Czechoslovak Mathematical Journal

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Czechoslovak Mathematical Journal, Vol. 56 (2006), No. 4, 1147–1163

Persistent URL: http://dml.cz/dmlcz/128136

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OPERATORS OF HANKEL TYPE

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(Received March 15, 2004)

Abstract. Hankel operators and their symbols, as generalized by V. Pták and P. Vrbová, are considered. The present note provides a parametric labeling of all the Hankel symbols of a given Hankel operator X by means of Schur class functions. The result includes uniqueness criteria and a Schur like formula. As a by-product, a new proof of the existence of Hankel symbols is obtained. The proof is established by associating to the data of the problem a suitable isometry V so that there is a bijective correspondence between the symbols of X and the minimal unitary extensions of V.

Keywords: Hankel operators, Hankel symbols

MSC 2000: 47B35, 47A20

1. Introduction

The intertwining relation that characterizes the classical Hankel operators has been exploited to study the symbols as well as the operators themselves, from the operator theory point of view rather than from the function theory standpoint. Under the commutant perspective, other intertwining operators may be thought of as abstract Hankel operators. That was the approach adopted by V. Pták and P. Vrbová [10], [11], [9] to introduce a wider class of Hankel operators.

If S is the shift operator on the space of the L^2 functions on the unit circle of the complex plane, H^2 the Hardy space and H^2_- its orthogonal complement in L^2, then we recall that a Hankel operator is a linear map $X\colon H^2 \to H^2_-$ such that $XS|_{H^2} = P_-SX$, with P_- the orthogonal projection from L^2 onto H^2_-. If we consider

The first author was partially supported by grant BFM2001-3735 (Ministerio de Ciencia y Tecnología, Spain.)
The third author was partially supported by grant G-97000668 (FONACIT, Venezuela.)

the contraction operators $T_1 := PS^*|_{H^2}$, with P the orthogonal projection from L^2 onto H^2, and $T_2 := P_-S|_{H^2_-}$, then the intertwining condition satisfied by X can be written as $XT_1^* = T_2X$.

The celebrated Nehari Theorem states that the Hankel operator X is bounded if, and only if, there exists an L^∞ function Φ, a symbol of X, such that $Xf = P_-\Phi f$ for all $f \in H^2$. Hence, in the classical case, the symbols are multiplication operators induced by L^∞ functions with prescribed antianalytic part or, from another point of view, operators that commute with S and have the same fixed component from H^2 into H^2_-. Since the unitary operators $V_1 := S^*$ and $V_2 := S$ are the corresponding minimal isometric dilations of the above defined contractions T_1 and T_2, we can conclude that the symbols of the given Hankel operator X are the intertwining dilations Z of X, namely, those linear operators $Z\colon\ L^2 \to L^2$ such that $P_-Z|_{H^2} = X$ and $ZV_1^* = V_2Z$.

The generalized Hankel operators introduced by Pták and Vrbová are linear maps X from a Hilbert space $\mathcal{H}_1$ into a Hilbert space $\mathcal{H}_2$ satisfying the intertwining relation $XT_1^* = T_2X$, for given contraction operators T_1 on $\mathcal{H}_1$ and T_2 on $\mathcal{H}_2$. In this framework, the operators that play the role of symbols might be the solutions Z of the commutant dilation problem $ZV_1^* = V_2Z$, where V_1 and V_2 are the minimal isometric dilations of T_1 and T_2, respectively. The investigations carried on by Pták and Vrbová indicate that the problem is solvable whenever X verifies certain boundedness condition that depends on the unitary parts of the Wold-Von Neumann decompositions of V_1 and V_2. Since the Wold-Von Neumann decomposition is trivial in the classical case, for S being unitary, the result includes the classical situation.

As counterpart of the classical case and with the aim of developing a full analogue of the theory of the Commutant Lifting Theorem, the problem of describing the symbols Z of any abstract Hankel operator X, for given contractions T_1 and T_2, turns out to be of greatest interest.

We show that there is a bijective correspondence between the symbols of X and the minimal unitary extensions of a Hilbert space isometry V determined by X, T_1 and T_2. Since any Hilbert space isometry has at least one minimal unitary extension, our approach provides a new proof of the existence of the symbols for the generalized Hankel operator on hand. The Arov-Grossman functional model [1] yields a complete description of the minimal unitary extensions of V, as it associates to each minimal unitary extension U of V a function θ_U in a suitable Schur class of operator valued functions, and to each function θ in the Schur class, an operator model U_θ which gives rise to a minimal unitary extension of V, in such a way that the outlined correspondence is bijective. Then the adopted method combined with the Arov-Grossman model gives in turn a bijective correspondence between the symbols of X and the Schur class. We show that the connection between the symbols and the Schur functions can be realized as a parametric description. We also include uniqueness criteria and a Schur-like formula.

In the framework of the Commutant Lifting Theorem, the methods were developed in [8] for the usual Hilbert space case and in [4] for the more general Kreĭn space case.

We point out that the line of investigations initiated by V. Pták and P. Vrbová has been pursued mainly by C. H. Mancera and P. J. Paúl [6], [7]. A few historical remarks connected with the original work of V. Pták and P. Vrbová can be found in [11]. For other interesting comments the reader is referred to the introductory pages of [6].

As a final remark, we mention that abstract Hankel operators can also be treated as bilinear forms defined in the even more general framework of the algebraic scattering systems as by M. Cotlar and C. Sadosky (see, for instance, [2], [3] and further references given therein.) The construction of the isometry V, which plays a key role in the proof of our main result, is in fact inspired by the Cotlar-Sadosky algebraic scattering systems methods.

The paper is organized in three sections. Section 1, this section, serves as an introduction. In Section 2 we fix the notation and state some known results needed in the rest of the paper. Our main result, along with some comments and remarks, is presented in Section 3.

2. Notation and preliminaries

We follow the standard notation, so $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{C}$ are, respectively, the set of natural, integer and complex numbers; $\mathbb{D}$ stands for the open unit disk and $\mathbb{T}$ for the unit circle, hence $\mathbb{D} := \{z \in \mathbb{C}\colon\ |z| < 1\}$ and $\mathbb{T} := \partial\mathbb{D}$.

Throughout this note, all Hilbert spaces are assumed to be complex and separable. If $\{\mathcal{G}_\iota\}_{\iota\in I}$ is a collection of linear subspaces of a Hilbert space $\mathcal{K}$ then $\bigvee\limits_{\iota\in I} \mathcal{G}_\iota$ is the least closed subspace of $\mathcal{K}$ containing all the subspaces $\mathcal{G}_\iota$.

As usual, $\mathcal{L}(\mathcal{H},\mathcal{K})$ denotes the space of all everywhere defined bounded linear operators on the Hilbert space $\mathcal{H}$ to the Hilbert space $\mathcal{K}$, and $\mathcal{L}(\mathcal{H})$ is used instead of $\mathcal{L}(\mathcal{H},\mathcal{H})$.

By 1 we indicate either the scalar unit or the identity operator, depending on the context.

If $\mathcal{G}$ is a closed linear subspace of a Hilbert space $\mathcal{K}$, then $P_{\mathcal{G}}^{\mathcal{K}}$ stands for the orthogonal projection from $\mathcal{K}$ onto $\mathcal{G}$.

If $T \in \mathcal{L}(\mathcal{H}, \mathcal{K})$ and $\|T\| \leqslant \beta$, then $D_T^\beta := (\beta^2 - T^*T)^{\frac{1}{2}}$ and $\mathcal{D}_T^\beta := \overline{D_T^\beta \mathcal{H}}$. When $\beta = 1$, we use the standard notation D_T and $\mathcal{D}_T$ for the defect operator and the defect space of T.

If $\mathcal{H}$ is a Hilbert space, $H^2(\mathcal{H})$ is the Hardy space of the $\mathcal{H}$-valued functions on $\mathbb{D}$. So, the elements of $H^2(\mathcal{H})$ are all the analytic functions $f\colon \mathbb{D} \to \mathcal{H}$, $f(z) = \sum\limits_{n=0}^{\infty} z^n h(n)$, $z \in \mathbb{D}$, $\{h(n)\}_{n=0}^{\infty} \subseteq \mathcal{H}$, such that $\sum\limits_{n=0}^{\infty} \|h(n)\|^2 < \infty$. The shift operator on $H^2(\mathcal{H})$ is denoted by S. Thus, $(Sf)(z) := zf(z)$, $f \in H^2(\mathcal{H})$, $z \in \mathbb{D}$.

