Eungil Ko; Hae-Won Nam; Young Oh Yang
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ON TOTALLY $*$-PARANORMAL OPERATORS

Eungil Ko, Seoul, Hae-Won Nam, Seoul, and Youngoh Yang, Cheju

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Abstract. In this paper we study some properties of a totally $*$-paranormal operator (defined below) on Hilbert space. In particular, we characterize a totally $*$-paranormal operator. Also we show that Weyl's theorem and the spectral mapping theorem hold for totally $*$-paranormal operators through the local spectral theory. Finally, we show that every totally $*$-paranormal operator satisfies an analogue of the single valued extension property for $W^2(D,H)$ and some of totally $*$-paranormal operators have scalar extensions.

Keywords: hyponormal, totally $*$-paranormal, hypercyclic, operators

MSC 2000: 47B20, 47B38

1. Introduction and preliminaries

Let H and K be complex Hilbert spaces and $\mathscr{L}(H,K)$ denote the space of all bounded linear operators from H to K. If $H = K$, we write $\mathscr{L}(H)$ instead of $\mathscr{L}(H,K)$. Recall that an operator $T \in \mathscr{L}(H)$ is said to be hyponormal if $TT^* \leqslant T^*T$, or equivalently, if $\|T^*h\| \leqslant \|Th\|$ for every $h \in H$. A larger class of operators related to hyponormal operators is the following: $T \in \mathscr{L}(H)$ is called $*$-paranormal if $\|T^*h\|^2 \leqslant \|T^2h\|\|h\|$ for every $h \in H$. It is known [2] that T is $*$-paranormal if and only if $T^{*2}T^2 - 2rTT^* + r^2 \geqslant 0$ for each positive number r. This class of operators was introduced and studied by S. M. Patel. In this paper we want to focus on a class of $*$-paranormal operators which has the translation invariance property. The notion of such operators is obviously inspired by the class of totally paranormal operators in [16].

Definition 1.1. An operator $T \in \mathscr{L}(H)$ is said to be *totally $*$-paranormal* if $\|(T-\lambda)^*h\|^2 \leqslant \|(T-\lambda)^2h\|\|h\|$ for all $h \in H$ and all $\lambda \in \mathbb{C}$.

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The class of totally $*$-paranormal operators forms a proper subclass of $*$-paranormal operators (see Example 2.5). Since hyponormal operators are contained in the class of totally $*$-paranormal operators (see Proposition 2.6) and the class of hyponormal operators is now fairly well understood, we think that the study of the class of totally $*$-paranormal operators has a bright future.

Recall that an operator $T \in \mathscr{L}(H)$ is semi-Fredholm if its range $\operatorname{ran} T$ is closed and either its null space $\ker T$ or $H/\operatorname{ran} T$ is finite dimensional. Also an operator $T \in \mathscr{L}(H)$ is Fredholm if its range $\operatorname{ran} T$ is closed and both its null space $\ker T$ and $H/\operatorname{ran} T$ is finite dimensional. The index of a semi-Fredholm operator T is defined as

$$\operatorname{index} T = \dim(\ker T) - \dim(H/\operatorname{ran} T).$$

The semi-Fredholm spectrum $\sigma_{sF}(T)$ of T is the set $\{\lambda \in \mathbb{C}\colon (T-\lambda)$ is not semi-Fredholm$\}$ and $\varrho_{sF}(T) = \mathbb{C} \setminus \sigma_{sF}(T)$.

An operator $T \in \mathscr{L}(H, K)$ is called Weyl if T is a Fredholm operator of index 0. The Weyl spectrum $\omega(T)$ of T is defined by

$$\omega(T) = \{\lambda \in C\colon\ T - \lambda \text{ is not Weyl}\}.$$

L. A. Coburn showed in [6] that Weyl's theorem holds for hyponormal operators, i.e., that

$$\sigma(T) - \omega(T) = \pi_{00}(T)$$

for each hyponormal operator T, where $\pi_{00}(T)$ denotes the set of isolated points of $\sigma(T)$ that are eigenvalues of finite multiplicity.

The paper is organized as follows. In section one, we give some preliminary facts. In section two, we study some properties of a totally $*$-paranormal operator on Hilbert space. In particular, we characterize a totally $*$-paranormal operator. Also we show that Weyl's theorem and the spectral mapping theorem hold for totally $*$-paranormal operators. Finally, we show that every totally $*$-paranormal operator satisfies an analogue of the single valued extension property for $W^2(D, H)$ and some of totally $*$-paranormal operators have scalar extensions.

2. Totally $*$-paranormal operators

In this section, we study some properties of totally $*$-paranormal operators. We start with the following lemma which summarizes the basic properties of such operators.

Lemma 2.1. *If T is totally $*$-paranormal, then* $\ker T \subset \ker T^*$, $\ker T = \ker T^2$, $r(T) = \|T\|$, *and* $T|_{\mathscr{M}}$ *is a totally $*$-paranormal operator, where $r(T)$ denotes the spectral radius of T and $\mathscr{M}$ is any invariant subspace for T.*

The following theorem gives a characterization of a totally $*$-paranormal operator.

