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HONEST SUBMODULES

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Abstract. Lattices of submodules of modules and the operators we can define on these lattices are useful tools in the study of rings and modules and their properties. Here we shall consider some submodule operators defined by sets of left ideals. First we focus our attention on the relationship between properties of a set of ideals and properties of a submodule operator it defines. Our second goal will be to apply these results to the study of the structure of certain classes of rings and modules. In particular some applications to the study and the structure theory of torsion modules are provided.

Keywords: closed submodules, honest submodules, topological filters

MSC 2000: 16W35

1. Introduction

The aim of this paper was originally to consider and extend the theory of honest subgroups, as it was developed by Abian and Rinehart in [1], to modules over noncommutative rings. In the mentioned paper the authors proved that if a nonzero subgroup $H \subseteq M$ is honest in a torsion Abelian group M, then: (1) M is p-primary for some prime integer number p, (2) H is a direct sum of copies of the cyclic Abelian group $\mathbb{Z}_p$, and (3) H is a direct summand of M. In our approach to the theory first we considered modules over a Dedekind domain, and showed that the same result holds. After that, we extended the notion of honest subgroup to honest submodule with respect to a set of ideals. In particular we considered the case of sets of ideals defined from sets of maximal ideals, and showed that the Abian-Rinehart result holds, in this case, over any commutative ring.

In order to develop a similar theory over non commutative rings we shall consider *certain* algebras over an algebraically closed field, for instance the field $\mathbb{C}$ of all

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complex numbers; examples of these algebras are, for instance, the enveloping algebra of a finite dimensional solvable Lie algebra or the coordinate algebra of certain quantum groups $\mathscr{O}_q(SL_n(\mathbb{C}))$, q not being a root of unity. These examples have the particularity that they are Noetherian algebras in which cofinite prime ideals have codimension one; hence, in particular, they are maximal as left and right ideals. In this case a result similar to Abian-Rinehart's result for Abelian groups holds.

To develop this theory in the mentioned examples we find that it is useful to realize a general study of submodule operators, defined by sets of ideals, and show that the properties of these submodule operators are directly induced by the properties of the sets of ideals and vice versa. This shall be our first goal: *the characterization of properties of submodule operators in terms of properties of the sets of left ideals that define them.*

Our first examples of submodule operators consists in considering the notion of closed submodule and define two submodule operators. The first one is more well known and consists in defining it as the intersection of all the closed submodules containing a given submodule. The second one is based in an *elementarywise* definition that shall allow us to study the relationship between properties of the set of left ideals and properties of the submodule operator.

As an extension of these two submodule operators we define the honest operator and study its properties. A rich behavior occurs when we consider sets of left ideals satisfying certain properties: *weak closed under intersections* or *inductive* and, more in general, *topological* or *linear filters*. When we consider these properties of the sets of left ideals we shall establish the corresponding properties of the honest operator. Finally, when we put together all these properties we shall obtain the announced results on honest submodules.

Honest submodules also allow us to characterize certain classes of rings. Thus, following the theory developed by Fay and Joubert in [3] we obtain the characterization of rings of quotients in terms of the honest operator. In fact, a ring R is a ring of quotients iff the honest operator with respect to the set of all regular elements is a closure operator iff the honest operator is the identity operator. The relationship between the different submodule operators also characterizes when the set of elements is a left Ore set.

The structure of this paper is as follows. The paper is divided into four sections. The first section is introductory. The second one deals with closed submodules, the closed operators and their relationship with sets of ideals. The third section is devoted to honest submodules and the honest operator; also the relationship with the sets of ideals used to define honest submodules is studied; with enough conditions on the set of ideals we obtain that a honest submodule is either torsion or closed. As a consequence, as the closed case is well understood, we shall focus our attention on the torsion case in the section devoted to examples. In the last section we study the forerunner examples of the theory. First we see, following Fay and Joubert, how the honest operator characterizes rings of quotients and extend the theory to characterize, in general, left Ore subsets. After that we deal, in the commutative case, with torsion modules having a nonzero honest submodule with respect to sets of ideals generated by maximal ideals and apply the theory to honest submodules over Dedekind domains. As a final example we show that the theory can be applied to a noncommutative framework. In fact, if we consider the set of all cofinite left ideals in the complex enveloping algebra of a finite dimensional solvable Lie algebra or in the quantum coordinate algebra of $SL_n(\mathbb{C})$, q not being a root of unity, then the same result holds for torsion modules having a nonzero honest submodule.