Given a contraction operator $T \in \mathcal{L}(\mathcal{H})$, we recall that the operator matrix

$$V_T := \begin{pmatrix} T & 0 \\ D_T & S \end{pmatrix} : \begin{pmatrix} \mathcal{H} \\ H^2(\mathcal{D}_T) \end{pmatrix} \to \begin{pmatrix} \mathcal{H} \\ H^2(\mathcal{D}_T) \end{pmatrix}$$

is the minimal isometric dilation of T. That is, V_T is an isometry everywhere defined on the Hilbert space $\mathcal{K}_T := \mathcal{H} \oplus H^2(\mathcal{D}_T)$ such that $T^n = P_{\mathcal{H}}^{\mathcal{K}_T} V_T^n|_{\mathcal{H}}$, for all $n \in \mathbb{N}$, and $\mathcal{K}_T = \bigvee\limits_{n=0}^{\infty} V_T^n \mathcal{H}$.

If $V \in \mathcal{L}(\mathcal{K})$ is an isometric operator, we denote by $\mathcal{R}$ the closed linear subspace of $\mathcal{K}$ that reduces V to its unitary part in the Wold-Von Neumann decomposition. In particular, $\mathcal{R} = \bigcap\limits_{n=0}^{\infty} V^n \mathcal{K}$ and $P_{\mathcal{R}}^{\mathcal{K}} = \lim\limits_{n\to\infty} V^n V^{*n}$.

Let $T_1 \in \mathcal{L}(\mathcal{H}_1)$ and $T_2 \in \mathcal{L}(\mathcal{H}_2)$ be two contractions with minimal isometric dilations $V_1 \in \mathcal{L}(\mathcal{K}_1)$ and $V_2 \in \mathcal{L}(\mathcal{K}_2)$, respectively. As it was introduced by V. Pták and P. Vrbová [10], [11], [9], an operator $X \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ is said to be a *Hankel operator* for T_1 and T_2 if, and only if, $XT_1^* = T_2X$ and, for some $\beta \geqslant 0$,

$$(1) \qquad |\langle Xh_1, h_2\rangle| \leqslant \beta \|P_{\mathcal{R}_1}^{\mathcal{K}_1} h_1\| \|P_{\mathcal{R}_2}^{\mathcal{K}_2} h_2\|, \quad \text{for all } h_1 \in \mathcal{H}_1 \text{ and } h_2 \in \mathcal{H}_2,$$

where $\mathcal{R}_j$ is the subspace of $\mathcal{K}_j$ which reduces the minimal isometric dilation V_j of T_j to the unitary part R_j of V_j ($j = 1, 2$). We define $\|X\|_{PV} := \inf \beta$, where β runs over all nonnegative numbers satisfying (1).

Given a Hankel operator X for T_1 and T_2, we say that $Z \in \mathcal{L}(\mathcal{K}_1, \mathcal{K}_2)$ is a *Hankel symbol* of X if, and only if, (i) $ZV_1^* = V_2Z$, (ii) $P_{\mathcal{H}_2}^{\mathcal{K}_2} Z|_{\mathcal{H}_1} = X$, and (iii) $\|Z\| = \|X\|_{PV}$.

As we already remarked in the introduction, the relation $XT_1^* = T_2X$ alone is not sufficient to grant the existence of symbols. The difficulty is overcome by means of the boundedness condition (1), since it turns out to be necessary and sufficient to ensure that there exist intertwining dilations Z of X ((i) and (ii)), which satisfy (iii). The reader is referred to [10], [11], [9] as the original sources. A result used therein, which we also require in our treatment, is the following:

Lemma 2.1 [10, Proposition 1.4]. *Let $T_1 \in \mathcal{L}(\mathcal{H}_1)$ and $T_2 \in \mathcal{L}(\mathcal{H}_2)$ be two contractions with minimal isometric dilations $V_1 \in \mathcal{L}(\mathcal{K}_1)$ and $V_2 \in \mathcal{L}(\mathcal{K}_2)$, respectively. Let X be a Hankel operator for T_1 and T_2 with $\beta := \|X\|_{PV}$. For $j = 1, 2$, set $\mathcal{E}_j := \overline{P^{\mathcal{K}_j}_{\mathcal{R}_j}\mathcal{H}_j}$, where $\mathcal{R}_j$ is the subspace of $\mathcal{K}_j$ which reduces the minimal isometric dilation V_j of T_j to the unitary part R_j of V_j. Then there exists a unique bounded linear operator $\widetilde{X}\colon \mathcal{E}_1 \to \mathcal{E}_2$ such that $X = (P^{\mathcal{K}_2}_{\mathcal{R}_2}|_{\mathcal{H}_2})^* \widetilde{X} P^{\mathcal{K}_1}_{\mathcal{R}_1}|_{\mathcal{H}_1}$ and $\|\widetilde{X}\| = \beta$.*

Though the proof is not included in the cited papers by Pták and Vrbová, we do not append it either. We just remark that the result can be obtained from Douglas' Lemma [5].

As the minimal unitary extensions of an isometry, on the one hand, and the so called Schur functions, on the other, play key roles in the description of the symbols of a given Hankel operator, we conclude this section with a few words about these objects.

If V is an isometric operator on a Hilbert space $\mathcal{H}$ with domain $\mathcal{D}(V)$ and range $\mathcal{R}(V)$, both closed linear subspaces of $\mathcal{H}$, then a minimal unitary extension of V is a unitary operator U acting on a Hilbert space $\mathcal{F}$ that contains $\mathcal{H}$ as a closed linear subspace such that $U|_{\mathcal{D}(V)} = V$ and $\mathcal{F} = \bigvee_{n\in\mathbb{Z}} U^n\mathcal{H}$. Two minimal unitary extensions of V, namely $U \in \mathcal{L}(\mathcal{F})$ and $U' \in \mathcal{L}(\mathcal{F}')$, are to be interpreted as indistinguishable whenever there exists an isometric isomorphism $\varphi\colon \mathcal{F} \to \mathcal{F}'$ such that $\varphi|_{\mathcal{H}} = 1$ and $\varphi U = U'\varphi$. As for the existence of minimal unitary extensions of any given isometry V, we remark that if $V_T \in \mathcal{L}(\mathcal{K}_T)$ is the minimal isometric dilation of the contraction $T := VP^{\mathcal{H}}_{\mathcal{D}(V)}$, then the minimal isometric dilation of V_T^*, say $W \in \mathcal{L}(\mathcal{F})$, is indeed a unitary operator such that $W^*|_{\mathcal{D}(V)} = V$ and $\mathcal{F} = \bigvee_{n\in\mathbb{Z}} W^n\mathcal{H}$, hence $U := W^*$ is a minimal unitary extension of V. The defect spaces of the isometry V are $\mathcal{N} := \mathcal{H} \ominus \mathcal{D}(V)$ and $\mathcal{M} := \mathcal{H} \ominus \mathcal{R}(V)$. If either $\mathcal{N} = \{0\}$ or $\mathcal{M} = \{0\}$, then V has a unique (up to isometric isomorphism) minimal unitary extension.

If $\mathcal{N}$ and $\mathcal{M}$ are given Hilbert spaces, then the Schur class $\mathcal{S}(\mathcal{N}, \mathcal{M})$ is the family of all analytic functions $\theta\colon \mathbb{D} \to \mathcal{L}(\mathcal{N}, \mathcal{M})$ such that $\sup_{z\in\mathbb{D}} \|\theta(z)\| \leqslant 1$.