Theorem 2.2. *If T is totally $*$-paranormal, then*

$$\|T^*h\|^2 + |\langle T^2h, h\rangle| \leqslant 2\|Th\|^2 \tag{1}$$

for every $h \in H$.

Proof. Since T is totally $*$-paranormal, $T - \lambda$ is $*$-paranormal for every $\lambda \in \mathbb{C}$. Therefore,

$$(T^* - \bar{\lambda})^2(T - \lambda)^2 - 2r(T - \lambda)(T^* - \bar{\lambda}) + r^2 \geqslant 0$$

for each positive number r.

Set $\lambda = \varrho e^{i\theta}$ for every $0 \leqslant \theta < 2\pi$ and $\varrho > 0$. Then for each positive ϱ

$$(T^* - \varrho e^{-i\theta})^2(T - \varrho e^{i\theta})^2 - 2\varrho^2(T - \varrho e^{i\theta})(T^* - \varrho e^{-i\theta}) + \varrho^4 \geqslant 0.$$

Letting $\varrho \to \infty$, we have

$$e^{i2\theta}T^{*2} + e^{-i2\theta}T^2 + 4T^*T - 2TT^* \geqslant 0$$

for every $0 \leqslant \theta < 2\pi$. If $h \in H$, then

$$4\|Th\|^2 \geqslant 2\|T^*h\|^2 - \langle e^{-i2\theta}T^2h, h\rangle - \overline{\langle e^{-i2\theta}T^2h, h\rangle} = 2\|T^*h\|^2 - 2\mathrm{Re}\,\langle e^{-i2\theta}T^2h, h\rangle.$$

Therefore, by the arbitrariness of θ,

$$2\|Th\|^2 \geqslant \|T^*h\|^2 + |\langle T^2h, h\rangle|$$

for every $h \in H$. $\square$

From Theorem 2.2 we can show that the class of totally $*$-paranormal operators forms a proper subclass of the class of $*$-paranormal operators.

Corollary 2.3. *Let T be a weighted shift with weights $\{\alpha_n\}_{n=0}^{\infty}$. If $2|\alpha_k|^2 < |\alpha_{k-1}|^2$ for some positive integer k, then T is not totally $*$-paranormal.*

Proof. Let $\{e_n\}_{n=0}^{\infty}$ be an orthonormal basis of a Hilbert space H. Then $Te_n = \alpha_n e_{n+1}$ and $T^*e_n = \overline{\alpha}_{n-1}e_{n-1}$. Since $2|\alpha_k|^2 < |\alpha_{k-1}|^2$ for some positive integer k, we have

$$\|T^*e_k\|^2 + |\langle T^2e_k, e_k\rangle| > 2\|Te_k\|^2.$$

Therefore by Theorem 2.2 T is not totally *-paranormal. $\square$

Corollary 2.4. *Let T be a weighted shift with weights $\{\alpha_n\}_{n=0}^{\infty}$. If $2|\alpha_k|^2 < |\alpha_{k-1}|^2$ for some positive integer k and $|\alpha_{n-1}|^2 \leqslant |\alpha_n||\alpha_{n+1}|$ for each positive integer n, then T is not totally $*$-paranormal, but is $*$-paranormal.*

P r o o f. Let $\{e_n\}_{n=0}^{\infty}$ be an orthonormal basis of a Hilbert space H. Then $Te_n = \alpha_n e_{n+1}$ and $T^* e_n = \overline{\alpha}_{n-1} e_{n-1}$. By Corollary 2.3, T is not totally $*$-paranormal. Therefore it suffices to prove that T is $*$-paranormal. Now

$$\begin{aligned} T \quad \text{is } *\text{-paranormal} &\Leftrightarrow {T^*}^2 T^2 - 2rTT^* + r^2 \geqslant 0 \\ &\Leftrightarrow |\alpha_n \alpha_{n+1}|^2 - 2r|\alpha_{n-1}|^2 + r^2 \geqslant 0 \end{aligned}$$

for each $r > 0$ and each positive integer n. Therefore, $|\alpha_{n-1}|^2 \leqslant |\alpha_n||\alpha_{n+1}|$ for each positive integer n. Thus T is $*$-paranormal. □

Next we give an example of Corollary 2.4.

Example 2.5. Let $\{e_n\}_{n=0}^{\infty}$ be an orthonormal basis of a Hilbert space H, and let T be a weighted shift defined as $Te_0 = \frac{1}{2}e_1$, $Te_1 = \frac{1}{3}e_2$, and $Te_i = e_{i+1}$ for $i \geqslant 2$. Then $\alpha_0 = \frac{1}{2}$, $\alpha_1 = \frac{1}{3}$, and $\alpha_i = 1$ for $i \geqslant 2$. Therefore $2|\alpha_1|^2 < |\alpha_0|^2$ and $|\alpha_{n-1}|^2 \leqslant |\alpha_n||\alpha_{n+1}|$ for each positive integer n. By Corollary 2.4, T is not totally $*$-paranormal, but is $*$-paranormal.

The following proposition shows that hyponormal operators are contained in the class of totally *-paranormal operators.