2. Closed submodules

Let $\mathscr{X}$ be a non empty set of left ideals of a ring R such that $0 \notin \mathscr{X}$ (this restriction is only to avoid the trivial case) and let $N \subseteq M$ be a submodule of a left R-module M. We say N is a *$\mathscr{X}$-closed submodule*, or N is *$\mathscr{X}$-closed in M*, if for any $I \in \mathscr{X}$ and any $m \in M$, if $Im \subseteq N$, then $m \in N$. We write $N \subseteq^c_{\mathscr{X}} M$.

The notion of $\mathscr{X}$-closed submodule was used by many authors under different names; let us recall some of them: *$\mathscr{X}$-pure submodule*, by J. S. Golan in [5], *$\mathscr{X}$-isolated submodule*, by T. H. Fay and S. V. Joubert in [3], *$\mathscr{X}$-super-honest submodule*, by S. V. Joubert and M. J. Schoeman in [8] (the last two when $\mathscr{X}$ is a set of elements of R); and, of course, we have the corresponding notion when $R = \mathbb{Z}$ and $\mathscr{X}$ is the set of all nonzero integer numbers, see [4].

In the literature there are different methods to study $\mathscr{X}$-closed submodules. But before we start this study we show, without proof, some of their elementary properties.

Lemma 2.1. *Let $H \subseteq N \subseteq M$ be submodules, then the following statements hold.*

(1) *If $H \subseteq N$ and $N \subseteq M$ are $\mathscr{X}$-closed, then $H \subseteq M$ is $\mathscr{X}$-closed, i.e.: $H \subseteq^c_{\mathscr{X}} N \subseteq^c_{\mathscr{X}} M \Rightarrow H \subseteq^c_{\mathscr{X}} M$.*

(2) *$N \subseteq M$ is $\mathscr{X}$-closed if and only if $N/H \subseteq M/H$ is $\mathscr{X}$-closed, i.e.: $N \subseteq^c_{\mathscr{X}} M \Leftrightarrow N/H \subseteq^c_{\mathscr{X}} M/H$.*

Lemma 2.2. *Let $\{N_\lambda\colon \lambda \in \Lambda\}$ be a family of $\mathscr{X}$-closed submodules of M, then $\bigcap_\lambda N_\lambda$ is $\mathscr{X}$-closed.*

Once we have established these results we may, using Lemma 2.2, define a submodule operator, $\mathrm{C}^M_{\mathscr{X}}(-)$, in such a way that the image $\mathrm{C}^M_{\mathscr{X}}(N)$ of any submodule $N \subseteq M$ is the smallest $\mathscr{X}$-closed submodule of M containing N, i.e.:

$$\mathrm{C}^M_{\mathscr{X}}(N) = \bigcap\{H \subseteq M\colon\ N \subseteq H \text{ and } H \text{ is } \mathscr{X}\text{-closed in } M\}.$$

Thus we have:

Lemma 2.3. *For any submodule $N \subseteq M$ we have that $\mathrm{C}^M_{\mathscr{X}}(N)$ is the smallest $\mathscr{X}$-closed submodule of M containing N.*

In fact $\mathrm{C}^M_{\mathscr{X}}(-)$ defines a *closure operator*, i.e., for any left R-module M we have an operator $\mathrm{C}^M_{\mathscr{X}}(-)$ on the lattice of all R-submodules of M satisfying the expansive and monotone properties, and in addition it is continuous.

(1) Expansive. $N \subseteq \mathrm{C}^M_{\mathscr{X}}(N)$ for any $N \subseteq M$.
(2) Monotone. If $N_1 \subseteq N_2$, then $\mathrm{C}^M_{\mathscr{X}}(N_1) \subseteq \mathrm{C}^M_{\mathscr{X}}(N_2)$ for any N_1, $N_2 \subseteq M$.
(3) Continuous. For any homomorphism $f\colon M \to M'$ and $N \subseteq M$ we have $f(\mathrm{C}^M_{\mathscr{X}}(N)) \subseteq \mathrm{C}^{M'}_{\mathscr{X}}(f(N))$.

Proof. We only need to show that if $f(N) \subseteq L \subseteq^c_{\mathscr{X}} M'$, then $N \subseteq f^{-1}(L) \subseteq^c_{\mathscr{X}} M$. $\square$

In addition this closure operator satisfies the following property:

(4) Idempotent. $\mathrm{C}^M_{\mathscr{X}}\mathrm{C}^M_{\mathscr{X}} = \mathrm{C}^M_{\mathscr{X}}$.