An example of Schur function, remarkable for the problem we are concerned with, is the following: Let V be an isometry on $\mathcal{H}$ with domain $\mathcal{D}(V)$, range $\mathcal{R}(V)$ and defect spaces $\mathcal{N}$ and $\mathcal{M}$. Let $U \in \mathcal{L}(\mathcal{F})$ be a minimal unitary extension of V. For $z \in \mathbb{D}$, define

$$\theta_U(z) := P^{\mathcal{F}}_{\mathcal{M}} U(1 - zP^{\mathcal{F}}_{\mathcal{F}\ominus\mathcal{H}}U)^{-1}|_{\mathcal{N}}.$$

Then $\theta_U \in \mathcal{S}(\mathcal{N}, \mathcal{M})$. Moreover:

Lemma 2.2. *For all $z \in \mathbb{D}$,*

$$\begin{aligned} &P^{\mathcal{F}}_{\mathcal{H}} U(1-zU)^{-1}|_{\mathcal{H}} \\ &\quad = (VP^{\mathcal{F}}_{\mathcal{D}(V)} + \theta_U(z)P^{\mathcal{F}}_{\mathcal{N}})[1 - z(VP^{\mathcal{F}}_{\mathcal{D}(V)} + \theta_U(z)P^{\mathcal{F}}_{\mathcal{N}})]^{-1}|_{\mathcal{H}} \\ &\quad = VP^{\mathcal{F}}_{\mathcal{D}(V)}(1 - zVP^{\mathcal{F}}_{\mathcal{D}(V)})^{-1} + [zVP^{\mathcal{F}}_{\mathcal{D}(V)}(1 - zVP^{\mathcal{F}}_{\mathcal{D}(V)})^{-1} + 1] \\ &\qquad \times \theta_U(z)[1 - zP^{\mathcal{F}}_{\mathcal{N}}(1 - zVP^{\mathcal{F}}_{\mathcal{D}(V)})^{-1}\theta_U(z)]^{-1}P^{\mathcal{F}}_{\mathcal{N}}(1 - zVP^{\mathcal{F}}_{\mathcal{D}(V)})^{-1}. \end{aligned}$$

The first equality stated in the above lemma is proved in [8, Lemma III.1]. The second one is derived from a straightforward but rather long computation we omit in the present discussion.

The Schur class $\mathcal{S}(\mathcal{N}, \mathcal{M})$ features the Arov-Grossman functional model, which is an essential tool in our investigations:

Theorem 2.3 [1]**.** *Let V be an isometric operator on a Hilbert space $\mathcal{H}$ with domain $\mathcal{D}(V)$, range $\mathcal{R}(V)$ and defect spaces $\mathcal{N}$ and $\mathcal{M}$. The map that to each minimal unitary extension $U \in \mathcal{L}(\mathcal{F})$ associates the function*

$$\theta_U(z) := P^{\mathcal{F}}_{\mathcal{M}} U(1 - zP^{\mathcal{F}}_{\mathcal{F}\ominus\mathcal{H}}U)^{-1}|_{\mathcal{N}}, \quad z \in \mathbb{D},$$

establishes a bijection between the family $\mathcal{U}(V)$ of all minimal unitary extensions of V and the Schur class $\mathcal{S}(\mathcal{N}, \mathcal{M})$.

3. Labeling of all the Hankel symbols for a given Hankel operator

We now turn our attention to the problem of describing the Hankel symbols of a given Hankel operator X. We first consider the case when $\|X\|_{PV} = 1$:

Theorem 3.1. *Let $T_1 \in \mathcal{L}(\mathcal{H}_1)$ and $T_2 \in \mathcal{L}(\mathcal{H}_2)$ be two contractions with minimal isometric dilations $V_1 \in \mathcal{L}(\mathcal{K}_1)$ and $V_2 \in \mathcal{L}(\mathcal{K}_2)$, respectively. For $j = 1, 2$, let $\mathcal{R}_j$ be the subspace of $\mathcal{K}_j$ which reduces V_j to its unitary part. Given X, a Hankel operator for T_1 and T_2, with $\|X\|_{PV} = 1$, let $\widetilde{X} \in \mathcal{L}(\mathcal{E}_1, \mathcal{E}_2)$ be the contraction operator uniquely determined by X as in Lemma 2.1. Then there is a bijection between the set $\mathcal{HS}(X)$ of all Hankel symbols of X and the Schur class $\mathcal{S}(\mathcal{N}, \mathcal{M})$, where*

$$\mathcal{N} := \mathcal{D}_{\widetilde{X}} \ominus D_{\widetilde{X}} V_1^* P^{\mathcal{K}_1}_{\mathcal{R}_1} \mathcal{H}_1$$

and

$$\mathcal{M} := \{(e_1, e_2) \in \mathcal{D}_{\widetilde{X}} \oplus \mathcal{E}_2 \colon\ T_2 P^{\mathcal{K}_2}_{\mathcal{H}_2} e_2 = 0 \text{ and } D_{\widetilde{X}} e_1 + \widetilde{X}^* e_2 = 0\}.$$

P r o o f. We proceed by steps.

S T E P 1. In the first step we build up a Hilbert space $\mathcal{H}$ and an isometry V acting on $\mathcal{H}$ whose defect spaces are the Hilbert spaces $\mathcal{N}$ and $\mathcal{M}$ in the statement of the theorem.

We recall that $\mathcal{E}_j := \overline{P^{\mathcal{K}_j}_{\mathcal{R}_j}\mathcal{H}_j}$, $j = 1, 2$, and that $\widetilde{X}$ is a contraction from $\mathcal{E}_1$ into $\mathcal{E}_2$ such that $X = (P^{\mathcal{K}_2}_{\mathcal{R}_2}|_{\mathcal{H}_2})^* \widetilde{X} P^{\mathcal{K}_1}_{\mathcal{R}_1}|_{\mathcal{H}_1}$.

We set $\mathcal{H} := \mathcal{D}_{\widetilde{X}} \oplus \mathcal{E}_2$ with the standard inner product of $\mathcal{E}_1 \oplus \mathcal{E}_2$ and we define a Hermitian sesquilinear form $[\cdot,\cdot]$ on $\mathcal{H}_1 \times \mathcal{H}_2$ by setting

$$[(h_1, h_2), (h_1', h_2')] := \langle P^{\mathcal{K}_1}_{\mathcal{R}_1} h_1, h_1'\rangle + \langle X h_1, h_2'\rangle + \langle h_2, X h_1'\rangle + \langle P^{\mathcal{K}_2}_{\mathcal{R}_2} h_2, h_2'\rangle.$$

Then, for all $(h_1, h_2) \in \mathcal{H}_1 \times \mathcal{H}_2$,

$$\begin{aligned}[(h_1, h_2), (h_1, h_2)] &= \|P^{\mathcal{K}_1}_{\mathcal{R}_1} h_1\|^2 + 2\mathrm{Re}\langle \widetilde{X} P^{\mathcal{K}_1}_{\mathcal{R}_1} h_1, P^{\mathcal{K}_2}_{\mathcal{R}_2} h_2\rangle + \|P^{\mathcal{K}_2}_{\mathcal{R}_2} h_2\|^2 \\ &= \|P^{\mathcal{K}_1}_{\mathcal{R}_1} h_1\|^2 - \|\widetilde{X} P^{\mathcal{K}_1}_{\mathcal{R}_1} h_1\|^2 + \|\widetilde{X} P^{\mathcal{K}_1}_{\mathcal{R}_1} h_1 + P^{\mathcal{K}_2}_{\mathcal{R}_2} h_2\|^2 \\ &= \|D_{\widetilde{X}} P^{\mathcal{K}_1}_{\mathcal{R}_1} h_1\|^2 + \|\widetilde{X} P^{\mathcal{K}_1}_{\mathcal{R}_1} h_1 + P^{\mathcal{K}_2}_{\mathcal{R}_2} h_2\|^2.\end{aligned}$$

Therefore, if σ is defined on $\mathcal{H}_1 \times \mathcal{H}_2$ by

$$\sigma(h_1, h_2) := (D_{\widetilde{X}} P^{\mathcal{K}_1}_{\mathcal{R}_1} h_1, \widetilde{X} P^{\mathcal{K}_1}_{\mathcal{R}_1} h_1 + P^{\mathcal{K}_2}_{\mathcal{R}_2} h_2), \quad h_1 \in \mathcal{H}_1, h_2 \in \mathcal{H}_2,$$

then σ is an isometry from $(\mathcal{H}_1 \times \mathcal{H}_2, [\cdot,\cdot])$ onto a dense subspace of $\mathcal{H}$.