Proposition 2.6. *Every hyponormal operator is totally $*$-paranormal.*

P r o o f. Let $T \in \mathscr{L}(H)$ be hyponormal. Since the class of hyponormal operators has the translation invariance property, we have

$$\|(T-\lambda)^* h\| \leqslant \|(T-\lambda)h\|$$

for all $h \in H$ and all $\lambda \in \mathbb{C}$. Therefore

$$\begin{aligned} \|(T-\lambda)^* h\|^2 &= \langle (T-\lambda)(T-\lambda)^* h, h\rangle \\ &\leqslant \langle (T-\lambda)^*(T-\lambda)h, h\rangle \\ &\leqslant \|(T-\lambda)^*(T-\lambda)h\|\|h\| \\ &\leqslant \|(T-\lambda)^2 h\|\|h\|. \end{aligned}$$

□

We remark here that there exists an M-hyponormal operator which is not totally $*$-paranormal.

Example 2.7. Let $\{e_n\}_{n=0}^{\infty}$ be an orthonormal basis of a Hilbert space H, and let T be a weighted shift defined as $Te_0 = e_1$, $Te_1 = 2e_2$, and $Te_i = e_{i+1}$ for $i \geqslant 2$. Then T is an M-hyponormal operator (see [22]). But since $2|\alpha_2|^2 < |\alpha_1|^2$, T is not totally $*$-paranormal from Corollary 2.3.

We say that an operator T has *finite ascent* if for every $\lambda \in \mathbb{C}$ there is an $n \in \mathbb{N}$ such that $\ker(T-\lambda)^n = \ker(T-\lambda)^{n+1}$ for every $\lambda \in \mathbb{C}$. Recall that an operator $T \in \mathscr{L}(H)$ is said to satisfy the single valued extension property if for any open subset U in $\mathbb{C}$, the function

$$T - \lambda\colon\ \mathscr{O}(U, H) \longrightarrow \mathscr{O}(U, H)$$

defined by the usual pointwise multiplication is one-to-one, where $\mathscr{O}(U, H)$ denotes the Frechet space of H-valued analytic functions in U with respect to uniform topology.

Proposition 2.8. *Every totally $*$-paranormal operator has the single valued extension property.*

Proof. By Lemma 2.1, $T - \lambda$ has finite ascent for each λ. Hence T has the single valued extension property by [16]. □

Recall that an operator $X \in \mathscr{L}(H, K)$ is called a quasiaffinity if it has trivial kernel and dense range. An operator $S \in \mathscr{L}(H)$ is said to be a quasiaffine transform of an operator $T \in \mathscr{L}(K)$ if there is a quasiaffinity $X \in \mathscr{L}(H, K)$ such that $XS = TX$.

Corollary 2.9 ([16, Proposition 1.8])**.** *Let T be any totally $*$-paranormal operator. If S is any quasiaffine transform of T, then S has the single valued extension property.*

Proof. Since $\ker(S-\lambda) \subset \ker(S-\lambda)^2$, it suffices to show that $\ker(S-\lambda)^2 \subset \ker(S-\lambda)$. If $x \in \ker(S-\lambda)^2$, then $(S-\lambda)^2 x = 0$. Let X be a quasiaffinity such that $XS = TX$. Then $X(S-\lambda)^2 x = 0$. Hence $(T-\lambda)^2 Xx = 0$. Thus $Xx \in \ker(T-\lambda)^2$. Since $\ker(T-\lambda) = \ker(T-\lambda)^2$ by the proof of Proposition 2.8, $Xx \in \ker(T-\lambda)$. Therefore, $X(S-\lambda)x = (T-\lambda)Xx = 0$. Since X is one-to-one, $(S-\lambda)x = 0$. Thus $x \in \ker(S-\lambda)$. □

Corollary 2.10. *Let T be any totally $*$-paranormal operator. If $f\colon G \longrightarrow \mathbb{C}$ is analytic function nonconstant on every component of G where G is open and $G \supset \sigma(T)$, then $f(T)$ has the single valued extension property.*

Proof. The proof follows from Proposition 2.8 and [7, Theorem 1.5]. □

If T has the single valued extension property, then for any $x \in H$ there exists a unique maximal open set $\varrho_T(x)(\supset \varrho(T))$ and a unique H-valued analytic function f defined in $\varrho_T(x)$ such that $(T-\lambda)f(\lambda) = x$, $\lambda \in \varrho_T(x)$. Moreover, if F is a closed set in $\mathbb{C}$ and $\sigma_T(x) = \mathbb{C} \setminus \varrho_T(x)$, then

$$H_T(F) = \{x \in H\colon \ \sigma_T(x) \subset F\}$$

is a linear subspace (not necessarily closed) of H (see [7]).

Corollary 2.11. *If T is totally $*$-paranormal, then*

$$H_T(\{\lambda\}) = \{x \in H\colon \ \lim_{n\to\infty} \|(T-\lambda)^n x\|^{1/n} = 0\}.$$

Proof. Since T has the single valued extension property by Proposition 2.8, the proof follows from [16]. □

Corollary 2.12. *Let S and T be totally $*$-paranormal operators in $\mathscr{L}(H)$. If $AS = TA$, then for any closed set $F \subset \mathbb{C}$,*

$$AH_S(F) \subset H_T(F).$$

Proof. Since S and T have the single valued extension property by Proposition 2.8, if $x \in H_S(F)$, then $\sigma_S(x) \subset F$. Hence $F^c \subset \varrho_S(x)$. So there exists an analytic H-valued function f defined on F^c such that

$$(S-\lambda)f(\lambda) \equiv x, \quad \lambda \in F^c.$$

Since $AS = TA$,

$$(T-\lambda)Af(\lambda) = A(S-\lambda)f(\lambda) \equiv Ax, \quad \lambda \in F^c.$$

Since $Af\colon F^c \to H$ is analytic, $F^c \subset \varrho_T(Ax)$, i.e., $\sigma_T(Ax) \subset F$. Hence $Ax \in H_T(F)$, i.e., $AH_S(F) \subset H_T(F)$. □

We consider Weyl's theorem for totally $*$-paranormal operators through the local spectral theory.