For any submodule $N \subseteq M$ we say N is $\mathrm{C}^M_{\mathscr{X}}$-*closed* if $N = \mathrm{C}^M_{\mathscr{X}}(N)$, and $\mathrm{C}^M_{\mathscr{X}}$-*dense* if $\mathrm{C}^M_{\mathscr{X}}(N) = M$.

2.1. Torsion and closure. The closure operator $\mathrm{C}^M_{\mathscr{X}}(-)$ has, in general, a wild behavior. In order to control it we shall study properties of the set of left ideals $\mathscr{X}$.

One of the traditional ways to study closed submodules was to use hereditary torsion theories or, equivalently, linear filters. In this section we start from a very general set of left ideals $\mathscr{X}$, non necessarily a linear filter, and define for any submodule $N \subseteq M$ a subset $\mathrm{Cl}^M_{\mathscr{X}}(N) \subseteq M$. We shall study properties of the "*operator*" $N \mapsto \mathrm{Cl}^M_{\mathscr{X}}(N)$ in terms of properties of the set $\mathscr{X}$. In fact it will be possible to characterize linear filters in terms of the operator $\mathrm{Cl}^M_{\mathscr{X}}$.

Let M be a left R-module, we define the $\mathscr{X}$-*torsion* of M as the subset

$$T_{\mathscr{X}}(M) = \{m \in M\colon \text{ there is } I \in \mathscr{X} \text{ such that } Im = 0\}.$$

A left R-module is $\mathscr{X}$-*torsion* if $T_{\mathscr{X}}(M) = M$ and $\mathscr{X}$-*torsionfree* if $T_{\mathscr{X}}(M) = 0$.

Let $N \subseteq M$ be a submodule, we define the $\mathscr{X}$-*closure* of N is M as

$$\mathrm{Cl}^M_{\mathscr{X}}(N) = \{m \in M\colon \text{ there is } I \in \mathscr{X} \text{ such that } Im \subseteq N\}.$$

Of course $T_{\mathscr{X}}(M) = \mathrm{Cl}^M_{\mathscr{X}}(0)$ and $T_{\mathscr{X}}(M/N) = \mathrm{Cl}^M_{\mathscr{X}}(N)/N$ (as sets!). Thus we obtain that a submodule $N \subseteq M$ is $\mathscr{X}$-closed (if $\mathrm{Cl}^M_{\mathscr{X}}(N) = N$) if and only if it is $\mathrm{C}^M_{\mathscr{X}}$-closed. A submodule N is named *$\mathscr{X}$-dense* if $\mathrm{Cl}^M_{\mathscr{X}}(N) = M$.

Remark 2.4. In general, for arbitrary sets $\mathscr{X}$, the sets $T_{\mathscr{X}}(M)$ and $\mathrm{Cl}^M_{\mathscr{X}}(N)$ are not submodules of M, and the notation $\mathrm{Cl}^M_{\mathscr{X}}(N)/N$ refers only to sets.

Lemma 2.5.

(1) *If $\mathscr{X}$ is* weak closed under intersection *(i.e., for any I_1, $I_2 \in \mathscr{X}$ there exists $J \in \mathscr{X}$ such that $J \subseteq I_1 \cap I_2$), then for any left R-module M and any submodule $N \subseteq M$ we have that $\mathrm{Cl}^M_{\mathscr{X}}(N)$ is a subgroup of M.*

(2) *$\mathscr{X}$ is weak closed closed under intersection if and only if $\mathrm{Cl}^M_{\mathscr{X}}(N_1) \cap \mathrm{Cl}^M_{\mathscr{X}}(N_2) = \mathrm{Cl}^M_{\mathscr{X}}(N_1 \cap N_2)$ for any submodules N_1, $N_2 \subseteq M$ and any left R-module M.*

(3) *If $\mathscr{X}$ is weak closed under intersection, then $\mathscr{X}$ is* left closed *(i.e., for any $r \in R$ and any $I \in \mathscr{X}$ there is $J \in \mathscr{X}$ such that $Jr \subseteq I$), if and only if $\mathrm{Cl}^M_{\mathscr{X}}(N)$ is a submodule of M for any submodule $N \subseteq M$ and any left R-module M.*

Proof. (1). Let x_1, $x_2 \in \mathrm{Cl}^M_{\mathscr{X}}(N)$, then there exist I_1, $I_2 \in \mathscr{X}$ such that $I_i x_i \subseteq N$, for $i = 1, 2$, then $(I_1 \cap I_2)(x_1 + x_2) \subseteq N$, then $x_1 + x_2 \in \mathrm{Cl}^M_{\mathscr{X}}(N)$.