For all $h_j \in \mathcal{H}_j$ $(j = 1, 2)$,

$$\|P^{\mathcal{K}_j}_{\mathcal{R}_j} T_j^* h_j\| = \|P^{\mathcal{K}_j}_{\mathcal{R}_j} V_j^* h_j\| = \|V_j^* P^{\mathcal{K}_j}_{\mathcal{R}_j} h_j\| = \|P^{\mathcal{K}_j}_{\mathcal{R}_j} h_j\|.$$

From this and the relation $XT_1^* = T_2 X$, it follows that, for all $h_1 \in \mathcal{H}_1$ and $h_2 \in \mathcal{H}_2$,

$$\begin{aligned}[(T_1^* h_1, h_2), (T_1^* h_1, h_2)] &= \|P^{\mathcal{K}_1}_{\mathcal{R}_1} T_1^* h_1\|^2 + 2\mathrm{Re}\langle X T_1^* h_1, h_2\rangle + \|P^{\mathcal{K}_2}_{\mathcal{R}_2} h_2\|^2 \\ &= [(h_1, T_2^* h_2), (h_1, T_2^* h_2)].\end{aligned}$$

Hence, the linear operator $V\colon \overline{\sigma(T_1^*\mathcal{H}_1 \times \mathcal{H}_2)} \to \overline{\sigma(\mathcal{H}_1 \times T_2^*\mathcal{H}_2)}$ densely defined by

$$V\sigma(T_1^* h_1, h_2) := \sigma(h_1, T_2^* h_2), \quad h_1 \in \mathcal{H}_1, h_2 \in \mathcal{H}_2,$$

is an isometry on $\mathcal{H}$ with domain $\mathcal{D}(V) := \overline{\sigma(T_1^*\mathcal{H}_1 \times \mathcal{H}_2)}$ and range $\mathcal{R}(V) := \overline{\sigma(\mathcal{H}_1 \times T_2^*\mathcal{H}_2)}$.

Let $\mathcal{N} := \mathcal{H} \ominus \mathcal{D}(V)$ and $\mathcal{M} := \mathcal{H} \ominus \mathcal{R}(V)$ be the defect spaces of V. Then a straightforward computation gives that

$$\mathcal{N} = \mathcal{D}_{\widetilde{X}} \ominus D_{\widetilde{X}} V_1^* P^{\mathcal{K}_1}_{\mathcal{R}_1} \mathcal{H}_1$$

and, also, using that $\mathcal{R}_1 = \mathcal{E}_1 \oplus (\mathcal{R}_1 \cap \mathcal{H}_1^{\perp})$ (cf. [9, Lemma 2.3],) that

$$\mathcal{M} = \left\{(e_1, e_2) \in \mathcal{D}_{\widetilde{X}} \oplus \mathcal{E}_2 \colon\ T_2 P^{\mathcal{K}_2}_{\mathcal{H}_2} e_2 = 0 \text{ and } D_{\widetilde{X}} e_1 + \widetilde{X}^* e_2 = 0\right\}.$$

S T E P 2. We show that each minimal unitary extension U of V gives rise to a Hankel symbol Z of X.

Let $U \in \mathcal{L}(\mathcal{F})$ be a minimal unitary extension of V. If $n \in \mathbb{N} \cup \{0\}$ and $h_1, h_1' \in \mathcal{H}_1$, then

$$\begin{aligned}\langle U^n \sigma(h_1, 0), \sigma(h_1', 0)\rangle_{\mathcal{F}} &= \langle \sigma(h_1, 0), U^{*n} \sigma(h_1', 0)\rangle_{\mathcal{F}} = \langle \sigma(h_1, 0), V^{*n} \sigma(h_1', 0)\rangle_{\mathcal{H}} \\ &= \langle \sigma(h_1, 0), \sigma(T_1^{*n} h_1', 0)\rangle_{\mathcal{H}} = \langle P^{\mathcal{K}_1}_{\mathcal{R}_1} h_1, T_1^{*n} h_1'\rangle \\ &= \langle P^{\mathcal{K}_1}_{\mathcal{R}_1} h_1, V_1^{*n} h_1'\rangle = \langle V_1^n P^{\mathcal{K}_1}_{\mathcal{R}_1} h_1, h_1'\rangle = \langle P^{\mathcal{K}_1}_{\mathcal{R}_1} V_1^n h_1, h_1'\rangle.\end{aligned}$$

Thus, for all $\{h_1(m)\}_{m=0}^{\infty} \subseteq \mathcal{H}_1$ and $M \in \mathbb{N} \cup \{0\}$,

$$\left\| \sum_{m=0}^{M} U^m \sigma(h_1(m), 0) \right\|_{\mathcal{F}} = \left\| P^{\mathcal{K}_1}_{\mathcal{R}_1} \sum_{m=0}^{M} V_1^m h_1(m) \right\|. \tag{2}$$

Analogously, for all $\{h_2(m)\}_{m=0}^{\infty} \subseteq \mathcal{H}_2$ and $M \in \mathbb{N} \cup \{0\}$,

$$\left\| \sum_{m=0}^{M} U^{*m} \sigma(0, h_2(m)) \right\|_{\mathcal{F}} = \left\| P^{\mathcal{K}_2}_{\mathcal{R}_2} \sum_{m=0}^{M} V_2^m h_2(m) \right\|. \tag{3}$$

We define $\varphi_1 \colon \mathcal{R}_1 \to \mathcal{F}$ and $\varphi_2 \colon \mathcal{R}_2 \to \mathcal{F}$ by means of the relations

$$\varphi_1 P^{\mathcal{K}_1}_{\mathcal{R}_1} V_1^n h_1 := U^n \sigma(h_1, 0), \quad h_1 \in \mathcal{H}_1, n \in \mathbb{N} \cup \{0\},$$

and

$$\varphi_2 P^{\mathcal{K}_2}_{\mathcal{R}_2} V_2^n h_2 := U^{*n} \sigma(0, h_2), \quad h_2 \in \mathcal{H}_2, n \in \mathbb{N} \cup \{0\}.$$

Then, according to (2) and (3), both φ_1 and φ_2 are isometries. Furthermore, if $h_1 \in \mathcal{H}_1$ then

$$\varphi_1 P^{\mathcal{K}_1}_{\mathcal{R}_1} V_1^* h_1 = \varphi_1 P^{\mathcal{K}_1}_{\mathcal{R}_1} T_1^* h_1 = \sigma(T_1^* h_1, 0) = U^* \sigma(h_1, 0) = U^* \varphi_1 P^{\mathcal{K}_1}_{\mathcal{R}_1} h_1$$

and, for all $n \in \mathbb{N}$,

$$\begin{aligned}\varphi_1 P^{\mathcal{K}_1}_{\mathcal{R}_1} V_1^* V_1^n h_1 = \varphi_1 P^{\mathcal{K}_1}_{\mathcal{R}_1} V_1^{n-1} h_1 = U^{n-1} \sigma(h_1, 0) = U^* U^n \sigma(h_1, 0) \\ = U^* \varphi_1 P^{\mathcal{K}_1}_{\mathcal{R}_1} V_1^n h_1.\end{aligned}$$

Hence,

$$\varphi_1 P^{\mathcal{K}_1}_{\mathcal{R}_1} V_1^* k_1 = U^* \varphi_1 P^{\mathcal{K}_1}_{\mathcal{R}_1} k_1, \quad \text{for all } k_1 \in \mathcal{K}_1. \tag{4}$$

In a similar way it can be proved that

$$\varphi_2 P^{\mathcal{K}_2}_{\mathcal{R}_2} V_2^* k_2 = U \varphi_2 P^{\mathcal{K}_2}_{\mathcal{R}_2} k_2, \quad \text{for all } k_2 \in \mathcal{K}_2. \tag{5}$$

For the given minimal unitary extension U of V acting on $\mathcal{F}$, define $Z\colon \mathcal{K}_1 \to \mathcal{K}_2$ by

$$\langle Zk_1, k_2\rangle := \langle \varphi_1 P^{\mathcal{K}_1}_{\mathcal{R}_1} k_1, \varphi_2 P^{\mathcal{K}_2}_{\mathcal{R}_2} k_2\rangle_{\mathcal{F}}, \quad k_1 \in \mathcal{K}_1, k_2 \in \mathcal{K}_2.$$

Then, for all $k_1 \in \mathcal{K}_1$ and $k_2 \in \mathcal{K}_2$,

$$|\langle Zk_1, k_2\rangle| \leqslant \|P^{\mathcal{K}_1}_{\mathcal{R}_1} k_1\| \|P^{\mathcal{K}_2}_{\mathcal{R}_2} k_2\| \leqslant \|k_1\| \|k_2\|.$$

This shows that Z is a bounded linear operator with $\|Z\| \leqslant 1$.