Lemma 2.13. *If T is totally $*$-paranormal, then it is isoloid (i.e.,* $\operatorname{iso}\sigma(T) \subset \sigma_p(T)$*).*

P r o o f. Since T has the translation invariance property, it suffices to show that if $0 \in \operatorname{iso}\sigma(T)$ then $0 \in \sigma_p(T)$. Choose $\varrho > 0$ sufficiently small that 0 is the only point of $\sigma(T)$ contained in or on the circle $|\lambda| = \varrho$. Define

$$E = \int_{|\lambda|=\varrho} (\lambda I - T)^{-1} \, \mathrm{d}\lambda$$

Then E is the Riesz idempotent corresponding to 0. So $\mathscr{M} = EH$ is an invariant subspace for T, $\mathscr{M} \neq \{0\}$, and $\sigma(T|_{\mathscr{M}}) = \{0\}$. Since $T|_{\mathscr{M}}$ is also totally $*$-paranormal, $T|_{\mathscr{M}} = 0$. Therefore, T is not one-to-one. Thus $0 \in \sigma_p(T)$. □

Theorem 2.14. *Weyl's theorem holds for any totally $*$-paranormal operator.*

P r o o f. If T is totally $*$-paranormal, then it has the single valued extension property from Proposition 2.8. By [9, Theorem 2], it suffices to show that $H_T(\{\lambda\})$ is finite dimensional for $\lambda \in \pi_{00}(T)$. If $\lambda \in \pi_{00}(T)$, then $\lambda \in \operatorname{iso}\sigma(T)$ and $0 < \dim\ker(T - \lambda) < \infty$. Since $\ker(T - \lambda)$ is a reducing subspace for $T - \lambda$, write $T - \lambda = 0 \oplus (T_1 - \lambda)$, where 0 denotes the zero operator on $\ker(T - \lambda)$ and $T_1 - \lambda = (T - \lambda)|_{(\ker(T-\lambda))^\perp}$ is injective. Therefore,

$$\sigma(T - \lambda) = \{0\} \cup \sigma(T_1 - \lambda).$$

If $T_1 - \lambda$ is not invertible, $0 \in \sigma(T_1 - \lambda)$. Since $\sigma(T - \lambda) = \{0\} \cup \sigma(T_1 - \lambda)$, $\sigma(T - \lambda) = \sigma(T_1 - \lambda)$. Since $\lambda \in \pi_{00}(T)$, $\lambda \in \operatorname{iso}\sigma(T_1)$. Since T is totally *-paranormal, it is easy to show that T_1 is totally *-paranormal. Since T_1 is isoloid by Lemma 2.13, $\lambda \in \sigma_p(T_1)$. Therefore, $\ker(T_1 - \lambda) \neq \{0\}$. So we have a contradiction. Thus $T_1 - \lambda$ is invertible. Therefore, $(T - \lambda)((\ker(T - \lambda))^\perp) = (\ker(T - \lambda))^\perp$. Thus $(\ker(T - \lambda))^\perp \subset \operatorname{ran}(T - \lambda)$. Since $\ker(T - \lambda) \subset \ker(T - \lambda)^* = (\operatorname{ran}(T - \lambda))^\perp$,

$$\operatorname{ran}(T - \lambda) \subset (\ker(T - \lambda))^\perp \subset \operatorname{ran}(T - \lambda).$$

Therefore, $\operatorname{ran}(T - \lambda) = (\ker(T - \lambda))^\perp$. Thus $\operatorname{ran}(T - \lambda)$ is closed. Since $\dim\ker(T - \lambda) < \infty$, $T - \lambda$ is semi-Fredholm. By [17, Lemma 1], $H_T(\{\lambda\})$ is finite dimensional. □

Next we show that the spectral mapping theorem holds for totally $*$-paranormal operators. Furthermore, Weyl's theorem holds for $f(T)$, where T is a totally $*$-paranormal operator and f is analytic in a neighborhood of $\sigma(T)$.

