Czechoslovak Mathematical Journal

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Czechoslovak Mathematical Journal, Vol. 56 (2006), No. 4, 1185–1192

Persistent URL: `http://dml.cz/dmlcz/128138`

Czechoslovak Mathematical Journal, 56 (131) (2006), 1185–1192

INVERTIBLE COMMUTATIVITY PRESERVERS OF MATRICES OVER MAX ALGEBRA

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(Received May 28, 2004)

Abstract. The *max algebra* consists of the nonnegative real numbers equipped with two binary operations, maximization and multiplication. We characterize the invertible linear operators that preserve the set of commuting pairs of matrices over a subalgebra of max algebra.

Keywords: max algebra, linear operator, pair of commuting matrices

MSC 2000: 15A03, 15A04

1. Introduction

Recently, there has been a great deal of interest in the algebraic system called "max-algebra" (see [3] and [4]). This system allows one to express, in a linear fashion, phenomena that are nonlinear in the conventional algebra. It has applications in many diverse areas such as parallel computation, transportation networks and scheduling.

The *max algebra* consists of the set $\mathbb{R}_{\max}$, where $\mathbb{R}_{\max}$ is the set of nonnegative real numbers equipped with two binary operations, denoted by $\oplus$ and $\cdot$, respectively. The operations are defined as follows:

$$a \oplus b = \max(a, b) \quad \text{and} \quad a \cdot b = ab.$$

That is, their sum is the maximum of a and b and their product is the usual product.

Our interest will be in describing the invertible commutativity preserver of matrices over this max algebra.

For a set $\mathscr{S}$ we denote by $\mathscr{M}_n(\mathscr{S})$ the set of $n \times n$ matrices over $\mathscr{S}$. The set of commuting pairs of matrices, $\mathscr{C}$, is the set of (unordered) pairs of matrices (A, B) such that $AB = BA$. A linear operator T on $\mathscr{M}_n(\mathscr{S})$ is said to *preserve* $\mathscr{C}$ (or simply T *preserves commutativity*) whenever $T(A)T(B) = T(B)T(A)$ if $AB = BA$.

Watkins [5] showed that if $n \geqslant 4$ and $\mathscr{S}$ is an algebraically closed field of characteristic 0, and L is a nonsingular linear operator on $\mathscr{M}_n(\mathscr{S})$ which preserves commutativity, then there exist an invertible matrix U, a nonzero scalar α and a linear functional $f\colon \mathscr{M}_n(\mathscr{S}) \to \mathscr{S}$ such that either

1. $T(X) = \alpha U X U^{-1} + f(X) I_n$ for all $X \in \mathscr{M}_n(\mathscr{S})$, or
2. $T(X) = \alpha U X^t U^{-1} + f(X) I_n$ for all $X \in \mathscr{M}_n(\mathscr{S})$,

where X^t denotes the transpose of X. In [1] Beasley extended this result to $n = 3$. In [2] Beasley and Pullman characterized the linear operators that preserve commutativity over subsemirings of a fuzzy semiring.

In this article we investigate the set of invertible linear operators on $\mathscr{M}_n(\mathbb{S}_{\max})$ which preserve commuting pairs of matrices, where $\mathbb{S}_{\max}$ is a subalgebra of the max algebra $\mathbb{R}_{\max}$.

2. Preliminaries

The max algebra $\mathbb{R}_{\max}$ is the set of nonnegative real numbers equipped with two binary operations $\oplus$ and $\cdot$. Throughout this paper, $\mathbb{S}_{\max}$ denotes a subalgebra of $\mathbb{R}_{\max}$ such as $\mathbb{R}_{\max}$, $\mathbb{Q}_{\max}$, $\mathbb{Z}_{\max}$, $\mathbb{Z}[\sqrt{2}]_{\max}$, etc., where $\mathbb{Q}$ denotes the rational numbers, $\mathbb{Z}$ the integers and $\mathbb{Z}[\sqrt{2}]$ the set of all values of the form $x\sqrt{2} + y$ with $x, y \in \mathbb{Z}$. In the corresponding subalgebras $\mathbb{S}_{\max}$, only the nonnegative values are considered.

