bigr| \leqslant \varepsilon$.

Suppose now that E is reflexive and let b be the weak accumulation point of the sequence $\{b_n\}$. Then $b \in \overline{\operatorname{span}}\{b_i\}$ and thus $\lim\langle f_j, b\rangle = 0$. On the other hand $\langle b_i, f_j\rangle = \beta_i + \sigma_{ij}$ for $i \geqslant j$. Having in mind that we suppose that $|\sigma_{ij}| \leqslant \sigma_i \longrightarrow 0$ we get $\langle b, f_j\rangle \geqslant \inf \beta_i > 0$ for all j—a contradiction.

Proposition 1 will be applied to the nonreflexive π-tensor product $E = X \widetilde{\otimes}_\pi Y$. We take benefit of the following observation namely that the elements b_i may be chosen of the form $b_i = x_i \otimes y_i$ which means that we may choose b_i in the $\otimes$ image of the Cartesian product $B_X \times B_Y$:

Proposition 3. *Let X and Y be reflexive Banach spaces. Suppose that $X \widetilde{\otimes}_\pi Y$ is not reflexive. Then there are weakly null basic sequences $\{x_n\} \subset B_X$, $\{y_n\} \subset B_Y$, $\{x_n^*\} \subset X^*$, $\{y_n^*\} \subset Y^*$ such that*

(a) *$\{x_n \otimes y_n\}$ has no weak accumulation point in $X \widetilde{\otimes}_\pi Y$*

(b) *$\{(x_n, x_n^*)\}$ is a bounded biorthogonal system*

(c) $\{(y_n, y_n^*)\}$ *is a bounded biorthogonal system*

(d) *there is* $f \in L(X, Y^*) = (X\widetilde{\otimes}_\pi Y)^*$ *such that* $\langle f, x_n \otimes y_n\rangle \geqslant 1$.

P r o o f. Suppose that any sequence $\{a_i \otimes b_i\}$ in $B = \{x \otimes y;\ x \in B_X, y \in B_Y\}$ has a subnet weakly convergent in $X\widetilde{\otimes}_\pi Y$. Then the weak convex closure $\overline{\mathrm{conv}}B$ of B in $X\widetilde{\otimes}_\pi Y$ is weakly compact in $X\widetilde{\otimes}_\pi Y$. By the definition of the π-tensor product we know [2], [5], [8] that $\overline{\mathrm{conv}}B$ equals the closed unit ball of $X\widetilde{\otimes}_\pi Y$. We conclude that $X\widetilde{\otimes}_\pi Y$ is reflexive which contradicts our assumption. We have thus shown that there is a sequence $\{a_i \otimes b_i\} \subset B$ with no weak accumulation points in $X\widetilde{\otimes}_\pi Y$.

Passing to a subsequence if necessary we may suppose that $a_i \longrightarrow a \in X$ weakly and $b_i \longrightarrow a \in Y$ weakly. This follows by the weak sequential compactness of the closed unit balls of X and Y. It is now a routine to check that neither $\{a_i - a\}$ nor $\{b_i - b\}$ are norm null sequences. Indeed, suppose for example that $\|a_i - a\| \to 0$. Then for any $f \in (X\widetilde{\otimes}_\pi Y)^*$ we have

$$|f(a_i, b_i) - f(a, b)| \leqslant |f(a_i - a, b_i)| + |f(a, b_i - b)| \xrightarrow{\ i\ } 0.$$

This would mean that $a \otimes b$ is a weak accumulation point of $\{a_i \otimes b_i\} \subset B$ which is a contradiction.

The Bessaga-Pełczyński selection theorem yields a basic subsequence $\{2^{-1}(a_{i_l} - a)\} = \{a'_l\} \subset X$ which we call $\{a'_l\}$. Evidently the sequence $\{a'_l \otimes b_{i_l}\}_l = \{a'_l \otimes b'_l\}_l = \{2^{-1}a_{i_l} \otimes b_{i_l} - 2^{-1}a \otimes b_{i_l}\}_l$ still has no weak accumulation point in $X\widetilde{\otimes}_\pi Y$ and $\{a'_l \otimes b'_l\} \subset B = B_X \times B_Y \subset X\widetilde{\otimes}_\pi Y$.

Let $\{a'^*_l\}$ be the Hahn-Banach extensions to X of the biorthogonal functionals to the basis $\{a'_l\}$ of $L = \overline{\mathrm{span}}\{a'_l\}$. Passing to a subsequence if necessary we may suppose that $a'^*_l \longrightarrow a^* \in X^*$ weakly. Again $\{a'^*_l - a^*\}$ is not a norm null sequence. Indeed, otherwise the restrictions to L would satisfy $\lim\limits_l a'^*_l|_L = a^*|_L$ in norm. But this is not possible because $a'^*_l|_L \longrightarrow 0 = a^*|_L$ weakly in L^* and the $a'^*_l|_L$'s are norm bounded from below because $a'^*_l(a'_l) = 1$.