Theorem 2.15. *If T is a totally $*$-paranormal operator and f is analytic on a neighborhood of $\sigma(T)$, then*

$$\omega(f(T)) = f(\omega(T)).$$

P r o o f. Suppose that p is any polynomial. Let $p(\lambda)-\mu = a(\lambda-\lambda_1)\dots(\lambda-\lambda_n)$. Then we have

$$p(T) - \mu I = a(T-\lambda_1 I)\dots(T-\lambda_n I).$$

Since T is totally $*$-paranormal, $T-\lambda_i I$ are commuting totally $*$-paranormal operators for each $i = 1, 2, \dots, n$. It follows that

$$\begin{aligned} \lambda \notin \omega(p(T)) &\Leftrightarrow p(T) - \mu I \quad \text{is Weyl} \\ &\Leftrightarrow a(T-\lambda_1 I)\dots(T-\lambda_n I) \quad \text{is Weyl} \\ &\Leftrightarrow T - \lambda_i I \quad \text{is Weyl for each } i = 1, 2, \dots, n \\ &\Leftrightarrow \lambda_i \notin \omega(T) \quad \text{for each } i = 1, 2, \dots, n \\ &\Leftrightarrow \mu \notin p(\omega(T)). \end{aligned}$$

Thus

$$\omega(p(T)) = p(\omega(T)). \tag{2}$$

Next suppose r is any rational function with poles off $\sigma(T)$. Write $r = \frac{p}{q}$, where p and q are polynomials and q has no zeros in $\sigma(T)$. Then

$$r(T) - \lambda I = (p-\lambda q)(T)(q(T))^{-1}.$$

By the proof of (2),

$$(p-\lambda q)(T) \text{ is Weyl} \Leftrightarrow p - \lambda q \text{ has no zeros in } \omega(T).$$

Thus we have

$$\begin{aligned} \lambda \notin \omega(r(T)) &\Leftrightarrow (p-\lambda q)(T) \quad \text{is Weyl} \\ &\Leftrightarrow p - \lambda q \quad \text{has no zeros in } \sigma(T) \\ &\Leftrightarrow (p-\lambda q)(x)q(x)^{-1} \neq 0 \quad \text{for any } x \in \sigma(T) \\ &\Leftrightarrow \lambda \notin r(\omega(T)), \end{aligned}$$

i.e. $\omega(r(T)) = r(\omega(T))$ for any rational function r with poles off $\sigma(T)$. If f is analytic on a neighborhood of $\sigma(T)$, then by Runge's theorem, there is a sequence $\{r_n\}$ of rational functions with poles off $\sigma(T)$. Note that the mapping $T \to \omega(T)$ is upper semi-continuous at T by [18, Theorem 1]. Since each $r_n(T)$ commutes with $f(T)$, it follows from [18] that

$$f(\omega(T)) = \lim_{n\to\infty} r_n(\omega(T)) = \lim_{n\to\infty} \omega(r_n(T)) = \omega(f(T)).$$

Thus $f(\omega(T)) = \omega(f(T))$. □

Corollary 2.16. *If T is totally $*$-paranormal, then Weyl's theorem holds for $f(T)$ where f is analytic on $\sigma(T)$.*

Proof. Since T is totally $*$-paranormal, it is isoloid by Lemma 2.13. Also Weyl's theorem holds for T by Theorem 2.14. If f is analytic in a neighborhood of $\sigma(T)$, it follows from [18] and [19], or [10] that

$$f(\omega(T)) = f(\sigma(T) - \pi_{00}(T)) = \sigma(f(T)) - \pi_{00}(f(T)).$$

Since $f(\omega(T)) = \omega(f(T))$ by Theorem 2.15,

$$\omega(f(T)) = \sigma(f(T)) - \pi_{00}(f(T)).$$

Therefore, $f(T)$ satisfies Weyl's theorem. □

C. Kitai showed (in [14]) that hyponormal operators are not hypercyclic. We generalize Kitai's theorem to the class of totally $*$-paranormal operators. Recall that if $T \in \mathscr{L}(H)$ and $x \in H$, then $\{T^n x\}_{n=0}^{\infty}$ is called the orbit of x under T, and is denoted by $\operatorname{orb}(T, x)$. If $\operatorname{orb}(T, x)$ is dense in H, then T is called a hypercyclic operator. Recall that an operator T is said to be semi-Fredholm if $\operatorname{ran} T$ is closed and either $\ker T$ or $H/\operatorname{ran} T$ is finite dimensional. The semi-Fredholm spectrum $\sigma_{sF}(T)$ of T is the set $\{\lambda \in \mathbb{C}\colon\ T - \lambda$ is not semi-Fredholm$\}$. At this point we cannot prove that every totally $*$-paranormal operator is not hypercyclic.

Proposition 2.17. *If T is a totally $*$-paranormal operator with $\sigma(T) \neq \sigma_{sF}(T)$, then it is not hypercyclic.*

Proof. If T is hypercyclic, $\sigma_p(T^*) = \emptyset$ by [14, Corollary 2.4]. By Lemma 2.1, $\sigma_p(T) = \emptyset$. So we have a contradiction. □

Corollary 2.18. *If T is an invertible totally $*$-paranormal operator with $\sigma(T) \neq \sigma_{sF}(T)$, then T and T^{-1} have a common nontrivial invariant closed set.*

Proof. Since T is not hypercyclic from Theorem 2.17, the proof follows from [14, Theorem 2.15]. □

Theorem 2.19. *Let T be a totally $*$-paranormal operator in $\mathscr{L}(H)$. If T^* is hypercyclic, then $\sigma(T|_{\mathscr{M}}) \cap (\mathbb{C} \setminus \overline{D}) \neq \emptyset$ for every hyperinvariant subspace $\mathscr{M} \neq (0)$ of T where D is the unit disk.*