For a nonzero element $a \in \mathbb{S}_{\max}$, a is called a *unit* if there exists a nonzero element $b \in \mathbb{S}_{\max}$ which is denoted $b = a^{-1}$, such that $ab = 1$. Thus, all nonzero elements of $\mathbb{R}_{\max}$ and $\mathbb{Q}_{\max}$ are units, while $\mathbb{Z}_{\max}$ has only the unit element 1. We remark that there are various unit elements in $\mathbb{Z}[\sqrt{2}]_{\max}$, such as 1, $\sqrt{2}+1$, $\sqrt{2}-1$, $5\sqrt{2}+7$, $5\sqrt{2}-7$ and others, which follows from equalities like $1 = (\sqrt{2}+1)(\sqrt{2}-1)$, $1 = (5\sqrt{2}+7)(5\sqrt{2}-7)$, $1 = (29\sqrt{2}+41)(29\sqrt{2}-41)$, $1 = (169\sqrt{2}+239)(169\sqrt{2}-239)$, $\ldots$.

If $A = [a_{i,j}]$ and $B = [b_{i,j}]$ are matrices in $\mathscr{M}_n(\mathbb{S}_{\max})$, then the sum of A and B is denoted by $A \oplus B$, which is the matrix with $a_{i,j} \oplus b_{i,j}$ as its (i,j)th entry. If $c \in \mathbb{S}_{\max}$, then cA is the matrix $[c\, a_{i,j}]$. The product of A and B is denoted by $A \otimes B$, which is the matrix with $\max\limits_r \{a_{i,r} b_{r,j}\}$ as its (i,j)th entry. The identity matrix of order n is denoted by I_n.

The following example shows that there is no relation between commuting pairs over max algebra and those over nonnegative reals.

Example 2.1. Let

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & 2 \\ 0 & 1 \end{bmatrix}. \tag{2.1}$$

Then we have $AB = BA$ over $\mathbb{R}_+$, but $A \otimes B \neq B \otimes A$ over $\mathbb{R}_{\max}$.

Let

$$C = \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 1 & 1 \\ \frac{1}{2} & 1 \end{bmatrix}. \tag{2.2}$$

Then we have $CD \neq DC$ over $\mathbb{R}_+$, but $C \otimes D = D \otimes C$ over $\mathbb{R}_{\max}$.

For a matrix $A \in \mathscr{M}_n(\mathbb{S}_{\max})$, A is called *invertible* if there exists a matrix $B \in \mathscr{M}_n(\mathbb{S}_{\max})$, denoted by $B = A^{-1}$, such that $A \otimes B = B \otimes A = I_n$. It is well known [4] that a matrix A in $\mathscr{M}_n(\mathbb{S}_{\max})$ is invertible if and only if $A = P \otimes D$, where P is a permutation matrix and D is a diagonal matrix, and all the diagonal entries of D are units in $\mathbb{S}_{\max}$.

Evidently, the following operations preserve the set of commuting pairs of matrices over $\mathbb{S}_{\max}$:

(a) transposition ($X \to X^t$);

(b) similarity ($X \to S \otimes X \otimes S^{-1}$ for a fixed invertible matrix S).

In Theorem 3.2 we show that the semigroup of invertible linear operators preserving commuting pairs of matrices is generated by transpositions and similarities over a subalgebra of max algebra.

3. Invertible commutativity preservers of matrices over $\mathbb{S}_{\max}$

A mapping $T\colon \mathscr{M}_n(\mathbb{S}_{\max}) \to \mathscr{M}_n(\mathbb{S}_{\max})$ is called a *linear operator* if $T(\alpha A \oplus \beta B) = \alpha T(A) \oplus \beta T(B)$ for all $A, B \in \mathscr{M}_n(\mathbb{S}_{\max})$ and for all $\alpha, \beta \in \mathbb{S}_{\max}$. An operator T on $\mathscr{M}_n(\mathbb{S}_{\max})$ is called *invertible* if it is surjective and injective. It can be shown by standard arguments that if T is linear, then the inverse operator T^{-1} is also linear.