Again the Bessaga-Pełczyński selection theorem yields a subsequence $\{a'^*_{l_j}\}$ such that $\{a'^*_{l_j} - a^*\} = \{x'^*_j\} \subset X^*$ is a weakly null basic sequence. Let us put $x'_j = a'_{l_j}$ and $y'_j = b'_{l_j}$. Then $\{x'_j \otimes y'_j\} \subset \{a'_l \otimes b'_l\}$ has no weak accumulation point in $X\widetilde{\otimes}_\pi Y$.

Since $a^*(a'_{l_j}) = 0$ we conclude that $\{x'_j\}$ and $\{x'^*_j\}$ are bounded biorthogonal basic sequences.

Proceeding now in a similar way with the b'_j's we arrive at a subsequence $\{x_n \otimes y_n\}$ of $\{a'_i \otimes b'_i\}$ such that the sequences $\{x_n\} \subset B_X$, $\{y_n\} \subset B_Y$, $\{x^*_n\} \subset X^*$, $\{y^*_n\} \subset Y^*$ satisfy (a), (b) and (c).

Because $\{x_n \otimes y_n\}$ is not weakly convergent to 0 there is an $f \in L(X, Y^*)$ such that $\langle f, x_n \otimes y_n\rangle$ does not converge to 0. We complete the proof of the proposition by passing again to suitable subsequences and a suitable multiple of f. $\square$

Proposition 4. *Let X and Y be reflexive Banach spaces. Suppose that $X\widetilde{\otimes}_\pi Y$ is not reflexive and let $\sigma = \{\sigma_i\}$ be a sequence of arbitrary positive numbers. Then there are weakly null basic sequences $\{x_i\} \subset B_X$, $\{y_i\} \subset B_Y$, an element $r \in (X\widetilde{\otimes}_\pi Y)^{**} \setminus X\widetilde{\otimes}_\pi Y$ and a sequence of functionals $\{f_j\} \subset (X\widetilde{\otimes}_\pi Y)^* = L(X, Y^*)$ such that $\{b_i\} = \{x_i \otimes y_i\}$ and $\{f_j\}$ form the generalized Pták system relative to (σ, r).*

Proof. Let $\{x_n\} \subset X$ and $\{y_n\} \subset Y$ be sequences satisfying (a), (b), (c) of Proposition 3. Proposition 1 allows to extract from the sequence $\{x_n \otimes y_n\}$ a subsequence $\{b_i\} = \{x_{n_i} \otimes y_{n_i}\}$ and a sequence of elements of $\{f_j\} \subset (X\widetilde{\otimes}_\pi Y)^* = L(X, Y^*)$ which form a generalized Pták system relative to (σ, r). The extracted subsequences $\{x_{n_i}\}$, $\{y_{n_i}\}$ are again basic sequences. $\square$

Remark 1. Note that $\{b_i\} \subset X_0 \otimes Y_0 \subset X\widetilde{\otimes}_\pi Y$ where the subspaces $X_0 \subset X$ and $Y_0 \subset Y$ have the approximation property.

Definition. Let $X_0 \subset X$ and $Y_0 \subset Y$ be subspaces. Let us denote by $X_0\bar{\otimes}_\pi Y_0$ the closure of $X_0 \otimes Y_0$ which we consider as a normed subspace of $X\widetilde{\otimes}_\pi Y$.

By $L(X, Y^*)_{X_0}$ we will denote the space of all restrictions $\{f|_{X_0};\ f \in L(X, Y^*)\}$ equipped with the factor norm $\|\cdot\|_{X_0}$ of the space $L(X, Y^*)$ given by the quotient map $\mathrm{Re}\colon L(X, Y^*) \longrightarrow L(X, Y^*)_{X_0}$. Here Re is the restriction map to the subspace X_0. Thus $L(X, Y^*)_{X_0}$ is the factor space $L(X, Y^*)/\operatorname{Re}^{-1}(0)$.

Lemma. a) *Suppose that $g \in L(X, Y^*)_{X_0}$ and that $\operatorname{Re} f = g$, where $f \in L(X, Y^*)$. Then*

$$\|g\|_{X_0} = \inf\{\|f + h\|;\ \operatorname{Re} h = 0, h \in L(X, Y^*)\}.$$

b) *$(X_0\bar{\otimes}_\pi Y)^* = L(X, Y^*)_{X_0}$ with the equality of norms.*

Proof. Indeed, for example the dual of $X_0\bar{\otimes}_\pi Y$ is given by the restrictions to $X_0 \otimes Y$ of all the elements of the dual of $X\bar{\otimes}_\pi Y$, that is, by the bilinear forms on $X_0 \times Y$ which are the restrictions of the continuous bilinear forms on $X \times Y$. These are evidently exactly the linear operators $f\colon X_0 \longrightarrow Y^*$ which are continuously extendable to all of X. $\square$