Proof. Suppose that T^* is hypercyclic. Let $S = T|_{\mathscr{M}}$ for every hyperinvariant subspace $\mathscr{M} \neq (0)$ of T. If x is a hypercyclic vector for T^*, then $P_{\mathscr{M}}x$ is hypercyclic for $S^* = P_{\mathscr{M}}T^*|_{\mathscr{M}}$. Since S is a totally $*$-paranormal operator, it is easy to show that S is normaloid, i.e., $r(S) = \|S\| = \|S^*\|$ where $r(S)$ denotes the spectral radius of S. Since S^* is hypercyclic, $\|S^*\| > 1$. Hence $r(S) > 1$. Thus $\sigma(T|_{\mathscr{M}}) \cap (\mathbb{C} \setminus \overline{D}) \neq \emptyset$. □

Recall that the joint point spectrum of T is defined by $\sigma_{jp}(T) = \{\lambda \in \mathbb{C}\colon$ there exists a non-zero vector f such that $Tf = \lambda f$ and $T^*f = \bar{\lambda} f\}$. Also the joint approximate point spectrum of T is defined by $\sigma_{jap}(T) = \{\lambda \in \mathbb{C}\colon$ there exists a sequence $\{f_n\}$ of unit vectors such that $\lim\limits_{n\to\infty} \|(T-\lambda)f_n\| = \lim\limits_{n\to\infty} \|(T-\lambda)^* f_n\| = 0\}$.

Lemma 2.20. *If T is totally *-paranormal, then $\sigma_{jap}(T) = \sigma_{ap}(T)$. In particular, $\sigma_{jp}(T) = \sigma_p(T)$.*

Proof. It is clear that $\sigma_{jap}(T) \subset \sigma_{ap}(T)$. Let $\lambda \in \sigma_{ap}(T)$. Then there exists a sequence $\{h_n\}$ of unit vectors such that

$$\lim_{n\to\infty} \|(T-\lambda)h_n\| = 0.$$

Since $\|(T-\lambda)^*h_n\|^2 \leqslant \|(T-\lambda)^2 h_n\|$, we get

$$\lim_{n\to\infty} \|(T-\lambda)^*h_n\| = 0.$$

Therefore, $\lambda \in \sigma_{jap}(T)$. □

Proposition 2.21. *If T is totally $*$-paranormal, then*

$$\sigma(T) = \sigma_{ap}(T^*)^*.$$

Proof. It is known that for any $T \in \mathscr{L}(H)$

$$\sigma(T) = \sigma_{ap}(T) \cup \sigma(T^*)^*$$

by [12, Problem 73]. It follows from Lemma 2.20 that $\sigma_{ap}(T) = \sigma_{jap}(T)$. From the definition of joint approximate point spectrum, it is clear that for any $T \in \mathscr{L}(H)$

$$\sigma_{jap}(T) = \sigma_{jap}(T^*)^* \subset \sigma_{ap}(T^*)^*.$$

Therefore,

$$\sigma_{ap}(T) = \sigma_{jap}(T) \subset \sigma_{ap}(T^*)^*.$$

Since $\sigma(T) = \sigma_{ap}(T) \cup \sigma(T^*)^*$, $\sigma(T) \subset \sigma_{ap}(T)$. On the other hand, $\sigma(T) = \sigma_{ap}(T) \cup \sigma(T^*)^* \supset \sigma(T^*)^*$. So we conclude that $\sigma(T) = \sigma_{ap}(T^*)^*$. □

Halmos showed in [12] that a partial isometry is subnormal if and only if it is the direct sum of an isometry and zero. We generalize this theorem to the case of a totally $*$-paranormal operator.

Proposition 2.22. *A partial isometry T is quasinormal (i.e., $(T^*T)T = T(T^*T)$) if and only if T is $*$-paranormal.*

Proof. Assume T is a partial isometry and $*$-paranormal operator. Since $\ker T$ is a reducing subspace for T, $T = 0 \oplus A$ where $A = T|_{(\ker T)^\perp}$ is isometry. Hence $(T^*T)T = T(T^*T)$.

The converse implication is trivial. □

The next result shows that every totally $*$-paranormal operator satisfies an analogue of the single valued extension property for $W^2(D,H)$. First of all, let us define a special Sobolev type space. Let D be a bounded open subset of $\mathbb{C}$. $W^2(D,H)$ with respect to $\bar\partial$ will be the space of those functions $f \in L^2(D,H)$ whose derivatives $\bar\partial f$, $\bar\partial^2 f$ in the sense of distributions still belong to $L^2(D,H)$. Endowed with the norm

$$\|f\|_{W^2}^2 = \sum_{i=0}^{2} \|\bar\partial^i f\|_{2,D}^2,$$

$W^2(D,H)$ becomes a Hilbert space contained continuously in $L^2(D,H)$.

Theorem 2.23. *Let D be an arbitrary bounded disk in $\mathbb{C}$. If T is totally $*$-paranormal, then the operator*

$$T-\lambda\colon\ W^2(D,H) \longrightarrow W^2(D,H)$$

is one-to-one.