In this section we characterize the invertible linear operators that preserve commuting pairs of matrices over $\mathbb{S}_{\max}$.

Let $\Delta_n = \{(i,j);\ 1 \leqslant i, j \leqslant n\}$. Then for any $(i,j) \in \Delta_n$, $E_{i,j}$ denotes the $n \times n$ matrix whose (i,j)th entry is 1 and all the other entries are 0. We call $E_{i,j}$ a *cell*.

For $A = [a_{i,j}]$ and $B = [b_{i,j}]$ in $\mathscr{M}_n(\mathbb{S}_{\max})$, *A dominates B*, denoted by $A \sqsupseteq B$ or $B \sqsubseteq A$, if $b_{i,j} \neq 0$ implies $a_{i,j} \neq 0$. It follows that if $A, B, C, D \in \mathscr{M}_n(\mathbb{S}_{\max})$ with $A \sqsubseteq B$ and $C \sqsubseteq D$, then we have

$$A \oplus C \sqsubseteq B \oplus D \quad \text{and} \quad \alpha C \sqsubseteq \beta D \tag{3.1}$$

for all $\alpha, \beta \in \mathbb{S}_{\max}$ with $\beta \neq 0$.

Let T be a linear operator on $\mathscr{M}_n(\mathbb{S}_{\max})$. Then for any $A = [a_{i,j}] = \bigoplus_{i,j=1}^{n} a_{i,j}E_{i,j} \in \mathscr{M}_n(\mathbb{S}_{\max})$ we have

$$T(A) = T\left(\bigoplus_{i,j=1}^{n} a_{i,j}E_{i,j}\right) = \bigoplus_{i,j=1}^{n} T(a_{i,j}E_{i,j}) = \bigoplus_{i,j=1}^{n} a_{i,j}T(E_{i,j})$$

by the linearity of T. If $A = [a_{i,j}]$ and $B = [b_{i,j}]$ are in $\mathscr{M}_n(\mathbb{S}_{\max})$ with $A \sqsubseteq B$, then we have $T(A) \sqsubseteq T(B)$ because

$$T(A) = \bigoplus_{i,j=1}^{n} a_{i,j}T(E_{i,j}) \sqsubseteq \bigoplus_{i,j=1}^{n} b_{i,j}T(E_{i,j}) = T(B)$$

by (3.1) for any linear operator T on $\mathscr{M}_n(\mathbb{S}_{\max})$.

Lemma 3.1. *For a linear operator T on $\mathscr{M}_n(\mathbb{S}_{\max})$, T is invertible if and only if there exist a permutation θ on Δ_n and units $b_{i,j} \in \mathbb{S}_{\max}$ such that $T(E_{i,j}) = b_{i,j}E_{\theta(i,j)}$ for all $(i, j) \in \Delta_n$.*

Proof. Suppose that T is invertible on $\mathscr{M}_n(\mathbb{S}_{\max})$. Let $E_{r,s}$ be an arbitrary cell in $\mathscr{M}_n(\mathbb{S}_{\max})$. Since T is surjective, there exists a nonzero matrix $X = [x_{i,j}] \in \mathscr{M}_n(\mathbb{S}_{\max})$ such that $T(X) = E_{r,s}$. Since T is linear, it follows that there exists $x_{i,j} \neq 0$ such that $T(E_{i,j}) \sqsubseteq E_{r,s}$. This shows that $T(E_{i,j}) = b_{i,j}E_{r,s}$ for some nonzero $b_{i,j} \in \mathbb{S}_{\max}$. Let

$$C_{r,s} = \{E_{i,j}\,;\ T(E_{i,j}) = b_{i,j}E_{r,s} \quad \text{for some nonzero} \quad b_{i,j} \in \mathbb{S}_{\max}\}.$$