On the other hand, for all $h_1 \in \mathcal{H}_1$ and $h_2 \in \mathcal{H}_2$,

$$\langle Zh_1, h_2\rangle = \langle \sigma(h_1, 0), \sigma(0, h_2)\rangle_{\mathcal{F}} = \langle Xh_1, h_2\rangle.$$

Hence, $P^{\mathcal{K}_2}_{\mathcal{H}_2} Z|_{\mathcal{H}_1} = X$. Moreover, as $\langle Zk_1, k_2\rangle = \langle ZP^{\mathcal{K}_1}_{\mathcal{R}_1} k_1, P^{\mathcal{K}_2}_{\mathcal{R}_2} k_2\rangle$, for all $k_1 \in \mathcal{K}_1$ and $k_2 \in \mathcal{K}_2$, $\|Z\| \geqslant \|X\|_{PV} = 1$, so that $\|Z\| = 1$.

In order to show that Z is a Hankel symbol of X it remains to show that $ZV_1^* = V_2 Z$. Let $k_1 \in \mathcal{K}_1$ and $k_2 \in \mathcal{K}_2$ be given. From (4) and (5) we get that

$$\begin{aligned}
\langle ZV_1^* k_1, k_2\rangle &= \langle \varphi_1 P^{\mathcal{K}_1}_{\mathcal{R}_1} V_1^* k_1, \varphi_2 P^{\mathcal{K}_2}_{\mathcal{R}_2} k_2\rangle_{\mathcal{F}} = \langle U^* \varphi_1 P^{\mathcal{K}_1}_{\mathcal{R}_1} k_1, \varphi_2 P^{\mathcal{K}_2}_{\mathcal{R}_2} k_2\rangle_{\mathcal{F}} \\
&= \langle \varphi_1 P^{\mathcal{K}_1}_{\mathcal{R}_1} k_1, U \varphi_2 P^{\mathcal{K}_2}_{\mathcal{R}_2} k_2\rangle_{\mathcal{F}} = \langle \varphi_1 P^{\mathcal{K}_1}_{\mathcal{R}_1} k_1, \varphi_2 P^{\mathcal{K}_2}_{\mathcal{R}_2} V_2^* k_2\rangle_{\mathcal{F}} \\
&= \langle Zk_1, V_2^* k_2\rangle = \langle V_2 Z k_1, k_2\rangle.
\end{aligned}$$

The above discussion provides a new proof of the existence of Hankel symbols of the given Hankel operator X, as to each minimal unitary extension U of V there corresponds a Hankel symbol Z of X.

It turns out that any Hankel symbol Z of X can be obtained as before from a minimal unitary extension U of the isometry V associated to X, T_1 and T_2. We prove that in the next step.

STEP 3. If Z is a Hankel symbol of X, then $Z = P^{\mathcal{K}_2}_{\mathcal{R}_2} Z = ZP^{\mathcal{K}_1}_{\mathcal{R}_1}$ (see, for instance, [11, Proposition 2.1].) Whence, by setting

$$[(k_1, k_2), (k_1', k_2')] := \langle P^{\mathcal{K}_1}_{\mathcal{R}_1} k_1, k_1'\rangle + \langle Zk_1, k_2'\rangle + \langle k_2, Zk_1'\rangle + \langle P^{\mathcal{K}_2}_{\mathcal{R}_2} k_2, k_2'\rangle,$$

for $k_1, k_1' \in \mathcal{K}_1$ and $k_2, k_2' \in \mathcal{K}_2$, we get a Hermitian sesquilinear form on $\mathcal{K}_1 \times \mathcal{K}_2$ such that, for all $(k_1, k_2) \in \mathcal{K}_1 \times \mathcal{K}_2$,

$$[(k_1, k_2), (k_1, k_2)] = \|D_Z P_{\mathcal{R}_1}^{\mathcal{K}_1} k_1\|^2 + \|Z k_1 + P_{\mathcal{R}_2}^{\mathcal{K}_2} k_2\|^2.$$

Therefore, if $\mathcal{F} := \mathcal{D}_Z \oplus \mathcal{R}_2$, with the standard inner product of $\mathcal{R}_1 \oplus \mathcal{R}_2$, and τ is defined on $\mathcal{K}_1 \times \mathcal{K}_2$ as

$$\tau(k_1, k_2) := (D_Z P_{\mathcal{R}_1}^{\mathcal{K}_1} k_1, Z k_1 + P_{\mathcal{R}_2}^{\mathcal{K}_2} k_2), \quad k_1 \in \mathcal{K}_1, k_2 \in \mathcal{K}_2,$$

then τ is an isometry from $(\mathcal{K}_1 \times \mathcal{K}_2, [\cdot, \cdot])$ onto a dense subspace of $\mathcal{F}$ such that

$$\tau(k_1, k_2) = \tau(P_{\mathcal{R}_1}^{\mathcal{K}_1} k_1, P_{\mathcal{R}_2}^{\mathcal{K}_2} k_2), \quad \text{for all } k_1 \in \mathcal{K}_1 \text{ and } k_2 \in \mathcal{K}_2. \tag{6}$$

Since $P_{\mathcal{H}_2}^{\mathcal{K}_2} Z|_{\mathcal{H}_1} = X$, it readily follows that, for all $h_1 \in \mathcal{H}_1$ and $h_2 \in \mathcal{H}_2$, $\|\sigma(h_1, h_2)\|_{\mathcal{H}} = \|\tau(h_1, h_2)\|_{\mathcal{F}}$. Then, via the isometric operator $\varrho\colon \mathcal{H} \to \mathcal{F}$, $\varrho\sigma := \tau|_{\mathcal{H}_1 \times \mathcal{H}_2}$, the Hilbert space $\mathcal{H}$ can be regarded as a closed linear subspace of the Hilbert space $\mathcal{F}$.

Set

$$U\tau(k_1, k_2) := \tau(V_1 k_1, V_2^* k_2), \quad k_1 \in \mathcal{K}_1, k_2 \in \mathcal{K}_2.$$

As $Z V_1^* = V_2 Z$, the operator U is shown to be isometric. On the other hand, if $k_1 \in \mathcal{K}_1$ and $k_2 \in \mathcal{K}_2$ are given, then, according to (6),

$$\begin{aligned}\tau(k_1, k_2) &= \tau(P_{\mathcal{R}_1}^{\mathcal{K}_1} k_1, P_{\mathcal{R}_2}^{\mathcal{K}_2} k_2)\tau(V_1 V_1^* P_{\mathcal{R}_1}^{\mathcal{K}_1} k_1, V_2^* V_2 P_{\mathcal{R}_2}^{\mathcal{K}_2} k_2) \\ &= U\tau(V_1^* P_{\mathcal{R}_1}^{\mathcal{K}_1} k_1, V_2 P_{\mathcal{R}_2}^{\mathcal{K}_2} k_2).\end{aligned}$$

It thus turns out that the extension of U to all of $\mathcal{F}$ is a surjective isometry, that is, a unitary operator.

For any $h_1 \in \mathcal{H}_1$ and $h_2 \in \mathcal{H}_2$,

$$\begin{aligned}U\varrho\sigma(T_1^* h_1, h_2) &= U\tau(T_1^* h_1, h_2) = U\tau(V_1^* h_1, h_2) \\ &= U\tau(P_{\mathcal{R}_1}^{\mathcal{K}_1} V_1^* h_1, P_{\mathcal{R}_2}^{\mathcal{K}_2} h_2) = U\tau(V_1^* P_{\mathcal{R}_1}^{\mathcal{K}_1} h_1, P_{\mathcal{R}_2}^{\mathcal{K}_2} h_2) \\ &= \tau(V_1 V_1^* P_{\mathcal{R}_1}^{\mathcal{K}_1} h_1, V_2^* P_{\mathcal{R}_2}^{\mathcal{K}_2} h_2) = \tau(P_{\mathcal{R}_1}^{\mathcal{K}_1} h_1, P_{\mathcal{R}_2}^{\mathcal{K}_2} V_2^* h_2) \\ &= \tau(h_1, V_2^* h_2) = \tau(h_1, T_2^* h_2) \\ &= \varrho\sigma(h_1, T_2^* h_2) = \varrho V\sigma(T_1^* h_1, h_2).\end{aligned}$$

Therefore, $U\varrho|_{\mathcal{D}(V)} = \varrho V$. In other words, by means of the isometry ϱ, U can be interpreted as a unitary extension of V.