Proof. Let $f \in W^2(D,H)$ be such that $(T-\lambda)f = 0$, i.e., $\|(T-\lambda)f\|_{W^2} = 0$. Then for $i = 1,2$, we have

$$\|(T-\lambda)\bar\partial^i f\|_{2,D} = 0.$$

Hence for $i = 1,2$, we get

$$\|(T-\lambda)^2\bar\partial^i f\|_{2,D}\|\bar\partial^i f\|_{2,D} = 0.$$

Since T is totally $*$-paranormal, for $i = 1,2$

$$\|(T-\lambda)^*\bar\partial^i f\|_{2,D}^2 = 0.$$

By [20, Proposition 2.1], we obtain $\|(I-P)f\|_{2,D} = 0$ where P denotes the orthogonal projection of $L^2(D,H)$ onto the Bergman space $A^2(D,H)$. Hence $(T-\lambda)Pf = (T-\lambda)f = 0$. Since T has the single valued extension property by Proposition 2.8, $f = Pf = 0$. Hence $T-\lambda$ is one-to-one. □

Corollary 2.24. *If an operator T is a nilpotent perturbation of a totally $*$-paranormal operator S, i.e., $T = S + N$ where S is totally $*$-paranormal, S and N commute, and $N^m = 0$, then $T - \lambda$ is one-to-one on $W^2(D, H)$.*

Proof. If $f \in W^2(D, H)$ is such that $(T - \lambda)f = 0$, then

$$(S - \lambda)f = -Nf. \tag{3}$$

Hence $(S - \lambda)N^{j-1}f = -N^j f$ for $j = 1, 2, \ldots, m$. We prove that $N^j f = 0$ for $j = 0, 1, \ldots, m - 1$ by induction. Since $N^m = 0$,

$$(S - \lambda)N^{m-1}f = -N^m f = 0.$$

Since $S - \lambda$ is one-to-one from Theorem 2.23, $N^{m-1}f = 0$. Assume it is true when $j = k$, i.e., $N^k f = 0$. From (3), we get

$$(S - \lambda)N^{k-1}f = -N^k f = 0.$$

Since $S - \lambda$ is one-to-one from Theorem 2.23, $N^{k-1}f = 0$. By induction, we have $f = 0$. Hence $T - \lambda$ is one-to-one. □

Corollary 2.25. *Let $T \in \mathscr{L}(H)$ be any totally $*$-paranormal operator. If $S = VTV^*$ where V is an isometry, then $S - \lambda\colon W^2(D, H) \to W^2(D, H)$ is one-to-one.*

Proof. If $f \in W^2(D, H)$ is such that $(S - \lambda)f = 0$, then

$$(T - \lambda)V^*\overline{\partial}^i f = 0$$

for $i = 0, 1, 2$. From Theorem 2.23, we get $V^*\overline{\partial}^i f = 0$ for $i = 0, 1, 2$. Hence $VTV^*\overline{\partial}^i f = S\overline{\partial}^i f = 0$ for $i = 0, 1, 2$. Thus $\lambda\overline{\partial}^i f = 0$ for $i = 0, 1, 2$. By an application of [20, Proposition 2.1] with $T = (0)$, we have

$$\|(I - P)f\|_{2,D} = 0,$$

where P denotes the orthogonal projection of $L^2(D, H)$ onto the Bergman space $A^2(D, H)$. Hence $\lambda f = \lambda P f = 0$. From [8, Corollary 10.7], there exists a constant $c > 0$ such that

$$c\|Pf\|_{2,D} \leqslant \|\lambda Pf\|_{2,D} = 0.$$

So $f = Pf = 0$. Thus $S - \lambda$ is one-to-one. □

An operator $S \in \mathscr{L}(H)$ is called scalar of order m if there exists a continuous unital homomorphism of $\Phi\colon C_0^m(\mathbb{C}) \to \mathscr{L}(H)$ such that $\Phi(z) = S$, where as usual z stands for the identity function on $\mathbb{C}$ and $C_0^m(\mathbb{C})$ is the space of compactly supported functions on $\mathbb{C}$, continuously differentiable of order m, with the topology of uniform convergence on compact subsets. An operator is called subscalar if it is, up to similarity, the restriction of a scalar operator to an invariant subspace.

Problem 2.26. *Is every totally $*$-paranormal operator subscalar?*

Next proposition shows that every quasinilpotent totally $*$-paranormal operator is subscalar.

Theorem 2.27. *Let $p(T) - \lambda$ be $*$-paranormal for all $\lambda \in \mathbb{C}$, where p is a nonconstant polynomial. If $\sigma(T) = \{0\}$, then T is subscalar.*

Proof. Since $p(T) - \lambda$ be $*$-paranormal for all $\lambda \in \mathbb{C}$, we may write

$$p(z) = cz^k(z - z_1)\ldots(z - z_m)$$

where $z_i \neq 0$ for $i = 1, \ldots, m$, $c \in \mathbb{C}$, and $k \geqslant 1$. Since $\sigma(T) = \{0\}$, by the spectral mapping theorem $\sigma(p(T)) = p(\sigma(T)) = \{0\}$. Since $p(T) - \lambda$ be $*$-paranormal for all $\lambda \in \mathbb{C}$ and $\sigma(p(T)) = \{0\}$,

$$p(T) = cT^k(T - z_1)\ldots(T - z_m) = 0.$$

Since $T - z_i$ are invertible for $i = 1, \ldots, m$, $T^k = 0$. By [15], T is subscalar. □