By the above, $C_{r,s} \neq \emptyset$ for all $(r, s) \in \Delta_n$. Suppose that $T(E_{k,l}) = b_{k,l}E_{r,s}$ for a cell $E_{k,l}$ different from $E_{i,j}$ and for some $b_{k,l} \neq 0$. Then we have

$$T(b_{k,l}E_{i,j}) = b_{k,l}T(E_{i,j}) = b_{k,l}b_{i,j}E_{r,s} = b_{i,j}T(E_{k,l}) = T(b_{i,j}E_{k,l}),$$

a contradiction to the fact that T is injective. Hence $C_{r,s}$ is a singleton set for all $(r, s) \in \Delta_n$. Therefore, there exists a permutation θ on Δ_n such that $T(E_{i,j}) = b_{i,j}E_{\theta(i,j)}$ for some nonzero $b_{i,j} \in \mathbb{S}_{\max}$. It remains to show that all $b_{i,j}$ are units. Since T is surjective and $T(E_{r,s}) \not\sqsubseteq E_{\theta(i,j)}$ for $(r, s) \neq (i, j)$, there exists a nonzero $c \in \mathbb{S}_{\max}$ such that $T(c\,E_{i,j}) = E_{\theta(i,j)}$. By the linearity of T, we have

$$T(c\,E_{i,j}) = c\,T(E_{i,j}) = c\,b_{i,j}E_{\theta(i,j)} = E_{\theta(i,j)}.$$

That is, $c\,b_{i,j} = 1$ and hence $b_{i,j}$ is a unit.

The converse is immediate. □

Lemma 3.2. *Let A be a matrix in $\mathscr{M}_n(\mathbb{S}_{\max})$ such that $A \otimes X = X \otimes A$ for all $X \in \mathscr{M}_n(\mathbb{S}_{\max})$. Then $A = \alpha I_n$ for some scalar $\alpha \in \mathbb{S}_{\max}$.*

Proof. Let $A = [a_{i,j}]$. Now, we will show that $a_{i,j} = 0$ for all $i, j = 1, \ldots, n$ with $i \neq j$. Consider a cell $E_{j,i}$ with $i \neq j$. Then the (i,i)th entries of $A \otimes E_{j,i}$ and $E_{j,i} \otimes A$ are $a_{i,j}$ and 0, respectively. It follows from $A \otimes E_{j,i} = E_{j,i} \otimes A$ that $a_{i,j} = 0$ for all $i, j = 1, \ldots, n$ with $i \neq j$. Let $a_{i,i}$ be any diagonal entry of A. Then the $(1,i)$th entries of $A \otimes (E_{1,i} \oplus E_{i,1})$ and $(E_{1,i} \oplus E_{i,1}) \otimes A$ are $a_{1,1}$ and $a_{i,i}$, respectively, and hence $a_{i,i} = a_{1,1}$ for all $i = 1, \ldots, n$ because $A \otimes (E_{1,i} \oplus E_{i,1}) = (E_{1,i} \oplus E_{i,1}) \otimes A$. Thus, we have $A = \alpha I_n$ for $\alpha = a_{1,1}$. □

The *Schur* (or *Hadamard*) product, $A \circ B$, of $A = [a_{i,j}]$ and $B = [b_{i,j}]$ in $\mathscr{M}_n(\mathbb{S}_{\max})$ is the matrix $[a_{i,j}\, b_{i,j}]$. This notation will be used in the proof of the next theorem.

Theorem 3.3. *Let T be a linear operator on $\mathscr{M}_n(\mathbb{S}_{\max})$. Then T is an invertible linear operator which preserves pairs of commuting matrices if and only if there exist a unit $\alpha \in \mathbb{S}_{\max}$ and an invertible matrix $U \in \mathscr{M}_n(\mathbb{S}_{\max})$ such that either*
(1) *$T(X) = \alpha U \otimes X \otimes U^{-1}$ for all $X \in \mathscr{M}_n(\mathbb{S}_{\max})$, or*
(2) *$T(X) = \alpha U \otimes X^t \otimes U^{-1}$ for all $X \in \mathscr{M}_n(\mathbb{S}_{\max})$.*