If $\varphi_1\colon \mathcal{R}_1 \to \mathcal{F}$ and $\varphi_2\colon \mathcal{R}_2 \to \mathcal{F}$ are defined as

$$\varphi_1 r_1 := \tau(r_1, 0), \quad r_1 \in \mathcal{R}_1 \quad \text{and} \quad \varphi_2 r_2 := \tau(0, r_2), \quad r_2 \in \mathcal{R}_2,$$

then φ_1 and φ_2 are isometries such that
(i) $\varphi_1\mathcal{R}_1 \vee \varphi_2\mathcal{R}_2 = \mathcal{F}$
and, for all $k_1 \in \mathcal{K}_1$ and $k_2 \in \mathcal{K}_2$,
(ii) $\langle \varphi_1 P^{\mathcal{K}_1}_{\mathcal{R}_1} k_1, \varphi_2 P^{\mathcal{K}_2}_{\mathcal{R}_2} k_2\rangle_{\mathcal{F}} = \langle Zk_1, k_2\rangle$.
Furthermore, for all $h_1 \in \mathcal{H}_1$, $h_2 \in \mathcal{H}_2$ and $n \in \mathbb{N} \cup \{0\}$,
(iii) $\varphi_1 P^{\mathcal{K}_1}_{\mathcal{R}_1} V_1^n h_1 = U^n \varphi_1 P^{\mathcal{K}_1}_{\mathcal{R}_1} h_1 = U^n \varrho\sigma(h_1, 0)$
and
(iv) $\varphi_2 P^{\mathcal{K}_2}_{\mathcal{R}_2} V_2^n h_2 = U^{*n} \varphi_2 P^{\mathcal{K}_2}_{\mathcal{R}_2} h_2 = U^{*n} \varrho\sigma(0, h_2)$.

From (iii) and (iv) we get that

$$\bigvee_{n=0}^{\infty} U^n \varrho\sigma(\mathcal{H}_1 \times \{0\}) = \bigvee_{n=0}^{\infty} U^n \varphi_1 P^{\mathcal{K}_1}_{\mathcal{R}_1}\mathcal{H}_1 = \bigvee_{n=0}^{\infty} \varphi_1 P^{\mathcal{K}_1}_{\mathcal{R}_1} V_1^n \mathcal{H}_1 \varphi_1\mathcal{R}_1$$

and

$$\bigvee_{n=0}^{\infty} U^{*n} \varrho\sigma(\{0\} \times \mathcal{H}_2) \bigvee_{n=0}^{\infty} U^{*n} \varphi_2 P^{\mathcal{K}_2}_{\mathcal{R}_2}\mathcal{H}_2 = \bigvee_{n=0}^{\infty} \varphi_2 P^{\mathcal{K}_2}_{\mathcal{R}_2} V_2^n \mathcal{H}_2 \varphi_2\mathcal{R}_2.$$

From the above relations and by applying (i), we conclude that $\mathcal{F} = \bigvee_{n\in\mathbb{Z}} U^n \varrho\mathcal{H}$, which shows that U is minimal.

Finally, (ii), (iii) and (iv) say that Z is given by U as in the correspondence we established in STEP 2.

Therefore, we can conclude that the correspondence which associates to each minimal unitary extension U of V a Hankel symbol Z of X is surjective.

S T E P 4. Next we show that the correspondence is injective.

Assume that $U \in \mathcal{L}(\mathcal{F})$ and $U' \in \mathcal{L}(\mathcal{F}')$ are two minimal unitary extensions of V and let Z and Z' be the corresponding Hankel symbols. If $Z = Z'$ then, for all $h_1 \in \mathcal{H}_1$, $h_2 \in \mathcal{H}_2$ and $n \in \mathbb{N} \cup \{0\}$,

$$\begin{aligned}\langle U^n \sigma(h_1, 0), \sigma(0, h_2)\rangle_{\mathcal{F}} &= \langle ZV_1^n h_1, h_2\rangle \\ &= \langle Z'V_1^n h_1, h_2\rangle = \langle {U'}^n \sigma(h_1, 0), \sigma(0, h_2)\rangle_{\mathcal{F}'}.\end{aligned}$$

This, together with the minimality condition satisfied by both U and U' and the fact that $\mathcal{H} = \sigma(\mathcal{H}_1 \times \{0\}) \vee \sigma(\{0\} \times \mathcal{H}_2)$, grants that U and U' are isometrically isomorphic, since, besides, when $n \in \mathbb{N} \cup \{0\}$,

$$\langle U^n \sigma(h_1, 0), \sigma(h_1', 0)\rangle_{\mathcal{F}} = \langle P^{\mathcal{K}_1}_{\mathcal{R}_1} V_1^n h_1, h_1'\rangle = \langle U'^n \sigma(h_1, 0), \sigma(h_1', 0)\rangle_{\mathcal{F}'},$$

for all $h_1, h_1' \in \mathcal{H}_1$, and

$$\langle U^n \sigma(0,h_2), \sigma(0,h_2')\rangle_{\mathcal{F}} = \langle P^{\mathcal{K}_2}_{\mathcal{R}_2} V_2^{*n} h_2, h_2'\rangle = \langle U'^n \sigma(0,h_2), \sigma(0,h_2')\rangle_{\mathcal{F}'},$$

for all $h_2, h_2' \in \mathcal{H}_2$.

As a summary of what we have achieved so far, let us point out that we have built up a Hilbert space $\mathcal{H}$ and an isometry V acting on $\mathcal{H}$ such that if $U \in \mathcal{L}(\mathcal{F})$ is a minimal isometric extension of V, then there exist two isometries $\varphi_1 \colon \mathcal{R}_1 \to \mathcal{F}$ and $\varphi_2 \colon \mathcal{R}_2 \to \mathcal{F}$ such that the operator $Z := P^{\mathcal{K}_2}_{\mathcal{R}_2} \varphi_2^* \varphi_1 P^{\mathcal{K}_1}_{\mathcal{R}_1}$ is a symbol of X, and the mapping $U \to Z$ is a bijection between the family $\mathcal{U}(V)$ of all minimal unitary extensions of V and the set $\mathcal{HS}(X)$ of all Hankel symbols of X.

At this point we are ready to consider the problem of labeling the set $\mathcal{HS}(X)$.

S T E P 5. Let $Z \in \mathcal{HS}(X)$ be given. It readily follows that $ZV_1 = V_2^* Z$ (cf. [11, Proposition 2.1].) Thus, for all $h_1 \in \mathcal{H}_1$ and $n \in \mathbb{N}$, $ZV_1^n h_1 = V_2^{*n} Z h_1$. From this, as $\mathcal{K}_1 = \bigvee_{n=0}^{\infty} V_1^n \mathcal{H}_1$, it follows that Z is fully determined by its restriction to $\mathcal{H}_1$. On the other hand, since $\mathcal{K}_2 = \mathcal{H}_2 \oplus \bigoplus_{n=0}^{\infty} V_2^n \mathcal{L}_2$, where $\mathcal{L}_2 := \overline{(V_2 - T_2)\mathcal{H}_2}$, it turns out that $Z|_{\mathcal{H}_1} = X + \sum_{n=0}^{\infty} P^{\mathcal{K}_2}_{V_2^n \mathcal{L}_2} Z|_{\mathcal{H}_1}$. Therefore, Z is indeed determined by the sequence of operators $\{P^{\mathcal{K}_2}_{V_2^n \mathcal{L}_2} Z|_{\mathcal{H}_1}\}_{n=0}^{\infty}$.