Proposition 5. *Let X, Y be Banach spaces. The following are equivalent:*

(1) *$X\widetilde{\otimes}_\pi Y$ is reflexive*

(2) *the subspace $X_0\bar{\otimes}_\pi Y \subset X\widetilde{\otimes}_\pi Y$ is reflexive for every subspace $X_0 \subset X$ such that X_0 has a Schauder basis*

(3) *the quotient space $L(X, Y^*)_{X_0}$ is reflexive for every subspace $X_0 \subset X$ such that X_0 has a Schauder basis*

(4) *the subspace* $X_0\bar{\otimes}_\pi Y_0 \subset X\widetilde{\otimes}_\pi Y$ *is reflexive for every subspace* $X_0 \subset X$ *such that* X_0 *has a Schauder basis and for every subspace* $Y_0 \subset Y$ *such that* Y_0 *has a Schauder basis*

(5) *the Banach spaces* X *and* Y *are reflexive and the following holds: Let* $\{x_i\} \subset B_X$ *and* $\{y_i\} \subset B_Y$ *be basic sequences,* $r \in (X\widetilde{\otimes}_\pi Y)^{**}$, $\{f_j\} \subset (X\widetilde{\otimes}_\pi Y)^* = L(X, Y^*)$ *a sequence of functionals and let* $\{\sigma_i\}$ *be a null sequence of positive numbers. Then* $\{b_i\} = \{x_i \otimes y_i\}$ *and* $\{f_j\}$ *do not form a generalized Pták system relative to* (σ, r)

(6) *the Banach spaces* X *and* Y *are reflexive and the following holds: Let* $\{x_i\} \subset B_X$ *and* $\{y_i\} \subset B_Y$ *be basic sequences,* $r \in (X\widetilde{\otimes}_\pi Y)^{**}$, $\{f_j\} \subset (X\widetilde{\otimes}_\pi Y)^* = L(X, Y^*)$ *a sequence of functionals. Then* $\{b_i\} = \{x_i \otimes y_i\}$ *and* $\{f_j\}$ *do not form a generalized Pták system relative to* (σ, r) *for some null sequence of positive numbers* $\sigma = \{\sigma_i\}$.

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (4) because $X_0\bar{\otimes}_\pi Y$ is a closed subspace of the reflexive space $X\widetilde{\otimes}_\pi Y$ and $X_0\bar{\otimes}_\pi Y_0$ is a closed subspace of $X_0\bar{\otimes}_\pi Y$.

(2) $\Leftrightarrow$ (3) because by the Lemma $(X_0\bar{\otimes}_\pi Y)^* = L_{X_0}(X, Y^*)$.

(4) $\Rightarrow$ (1): First we note that (4) implies that X and Y are reflexive. Indeed, let us choose $Y_0 \subset Y$ to be any subspace of the dimension one. Then we get that $X_0 \subset X$ which is supposed to have a Schauder basis is isomorphic to $X_0\bar{\otimes}_\pi Y_0 \subset X\widetilde{\otimes}_\pi Y$ and thus reflexive. Now X is reflexive by Pełczyński's characterization [9] of the reflexivity of X by means of reflexivity of subspaces with a Schauder basis. Similarly we conclude that under the assumption (4) the Banach space Y is reflexive.

Suppose now that (1) does not hold. Then the Proposition 3 yields basic sequences $\{x_n\} \subset X$, $\{y_n\} \subset Y$ such that $\{x_n \otimes y_n\}$ has no weak accumulation point in $X\widetilde{\otimes}_\pi Y$. Then $\{x_n \otimes y_n\}$ has no weak accumulation point in the subspace $X_0\bar{\otimes}_\pi Y_0$ where $X_0 = \overline{\operatorname{span}}\{x_n\} \subset X$ and $Y_0 = \overline{\operatorname{span}}\{y_n\} \subset Y$. Thus $X_0\bar{\otimes}_\pi Y_0 \subset X\widetilde{\otimes}_\pi Y$ is not reflexive.

(1) $\Rightarrow$ (5): The Banach spaces X and Y, being isomorphic to subspaces of $X\widetilde{\otimes}_\pi Y$, are reflexive. The rest follows by Proposition 2.

(5) $\Rightarrow$ (6) is trivial and

(6) $\Rightarrow$ (1) is consequence of Proposition 4.

Remark 2. In [6] we observed that the equivalent conditions in the above Proposition hold if

$$X_0\tilde{\otimes}_\pi Y \text{ is reflexive for every subspace } X_0 \subset X \text{ such that } X_0 \text{ has a Schauder basis.} \tag{2'}$$

In a subsequent paper we show that the condition (2′) is in fact equivalent to the conditions expressed in Proposition 5.

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