Proof. Let T be an invertible linear operator which preserves pairs of commuting matrices over $\mathbb{S}_{\max}$. Since $I_n \otimes X = X \otimes I_n$ for all $X \in \mathscr{M}_n(\mathbb{S}_{\max})$, we have $T(I_n) \otimes T(X) = T(X) \otimes T(I_n)$ for all $X \in \mathscr{M}_n(\mathbb{S}_{\max})$ because T preserves pairs of commuting matrices. Let Y be an arbitrary matrix in $\mathscr{M}_n(\mathbb{S}_{\max})$. Since T is surjective, $Y = T(X)$ for some $X \in \mathscr{M}_n(\mathbb{S}_{\max})$. Thus, we have that $T(I_n) \otimes Y = Y \otimes T(I_n)$ for all $Y \in \mathscr{M}_n(\mathbb{S}_{\max})$. By Lemma 3.2, $T(I_n) = \alpha I_n$ for some $\alpha \in \mathbb{S}_{\max}$. Furthermore, there exists a matrix C in $\mathscr{M}_n(\mathbb{S}_{\max})$ such that $T(C) = I_n$ (equivalently, $T(\alpha C) = \alpha I_n$). Since T is injective, $\alpha C = I_n$. That is, α is a unit.

Since T is invertible, there exist a permutation θ on Δ_n and units $b_{i,j}$ in $\mathbb{S}_{\max}$ such that $T(E_{i,j}) = b_{i,j} E_{\theta(i,j)}$ for all $(i,j) \in \Delta_n$ by Lemma 3.1. It follows from $T(I_n) = \alpha I_n$ that there is a permutation σ of $\{1, \ldots, n\}$ such that $T(E_{i,i}) = \alpha E_{\sigma(i),\sigma(i)}$ for each $i = 1, \ldots, n$. Define $L\colon \mathscr{M}_n(\mathbb{S}_{\max}) \to \mathscr{M}_n(\mathbb{S}_{\max})$ by $L(X) = P \otimes T(X) \otimes P^t$, where P is the permutation matrix corresponding to σ so that

$$L(E_{i,i}) = \alpha E_{i,i} \tag{3.2}$$

for each $i = 1, \ldots, n$. Then we can easily show that L is an invertible linear operator on $\mathscr{M}_n(\mathbb{S}_{\max})$ which preserves pairs of commuting matrices. By Lemma 3.1, for any $(r,s) \in \Delta_n$ there exist $(p,q) \in \Delta_n$ and a unit $m_{r,s} \in \mathbb{S}_{\max}$ such that $L(E_{r,s}) = m_{r,s} E_{p,q}$.

Suppose that $r \neq s$. Since L is injective, we have $p \neq q$ because $L(E_{i,i}) = \alpha E_{i,i}$ for each $i = 1, \ldots, n$. Assume that $p \neq r$ and $p \neq s$. Then

$$E_{r,s} \otimes (E_{r,r} \oplus E_{s,s} \oplus E_{p,p}) = (E_{r,r} \oplus E_{s,s} \oplus E_{p,p}) \otimes E_{r,s}$$

so that

$$L(E_{r,s}) \otimes L(E_{r,r} \oplus E_{s,s} \oplus E_{p,p}) = L(E_{r,r} \oplus E_{s,s} \oplus E_{p,p}) \otimes L(E_{r,s})$$

or

$$(m_{r,s}E_{p,q}) \otimes (\alpha E_{r,r} \oplus \alpha E_{s,s} \oplus \alpha E_{p,p}) = (\alpha E_{r,r} \oplus \alpha E_{s,s} \oplus \alpha E_{p,p}) \otimes (m_{r,s}E_{p,q}),$$

equivalently

$$E_{p,q} \otimes (E_{r,r} \oplus E_{s,s} \oplus E_{p,p}) = (E_{r,r} \oplus E_{s,s} \oplus E_{p,p}) \otimes E_{p,q}.$$