(2) ($\Rightarrow$). We always have $\mathrm{Cl}^M_{\mathscr{X}}(N_1 \cap N_2) \subseteq \mathrm{Cl}^M_{\mathscr{X}}(N_1) \cap \mathrm{Cl}^M_{\mathscr{X}}(N_2)$; otherwise if $x \in \mathrm{Cl}^M_{\mathscr{X}}(N_1) \cap \mathrm{Cl}^M_{\mathscr{X}}(N_2)$ there exist I_1, $I_2 \in \mathscr{X}$ such that $I_i x \subseteq N_i$, then there exists $J \in \mathscr{X}$ such that $J \subseteq I_1 \cap I_2$, hence $Jx \subseteq N_1 \cap N_2$, and $x \in \mathrm{Cl}^M_{\mathscr{X}}(N_1 \cap N_2)$.

($\Leftarrow$). Let I_1, $I_2 \in \mathscr{X}$, then $1 \in \mathrm{Cl}^R_{\mathscr{X}}(I_i)$ and then $1 \in \mathrm{Cl}^R_{\mathscr{X}}(I_1 \cap I_2)$, hence there exists $J \in \mathscr{X}$ such that $J \subseteq I_1 \cap I_2$.

(3) ($\Rightarrow$). Let $x \in \mathrm{Cl}^M_{\mathscr{X}}(N)$ and $r \in R$, then there exists $I \in \mathscr{X}$ such that $Ix \subseteq N$, since there exists $J \in \mathscr{X}$ such that $Jr \subseteq I$ we have $Jrx \subseteq Ix \subseteq N$, hence $rx \in \mathrm{Cl}^M_{\mathscr{X}}(N)$.

($\Leftarrow$). Let $I \in \mathscr{X}$ and $r \in R$, then $\mathrm{Cl}^R_{\mathscr{X}}(I) = R$, hence $r \in \mathrm{Cl}^R_{\mathscr{X}}(I)$ and there is $J \in \mathscr{X}$ such that $Jr \subseteq I$. □

A sufficient condition for left closedness is the following: *for any element $r \in R$ and any ideal $I \in \mathscr{X}$ we have $(I : r) \in \mathscr{X}$.*

We say $\mathscr{X}$ is *inductive* if for any $I \in \mathscr{X}$ and any left ideal $J \supseteq I$, we have $J \in \mathscr{X}$.

A set of left ideals is called a *topological filter* if it is closed under intersections, inductive and left closed.

Lemma 2.6. *Let $\mathscr{X}$ be an inductive set of left ideals, then the following statements are equivalent:*

(a) *$\mathscr{X}$ is a topological filter.*

(b) *$\mathrm{Cl}^M_{\mathscr{X}}(N)$ is a submodule for any submodule $N \subseteq M$.*

P r o o f. We only need to show the implication (b) $\Rightarrow$ (a). As a matter of fact, $\mathscr{X}$ is weak closed under intersections and left closed as $\mathrm{Cl}_{\mathscr{X}}^{M}(N)$ is a submodule for any submodule $N \subseteq M$. Then $\mathscr{X}$ is a topological filter because it is inductive, therefore it is closed under intersections. □

A topological filter is a *linear filter* whenever it satisfies: *if* $I \subseteq R$ *and* $J \in \mathscr{T}$ *satisfy* $(I : y) \in \mathscr{T}$ *for any* $y \in J$*, then* $I \in \mathscr{T}$.

In the following linear filters will be denoted by the letter $\mathscr{L}$ and topological filters will be denoted by the letter $\mathscr{T}$.

Proposition 2.7. *Let* $\mathscr{T}$ *be a topological filter, the following statements are equivalent:*

(a) $\mathscr{T}$ *is a linear filter.*

(b) $\mathrm{Cl}_{\mathscr{T}}^{M}\mathrm{Cl}_{\mathscr{T}}^{M} = \mathrm{Cl}_{\mathscr{T}}^{M}$ *for any left* R*-module* M.