Theorem 2.28. *Let $T \in \mathscr{L}(H)$ be any totally $*$-paranormal operator. If T has the property that $\sup\limits_n \|f_n\|_{2,D} < \infty$ whenever $\|(T - \lambda)f_n\|_{2,D} \to 0$ as $n \to \infty$, then T is subscalar of order 2.*

Proof. Consider an arbitrary bounded open disk D in $\mathbb{C}$ which contains $\sigma(T)$ and the quotient space

$$H(D) = W^2(D,H)/\overline{(T - \lambda)W^2(D,H)}$$

endowed with the Hilbert space norm. The class of a vector f or an operator A on $H(D)$ will be denoted by $\tilde{f}$, respectively $\tilde{A}$. Let M be the operator of multiplication by z on $W^2(D,H)$. Then M is a scalar operator of order 2 and has a spectral distribution Φ. Let $S = \tilde{M}$. Since $\overline{(T - \lambda)W^2(D,H)}$ is invariant under every operator M_f, $f \in C_0^2(\mathbb{C})$, we infer that S is a scalar operator of order 2 with spectral distribution $\tilde{\Phi}$.

Consider the natural map $V\colon H \to H(D)$ defined by $Vh = \widetilde{1 \otimes h}$, for $h \in H$, where $1 \otimes h$ denotes the constant function sending any $z \in D$ to h. Then $VT = SV$. In particular $\operatorname{ran} V$ is an invariant subspace for S. In order to complete the proof, it suffices to show that V is one-to-one and has closed range.

Let $h_n \in H$ and $f_n \in W^2(D, H)$ be sequences such that

$$\lim_{n\to\infty} \|(T-\lambda)f_n + 1 \otimes h_n\|_{W^2} = 0. \tag{4}$$

It suffices to show that $\lim\limits_{n\to\infty} h_n = 0$. By the definition of the norm of Sobolev space (4) implies

$$\lim_{n\to\infty} \|(T-\lambda)\overline{\partial}^i f_n\|_{2,D} = 0 \tag{5}$$

for $i = 1, 2$. Since T has the property that $\sup\limits_n \|f_n\|_{2,D} < \infty$ whenever $\|(T-\lambda)f_n\|_{2,D} \to 0$ as $n \to \infty$,

$$\lim_{n\to\infty} \|(T-\lambda)^2\overline{\partial}^i f_n\|_{2,D}\|\overline{\partial}^i f_n\|_{2,D} = 0 \tag{6}$$

for $i = 1, 2$. Since T is totally $*$-paranormal,

$$\lim_{n\to\infty} \|(T-\lambda)^*\overline{\partial}^i f_n\|_{2,D}^2 = 0$$

for $i = 1, 2$. By [20, Proposition 2.1],

$$\lim_{n\to\infty} \|(I-P)f_n\|_{2,D} = 0 \tag{7}$$

where P denotes the orthogonal projection of $L^2(D, H)$ onto the Bergman space $A^2(D, H)$. Substituting (7) into (4), we obtain

$$\lim_{n\to\infty} \|(T-\lambda)Pf_n + 1 \otimes h_n\|_{2,D} = 0.$$

Let Γ be a curve in D surrounding $\sigma(T)$. Then for $z \in \Gamma$

$$\lim_{n\to\infty} \|Pf_n(\lambda) + (T-\lambda)^{-1}(1 \otimes h_n)\| = 0$$

uniformly. Hence by Riesz-Dunford functional calculus,

$$\lim_{n\to\infty} \left\| \frac{1}{2\pi \mathrm{i}} \int_\Gamma Pf_n(\lambda)\,\mathrm{d}\lambda + h_n \right\| = 0.$$

But since $(2\pi\mathrm{i})^{-1}\int_\Gamma Pf_n(\lambda)\,\mathrm{d}\lambda = 0$ by Cauchy's theorem, $\lim\limits_{n\to\infty} h_n = 0$. Thus V is one-to-one and has closed range. $\square$

Recall that if U is a nonempty open set in $\mathbb{C}$ and if $\Omega \subset U$ has the property that

$$\sup_{\lambda \in \Omega} |f(\lambda)| = \sup_{\beta \in U} |f(\lambda)|$$

for all f bounded and analytic on U, then Ω is said to be dominating for U. If we apply Theorem 2.28 and [11], we obtain the following.

Corollary 2.29. *Let $T \in \mathscr{L}(H)$ be any totally $*$-paranormal operator. If T has the properties that $\sup_n \|f_n\|_{2,D} < \infty$ whenever $\|(T-\lambda)f_n\|_{2,D} \to 0$ as $n \to \infty$ and that $\sigma(T) \cap U$ is dominating for some nonempty open set $U \subset \mathbb{C}$, then T has a nontrivial invariant subspace.*

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Authors' addresses: Eungil Ko, Hae-Won Nam, Department of mathematics, Ewha Women's University, Seoul 120-750, Korea, e-mail: `eiko@ewha.ac.kr`; Youngoh Yang, Department of mathematics, Cheju National University, Cheju 690-750, Korea, e-mail: `yangyo@cheju.cheju.ac.kr`.