It follows that $q = r$ or $q = s$. Since $E_{r,s} \otimes (E_{r,r} \oplus E_{s,s}) = (E_{r,r} \oplus E_{s,s}) \otimes E_{r,s}$, we have

$$L(E_{r,s}) \otimes L(E_{r,r} \oplus E_{s,s}) = L(E_{r,r} \oplus E_{s,s}) \otimes L(E_{r,s}),$$

equivalently

$$E_{p,q} \otimes (E_{r,r} \oplus E_{s,s}) = (E_{r,r} \oplus E_{s,s}) \otimes E_{p,q}.$$

Since $q = r$ or $q = s$, we have $E_{p,q} \otimes (E_{r,r} \oplus E_{s,s}) = E_{p,r}$ or $E_{p,s}$, but $(E_{r,r} \oplus E_{s,s}) \otimes E_{p,q} = 0$, a contradiction. Hence we have $p = r$ or $p = s$. Similarly we obtain $q = r$ or $q = s$. Therefore, for each $(r, s) \in \Delta_n$ there exists a unit $m_{r,s} \in \mathbb{S}_{\max}$ such that

$$L(E_{r,s}) = m_{r,s}E_{r,s} \quad \text{or} \quad L(E_{r,s}) = m_{r,s}E_{s,r}. \tag{3.3}$$

Let $L(E_{r,s}) = m_{r,s}E_{r,s}$ for some fixed $(r, s) \in \Delta_n$ with $r \neq s$. Suppose that $L(E_{r,t}) = m_{r,t}E_{t,r}$ for some $t \neq r, s$. By (3.3), we have

$$L(E_{s,t} \oplus E_{t,s}) = \mu E_{s,t} \oplus \xi E_{t,s}$$

for some units μ and ξ, where $\{\mu, \xi\} = \{m_{s,t}, m_{t,s}\}$. Let $A = E_{r,r} \oplus E_{s,t} \oplus E_{t,s}$ so that $L(A) = \alpha E_{r,r} \oplus \mu E_{s,t} \oplus \xi E_{t,s}$. Then $(E_{r,s} \oplus E_{r,t}) \otimes A = A \otimes (E_{r,s} \oplus E_{r,t})$, and hence

$$L(E_{r,s} \oplus E_{r,t}) \otimes L(A) = L(A) \otimes L(E_{r,s} \oplus E_{r,t}).$$

But

$$L(E_{r,s} \oplus E_{r,t}) \otimes L(A) = m_{r,s}\mu E_{r,t} \oplus m_{r,t}\alpha E_{t,r}$$

while

$$L(A) \otimes L(E_{r,s} \oplus E_{r,t}) = \alpha m_{r,s} E_{r,s} \oplus \mu m_{r,t} E_{s,r}.$$

Thus we have $t = s$, a contradiction. That is, $T(E_{r,t}) = m_{r,t} E_{r,t}$ for all $t = 1, \dots, n$. Similarly, we obtain $T(E_{t,s}) = m_{t,s} E_{t,s}$ for all $t = 1, \dots, n$. Let $E_{i,j}$ be an arbitrary cell. Since $T(E_{i,s}) = m_{i,s} E_{i,s}$, by method similar to the above, we have $T(E_{i,j}) = m_{i,j} E_{i,j}$. Therefore, we have established that if $L(E_{r,s}) = m_{r,s} E_{r,s}$ for some fixed $(r, s) \in \Delta_n$ with $r \neq s$, then

$$L(E_{i,j}) = m_{i,j} E_{i,j} \tag{3.4}$$

for all $(i, j) \in \Delta_n$. The parallel argument shows that if $L(E_{r,s}) = m_{r,s} E_{s,r}$ for some fixed $(r, s) \in \Delta_n$ with $r \neq s$, then

$$L(E_{i,j}) = m_{i,j} E_{j,i} \tag{3.5}$$

for all $(i, j) \in \Delta_n$.