P r o o f. (a) $\Rightarrow$ (b). Indeed, if $N \subseteq M$ is a submodule and $x \in \mathrm{Cl}_{\mathscr{T}}^{M}\mathrm{Cl}_{\mathscr{T}}^{M}(N)$, there exists $I \in \mathscr{T}$ such that $Ix \subseteq \mathrm{Cl}_{\mathscr{T}}^{M}(N)$, then to any $y \in I$ there exists $I_y \in \mathscr{T}$ such that $I_y yx \subseteq N$, hence $I_y y \subseteq (N : x)$, i.e., for any $y \in I$ the ideal $((N : x) : y)$ belongs to $\mathscr{T}$, therefore $(N : x) \in \mathscr{T}$ as $\mathscr{T}$ is a linear filter.

(b) $\Rightarrow$ (a). Let $J \in \mathscr{T}$ and $I \subseteq R$ be such that for any $y \in J$ the ideal $(I : y) \in \mathscr{T}$; then $J \subseteq \mathrm{Cl}_{\mathscr{T}}^{R}(I)$, hence $R = \mathrm{Cl}_{\mathscr{T}}^{R}(J) \subseteq \mathrm{Cl}_{\mathscr{T}}^{R}\mathrm{Cl}_{\mathscr{T}}^{R}(I) = \mathrm{Cl}_{\mathscr{T}}^{R}(I)$, therefore $\mathrm{Cl}_{\mathscr{T}}^{R}(I) = R$ and we have $I \in \mathscr{T}$. □

Remark 2.8. When $\mathscr{T}$ is only a topological filter, in general we have that $\mathrm{Cl}_{\mathscr{T}}^{M}(N)$ is a submodule that is not necessarily $\mathscr{T}$-closed in M; to be $\mathscr{T}$-closed we need that $\mathscr{T}$ is a linear filter.

Let us summarize the properties of $\mathrm{Cl}_{\mathscr{X}}^{M}$ in terms of properties of $\mathscr{X}$. First we observe that in general, without any assumption on $\mathscr{X}$, we have the following properties:

(1) Expansive. $N \subseteq \mathrm{Cl}_{\mathscr{X}}^{M}(N)$ for any $N \subseteq M$.

(2) Monotone. If $N_1 \subseteq N_2$, then $\mathrm{Cl}_{\mathscr{X}}^{M}(N_1) \subseteq \mathrm{Cl}_{\mathscr{X}}^{M}(N_2)$ for any N_1, $N_2 \subseteq M$.

(3) Continuous. For any homomorphism $f\colon M \to M'$ and $N \subseteq M$ we have $f(\mathrm{Cl}_{\mathscr{X}}^{M}(N)) \subseteq \mathrm{Cl}_{\mathscr{X}}^{M'}(f(N))$.

(4) Hereditary. $\mathrm{Cl}_{\mathscr{X}}^{N}(X) = N \cap \mathrm{Cl}_{\mathscr{X}}^{M}(X)$ for any $X \subseteq N \subseteq M$.

As a consequence $\mathrm{Cl}_{\mathscr{X}}^{M}$ is a closure operator when $\mathscr{X}$ is weak closed under intersections and left closed, i.e., when for any submodule $N \subseteq M$ we have that $\mathrm{Cl}_{\mathscr{X}}^{M}(N)$ is a submodule; in particular when $\mathscr{X}$ is a topological filter.

Let us now assume that $\mathscr{T}$ is a topological filter.

(5) Relationship between $Cl_{\mathscr{T}}^{M}$ and $\mathrm{C}_{\mathscr{T}}^{M}$.

For any submodule $N \subseteq M$ we have $\mathrm{Cl}^M_{\mathscr{T}}(N) \subseteq \mathrm{C}^M_{\mathscr{T}}(N)$, as if $m \in \mathrm{Cl}^M_{\mathscr{T}}(N)$, there exists $I \in \mathscr{T}$ such that $Im \subseteq N \subseteq \mathrm{C}^M_{\mathscr{T}}(N)$, hence $m \in \mathrm{C}^M_{\mathscr{T}}(N)$. Otherwise we have the equality $\mathrm{Cl}^M_{\mathscr{T}}(N) = \mathrm{C}^M_{\mathscr{T}}(N)$ if and only if $\mathscr{T}$ is a linear filter. Indeed we have the following equivalences for any left R-module M and any submodule $N \subseteq M$:

$\mathrm{Cl}^M_{\mathscr{T}}(N) = \mathrm{C}^M_{\mathscr{T}}(N)$ if and only if
$\mathrm{Cl}^M_{\mathscr{T}}(N)$ is $\mathscr{T}$-closed if and only if
$\mathrm{Cl}^M_{\mathscr{T}}(\mathrm{Cl}^M_{\mathscr{T}}(N)) = \mathrm{Cl}^M_{\mathscr{T}}(N)$ if and only if
$\mathscr{T}$ is linear.