To each $Z \in \mathcal{HS}(X)$ we associate the power series

$$S_Z(z) := \sum_{n=0}^{\infty} z^n \widehat{S_Z}(n), \quad z \in \mathbb{D}, \quad \widehat{S_Z}(n) := V_2^{*n} P^{\mathcal{K}_2}_{V_2^n \mathcal{L}_2} Z|_{\mathcal{H}_1}, \quad n \geqslant 0,$$

so that S_Z is an $\mathcal{L}(\mathcal{H}_1, \mathcal{L}_2)$-valued function defined and analytic on $\mathbb{D}$. We get that, for all $h_1 \in \mathcal{H}_1$ and $h_2 \in \mathcal{H}_2$,

$$\begin{aligned}
\langle S_Z(z)h_1, (V_2 - T_2)h_2\rangle &= \sum_{n=0}^{\infty} z^n \langle \widehat{S_Z}(n)h_1, (V_2 - T_2)h_2\rangle \\
&= \sum_{n=0}^{\infty} z^n \left[\langle Zh_1, V_2^{n+1} h_2\rangle - \langle Zh_1, V_2^n T_2 h_2\rangle\right] \\
&= \sum_{n=0}^{\infty} z^n \left[\langle \sigma(h_1,0), U^{*n+1}\sigma(0,h_2)\rangle_{\mathcal{F}} - \langle \sigma(h_1,0), U^{*n+1}\sigma(0,T_2^* T_2 h_2)\rangle_{\mathcal{F}}\right] \\
&= \sum_{n=0}^{\infty} z^n \langle \sigma(h_1,0), U^{*n+1}\sigma(0, h_2 - T_2^* T_2 h_2)\rangle_{\mathcal{F}} \\
&= \langle U(1 - zU)^{-1}\sigma(h_1,0), \sigma(0, h_2 - V_2^* T_2 h_2)\rangle_{\mathcal{F}}.
\end{aligned}$$

Set $\mathcal{L} := \overline{\{\sigma(0, h_2 - V_2^* T_2 h_2)\colon\ h_2 \in \mathcal{H}_2\}}$ and define $\varrho\colon\ \mathcal{L} \to \mathcal{R}_2$ by

$$\varrho\sigma(0, h_2 - V_2^* T_2 h_2) := P_{\mathcal{R}_2}^{\mathcal{K}_2}(V_2 - T_2)h_2, \quad h_2 \in \mathcal{H}_2.$$

It can be easily seen that ϱ is an isometry. Thus, for all $h_1 \in \mathcal{H}_1$ and $h_2 \in \mathcal{H}_2$,

$$\langle S_Z(z)h_1, (V_2 - T_2)h_2\rangle = \langle \varrho P_{\mathcal{L}}^{\mathcal{F}} U(1 - zU)^{-1}\sigma(h_1, 0), (V_2 - T_2)h_2\rangle.$$

Therefore, setting $\varsigma h_1 := \sigma(h_1, 0)$ for $h_1 \in \mathcal{H}_1$, we get that

$$S_Z(z) = \varrho P_{\mathcal{L}}^{\mathcal{F}} U(1 - zU)^{-1}\varsigma. \tag{7}$$

On the other hand, if

$$\theta_U(z) := P_{\mathcal{M}}^{\mathcal{F}} U(1 - zP_{\mathcal{F}\ominus\mathcal{H}}^{\mathcal{F}} U)^{-1}|_{\mathcal{N}}, \quad z \in \mathbb{D},$$

is the $\mathcal{S}(\mathcal{N}, \mathcal{M})$ function associated to U via the Arov-Grossmann functional model (Theorem 2.3), then, according to Lemma 2.2,

$$\begin{aligned} &P_{\mathcal{H}}^{\mathcal{F}} U(1 - zU)^{-1}|_{\mathcal{H}} \\ &\quad = (VP_{\mathcal{D}(V)}^{\mathcal{F}} + \theta_U(z)P_{\mathcal{N}}^{\mathcal{F}})[1 - z(VP_{\mathcal{D}(V)}^{\mathcal{F}} + \theta_U(z)P_{\mathcal{N}}^{\mathcal{F}})]^{-1}|_{\mathcal{H}} \\ &\quad = VP_{\mathcal{D}(V)}^{\mathcal{F}}(1 - zVP_{\mathcal{D}(V)}^{\mathcal{F}})^{-1} + [zVP_{\mathcal{D}(V)}^{\mathcal{F}}(1 - zVP_{\mathcal{D}(V)}^{\mathcal{F}})^{-1} + 1] \\ &\qquad \times \theta_U(z)[1 - zP_{\mathcal{N}}^{\mathcal{F}}(1 - zVP_{\mathcal{D}(V)}^{\mathcal{F}})^{-1}\theta_U(z)]^{-1}P_{\mathcal{N}}^{\mathcal{F}}(1 - zVP_{\mathcal{D}(V)}^{\mathcal{F}})^{-1}. \end{aligned}$$

From this expression and (7), we get that, for all $z \in \mathbb{D}$,

$$S_Z(z) = a(z) + b(z)\theta(z)(1 - c(z)\theta(z))^{-1}d(z), \tag{8}$$

where $\theta \equiv \theta_U$ and

$$\begin{aligned} a(z) &:= \varrho P_{\mathcal{L}}^{\mathcal{H}} VP_{\mathcal{D}(V)}^{\mathcal{H}}(1 - zVP_{\mathcal{D}(V)}^{\mathcal{H}})^{-1}\varsigma \in \mathcal{L}(\mathcal{H}_1, \mathcal{R}_2), \\ b(z) &:= \varrho P_{\mathcal{L}}^{\mathcal{H}}[1 + zVP_{\mathcal{D}(V)}^{\mathcal{H}}(1 - zVP_{\mathcal{D}(V)}^{\mathcal{H}})^{-1}]|_{\mathcal{M}} \in \mathcal{L}(\mathcal{M}, \mathcal{R}_2), \\ c(z) &:= zP_{\mathcal{N}}^{\mathcal{H}}(1 - zVP_{\mathcal{D}(V)}^{\mathcal{H}})^{-1}|_{\mathcal{M}} \in \mathcal{L}(\mathcal{M}, \mathcal{N}), \\ d(z) &:= P_{\mathcal{N}}^{\mathcal{H}}(1 - zVP_{\mathcal{D}(V)}^{\mathcal{H}})^{-1}\varsigma \in \mathcal{L}(\mathcal{H}_1, \mathcal{N}). \end{aligned}$$

The Schur-like formula (8) establishes the direct connection between $\mathcal{S}(\mathcal{N}, \mathcal{M})$ and $\{S_Z\colon\ Z \in \mathcal{HS}(X)\}$. Finally, the map

$$\theta \longrightarrow S_Z$$

determined by (8) is a bijection between $\mathcal{S}(\mathcal{N},\mathcal{M})$ and $\mathcal{HS}(X)$, since the mappings

$$\begin{array}{ccc} U \in \mathcal{U}(V) & \longrightarrow & Z \in \mathcal{HS}(X) \\ \downarrow & & \downarrow \\ \theta \in \mathcal{S}(\mathcal{N},\mathcal{M}) & & \{\widehat{S_Z}(n)\} \\ & & \downarrow \\ & & S_Z \end{array}$$

are all bijections (up to isomorphism as far as $U \in \mathcal{U}(V)$ is concerned). This completes the proof. $\square$

Remark 3.2. The hypothesis that $\|X\|_{PV} = 1$ can be dropped as long as we deal with symbols Z of X such that $\|Z\| = \|X\|_{PV}$. Clearly, if X is a Hankel operator for T_1 and T_2 with $\|X\|_{PV} = \beta > 0$, then $X' := \beta^{-1}X$ is a Hankel operator for T_1 and T_2 with $\|X'\|_{PV} = 1$. Furthermore, $Z' \in \mathcal{HS}(X')$ if, and only if, $\beta Z' \in \mathcal{HS}(X)$. Also, the arguments in the proof of Theorem 3.1 can be slightly modified to replace $\mathcal{N}$ and $\mathcal{M}$ by

$$\mathcal{N} := \mathcal{D}^{\beta}_{\widetilde{X}} \ominus D^{\beta}_{\widetilde{X}} V_1^* P^{\mathcal{K}_1}_{\mathcal{R}_1} \mathcal{H}_1$$

and

$$\mathcal{M} := \{(e_1, e_2) \in \mathcal{D}^{\beta}_{\widetilde{X}} \oplus \mathcal{E}_2 \colon\ T_2 P^{\mathcal{K}_2}_{\mathcal{H}_2} e_2 = 0 \text{ and } D^{\beta}_{\widetilde{X}} e_1 + \widetilde{X}^* e_2 = 0\},$$

in order to get a bijective correspondence between the set $\mathcal{HS}(X)$ and the corresponding Schur class $\mathcal{S}(\mathcal{N},\mathcal{M})$. It can even be assumed that β is any fixed nonnegative number such that $\beta \geqslant \|X\|_{PV}$. If so, the bijection is established between $\mathcal{S}(\mathcal{N},\mathcal{M})$ and the larger set $\mathcal{HS}_\beta(X)$ of intertwining dilations Z of X satisfying $\|Z\| \leqslant \beta$.