Assume that (3.4) is satisfied. That is, $L(E_{i,j}) = m_{i,j} E_{i,j}$ for all $(i, j) \in \Delta_n$ and all $m_{i,j}$ are units. Let $M = [m_{i,j}]$. By (3.2), we have $m_{i,i} = \alpha$ for all $i = 1, \dots, n$. Consider an arbitrary matrix $X = [x_{i,j}] = \bigoplus_{i,j=1}^{n} x_{i,j} E_{i,j}$ in $\mathscr{M}_n(\mathbb{S}_{\max})$. Then by the linearity of L we have

$$L(X) = L\left(\bigoplus_{i,j=1}^{n} x_{i,j} E_{i,j} \right) = \bigoplus_{i,j=1}^{n} x_{i,j} L(E_{i,j}) = \bigoplus_{i,j=1}^{n} x_{i,j} m_{i,j} E_{i,j} = X \circ M.$$

Let $P_{i,j}$ denote the permutation matrix corresponding to the transposition (i, j) and let J be the matrix whose all entries are 1. Then $P_{i,j} \otimes J = J \otimes P_{i,j}$ so that $L(P_{i,j}) \otimes L(J) = L(J) \otimes L(P_{i,j})$. Thus, for $(i, j) \in \Delta_n$ with $i \neq j$ and $k \neq i, j$, we have

$$m_{i,j} m_{j,k} = m_{i,k} m_{k,k} = m_{i,k} \alpha \quad \text{and} \quad m_{j,i} m_{i,k} = m_{j,k} m_{k,k} = m_{j,k} \alpha \tag{3.6}$$

by considering the (i, k)th and (j, k)th entries of $L(P_{i,j}) \otimes L(J)$ and $L(J) \otimes L(P_{i,j})$. Thus, we have

$$m_{i,j} m_{j,k} m_{j,i} m_{i,k} = m_{i,k} m_{j,k} \alpha, \quad \text{equivalently} \quad m_{i,j} m_{j,i} = \alpha^2. \tag{3.7}$$

Let $D = [d_{i,j}]$ be the diagonal matrix with $d_{i,i} = \alpha^{-1} m_{i,2}$ for each $i = 1, \dots, n$. Then $D^{-1} = [e_{i,j}]$ is the diagonal matrix with $e_{j,j} = \alpha^{-1} m_{2,j}$ for each $j = 1, \dots, n$ by (3.7). Now for any $X = [x_{i,j}] \in \mathscr{M}_n(\mathbb{S}_{\max})$, the (i, j)th entry of $\alpha D \otimes X \otimes E$ is

$$\alpha d_{i,i} x_{i,j} e_{j,j} = \alpha \alpha^{-1} m_{i,2} x_{i,j} \alpha^{-1} m_{2,j} = \alpha^{-1} m_{i,2} m_{2,j} x_{i,j} = \alpha^{-1} m_{i,j} \alpha x_{i,j} = m_{i,j} x_{i,j}$$

by (3.6), which is the (i,j)th entry of $X \circ M$. Thus, we have that $L(X) = \alpha D \otimes X \otimes D^{-1}$ for all $X \in \mathscr{M}_n(\mathbb{S}_{\max})$. Since $L(X) = P^{-1} \otimes T(X) \otimes P$, we have that $T(X) = P \otimes L(X) \otimes P^{-1} = \alpha(P \otimes D) \otimes X \otimes (P \otimes D)^{-1}$. If $U = P \otimes D$, then we have $T(X) = \alpha U \otimes X \otimes U^{-1}$ for all $X \in \mathscr{M}_n(\mathbb{S}_{\max})$.

Similarly, if (3.5) is satisfied, then we obtain that $T(X) = \alpha U \otimes X^t \otimes U^{-1}$ for all $X \in \mathscr{M}_n(\mathbb{S}_{\max})$.

The converse is immediate.

Thus we have characterized the linear operators that preserve commuting pairs of matrices over a subalgebra of the max algebra.

Acknowledgement. The authors would like to thank the referee for his suggestions improving the original manuscript of this paper.

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