We finally study the problem of determining whether the set $\mathcal{HS}(X)$ has a single element. From the remark it is clear that we may assume that $\|X\|_{PV} = 1$:

Theorem 3.3. *Let $T_1 \in \mathcal{L}(\mathcal{H}_1)$ and $T_2 \in \mathcal{L}(\mathcal{H}_2)$ be two contractions with minimal isometric dilations $V_1 \in \mathcal{L}(\mathcal{K}_1)$ and $V_2 \in \mathcal{L}(\mathcal{K}_2)$, respectively. For $j = 1, 2$, let $\mathcal{R}_j$ be the subspace of $\mathcal{K}_j$ which reduces V_j to its unitary part. Let X be a Hankel operator for T_1 and T_2 such that $\|X\|_{PV} = 1$. On the Hilbert space $\mathcal{H}_1 \oplus \mathcal{H}_2$, with the standard inner product, consider the 2×2 block matrix operators*

$$\widetilde{T}_1 := \begin{pmatrix} T_1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \widetilde{T}_2 := \begin{pmatrix} 1 & 0 \\ 0 & T_2 \end{pmatrix}$$

and

$$E := \begin{pmatrix} P^{\mathcal{K}_1}_{\mathcal{H}_1} P^{\mathcal{K}_1}_{\mathcal{R}_1}|_{\mathcal{H}_1} & X^* \\ X & P^{\mathcal{K}_2}_{\mathcal{H}_2} P^{\mathcal{K}_2}_{\mathcal{R}_2}|_{\mathcal{H}_2} \end{pmatrix}.$$

Then X has a unique Hankel symbol if, and only if, either

(a) $$\textit{kernel}\,(\widetilde{T}_1 E) \subseteq \textit{kernel}\, E$$

or

(b) $$\textit{kernel}\,(\widetilde{T}_2 E) \subseteq \textit{kernel}\, E.$$

Proof. We follow the notation introduced in the proof of Theorem 3.1. In particular, if $[\cdot,\cdot]$ is the inner product on $\mathcal{H}_1 \times \mathcal{H}_2$ and σ is the isometry from $(\mathcal{H}_1 \times \mathcal{H}_2, [\cdot,\cdot])$ onto a dense subspace of $\mathcal{H}$ as defined therein, then, for all $h_1, h_1' \in \mathcal{H}_1$ and $h_2, h_2' \in \mathcal{H}_2$,

(9) $$\langle \sigma(h_1,h_2), \sigma(h_1',h_2')\rangle_{\mathcal{H}} = [(h_1,h_2),(h_1',h_2')] = \langle E(h_1,h_2),(h_1',h_2')\rangle.$$

Therefore, if $\mathcal{A}$ is a subset of $\mathcal{H}_1 \times \mathcal{H}_2$, then $\sigma\mathcal{A} = \{0\}$ if, and only if, $\mathcal{A}$, regarded as a subset of $\mathcal{H}_1 \oplus \mathcal{H}_2$, is contained in the null space of E, that is, $\mathcal{A} \subseteq \operatorname{kernel} E$. Besides, on the other hand, if $\mathcal{A},\mathcal{B}$ are subsets of $\mathcal{H}_1 \times \mathcal{H}_2$ then $\overline{\sigma\mathcal{A}} \subseteq \overline{\sigma\mathcal{B}}$ if, and only if, $\overline{\operatorname{range}(E|_{\mathcal{A}})} \subseteq \overline{\operatorname{range}(E|_{\mathcal{B}})}$, with $\mathcal{A},\mathcal{B}$ interpreted as subsets of $\mathcal{H}_1 \oplus \mathcal{H}_2$.

We also recall that the Hilbert spaces $\mathcal{N}$ and $\mathcal{M}$ given in the statement of Theorem 3.1 are the defect spaces of the isometry V built up in its proof. Therefore, they can be expressed as
$$\mathcal{N} = \mathcal{H} \ominus \sigma(T_1^*\mathcal{H}_1 \times \mathcal{H}_2)$$
and
$$\mathcal{M} = \mathcal{H} \ominus \sigma(\mathcal{H}_1 \times T_2^*\mathcal{H}_2),$$
as $\overline{\sigma(T_1^*\mathcal{H}_1 \times \mathcal{H}_2)}$ and $\overline{\sigma(\mathcal{H}_1 \times T_2^*\mathcal{H}_2)}$ are, respectively, the domain and the range of V.

As the set $\mathcal{HS}(X)$ of the Hankel symbols of X is in bijective correspondence with the Schur class $\mathcal{S}(\mathcal{N},\mathcal{M})$, it is clear that $\mathcal{HS}(X)$ has a single element if, and only if, either $\mathcal{N} = \{0\}$ or $\mathcal{M} = \{0\}$. Thus, the theorem is proved if (a) and (b) are shown to be necessary and sufficient conditions for $\mathcal{N}$ and $\mathcal{M}$, respectively, to be trivial.

From (9) we get that if $(h_1,h_2) \in \operatorname{kernel}(\widetilde{T}_1 E)$ then, for all $h_1' \in \mathcal{H}_1$ and $h_2' \in \mathcal{H}_2$,

$$\begin{aligned}\langle \sigma(h_1,h_2), \sigma(T_1^* h_1', h_2')\rangle_{\mathcal{H}} &= [(h_1,h_2),(T_1^*h_1',h_2')] \\ &= \langle E(h_1,h_2), \widetilde{T}_1^*(h_1',h_2')\rangle \\ &= \langle \widetilde{T}_1 E(h_1,h_2),(h_1',h_2')\rangle = 0.\end{aligned}$$

Whence $\sigma(\operatorname{kernel}(\widetilde{T}_1 E)) \subseteq \mathcal{N}$. In a similar fashion it can be seen that $\sigma(\operatorname{kernel}(\widetilde{T}_2 E)) \subseteq \mathcal{M}$.

So, if $\mathcal{N} = \{0\}$, then $\text{kernel}\,(\widetilde{T}_1 E) \subseteq \text{kernel}\, E$, as $\sigma(\text{kernel}\,\overline{(\widetilde{T}_1 E))} = \{0\}$. On the other hand, if $\text{kernel}\,(\widetilde{T}_1 E) \subseteq \text{kernel}\, E$, then $\overline{\text{range}\, E} \subseteq \overline{\text{range}\,(E\widetilde{T}_1^*)}$, hence, $\mathcal{H} = \overline{\sigma(\mathcal{H}_1 \times \mathcal{H}_2)} \subseteq \overline{\sigma(T_1^* \mathcal{H}_1 \times \mathcal{H}_2)}$ and $\mathcal{N} = \{0\}$. This shows that $\mathcal{N} = \{0\}$ if, and only if, (a) holds. Similar arguments lead to the conclusion that $\mathcal{M} = \{0\}$ if, and only if, (b) holds true. This completes the proof. □

Remark 3.4. It is convenient to remark that, for a fixed pair of contractions $T_1 \in \mathcal{L}(\mathcal{H}_1)$ and $T_2 \in \mathcal{L}(\mathcal{H}_2)$, any Hankel operator X for T_1 and T_2 has a unique Hankel symbol, say Z_X, if either $T_1 P_{\mathcal{H}_1}^{\mathcal{K}_1}|_{\mathcal{E}_1}$ is injective or $T_2 P_{\mathcal{H}_2}^{\mathcal{K}_2}|_{\mathcal{E}_2}$ is injective.

Indeed, if $T_2 P_{\mathcal{H}_2}^{\mathcal{K}_2}|_{\mathcal{E}_2}$ is assumed to be injective, then

$$\mathcal{M} = \{(e_1, e_2) \in \mathcal{D}_{\widetilde{X}} \oplus \mathcal{E}_2 \colon\ T_2 P_{\mathcal{H}_2}^{\mathcal{K}_2} e_2 = 0 \text{ and } D_{\widetilde{X}} e_1 + \widetilde{X}^* e_2 = 0\} = \{0\}.$$

In a similar way one can show that if $T_1 P_{\mathcal{H}_1}^{\mathcal{K}_1}|_{\mathcal{E}_1}$ is injective, then $\mathcal{N} = \{0\}$. Therefore, if either of the conditions holds true, then any given Hankel operator X for T_1 and T_2 has a unique Hankel symbol Z_X.

Acknowledgement. A part of the first named author's work on this note was done while he was visiting the Instituto Venezolano de Investigaciones Científicas as graduate student fellow. The hospitality of the Department of Mathematics is gratefully acknowledged.

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