3. Honest submodules

A parallel notion to $\mathscr{X}$-closed submodule is the notion of $\mathscr{X}$-honest submodule; in this case the zero submodule will be always an $\mathscr{X}$-honest submodule and, as we will see later, the existence of nonzero $\mathscr{X}$-honest submodules induces some particular properties on the structure of the module. In addition, every $\mathscr{X}$-closed submodule is $\mathscr{X}$-honest.

If $\mathscr{X}$ is a nonempty set of ideals such that $0 \notin \mathscr{X}$, a submodule $N \subseteq M$ of a left R-module M is said to be an *$\mathscr{X}$-honest submodule*, or N is *$\mathscr{X}$-honest in M*, if for any $I \in \mathscr{X}$ and any $m \in M$, if $0 \neq Im \subseteq N$, then $m \in N$, and we write $N \subseteq^h_{\mathscr{X}} M$. Of course, if N is $\mathscr{X}$-closed in M, then N is $\mathscr{X}$-honest in M.

Contrary to the notion of $\mathscr{X}$-closed, the $\mathscr{X}$-honest notion was used only by a reduced number of authors; let us recall some of them: [1] where the authors define *honest subgroups* using instead of $\mathscr{X}$ the set of all nonzero integer numbers, and [3] where the authors define *honest submodules* using a set of elements of the ring.

As a matter of fact we collect the basic results on the behavior of honest submodules.

Lemma 3.1. *Let $H \subseteq N \subseteq M$ be submodules, then the following statements hold:*

(1) *If $H \subseteq N$ and $N \subseteq M$ are $\mathscr{X}$-honest submodules, then $H \subseteq M$ is $\mathscr{X}$-honest, i.e.: $H \subseteq^h_{\mathscr{X}} N \subseteq^h_{\mathscr{X}} M \Rightarrow H \subseteq^h_{\mathscr{X}} M$.*
(2) *If $N \subseteq M$ is $\mathscr{X}$-honest, then $N/H \subseteq M/H$ is $\mathscr{X}$-honest, i.e.: $N \subseteq^h_{\mathscr{X}} M \Rightarrow N/H \subseteq^h_{\mathscr{X}} M/H$.*
(3) *If $H \subseteq M$ and $N/H \subseteq M/H$ are $\mathscr{X}$-honest, then $N \subseteq M$ is $\mathscr{X}$-honest, i.e.: $H \subseteq^h_{\mathscr{X}} M$ and $N/H \subseteq^h_{\mathscr{X}} M/H \Rightarrow N \subseteq^h_{\mathscr{X}} M$.*

Proof. (1) Let $m \in M$, $I \in \mathscr{X}$ be such that $0 \neq Im \subseteq H$, then $0 \neq Im \subseteq N$, hence $m \in N$, and we see that $m \in H$.

(2) Let $m \in M$ and $I \in \mathscr{X}$ be such that $0 \neq I(m+H) \subseteq N/H$, then $0 \neq Im \subseteq N$, hence $m \in N$ and we obtain $m + H \in N/H$.

(3) Let $m \in M$ and $I \in \mathscr{X}$ be such that $0 \neq Im \subseteq N$. If $Im \subseteq H$, then $m \in H \subseteq N$. If $Im \nsubseteq H$, then $0 \neq I(m+H) \subseteq N/H$, hence $m + H \in N/H$, and we obtain $m \in N$. □

Lemma 3.2. *Let $\{N_\lambda \colon \lambda \in \Lambda\}$ be a family of $\mathscr{X}$-honest submodules of M, then $\bigcap_\lambda N_\lambda$ is $\mathscr{X}$-honest.*

Proof. Let $m \in M$ and $I \in \mathscr{X}$ such that $0 \neq Im \subseteq \bigcap_\lambda N_\lambda$, then $m \in \bigcap_\lambda N_\lambda$. □

We saw that $\mathscr{X}$-closed submodules are $\mathscr{X}$-honest submodules, but this is only part of a more close relationship between the $\mathscr{X}$-torsion submodule of M and $\mathscr{X}$-honest submodules.

Proposition 3.3. *Let $N \subseteq M$ be a submodule, then the following statements are equivalent:*

(a) *N is $\mathscr{X}$-honest in M. If, in addition, $\mathscr{X}$ is inductive, then the above statements are also equivalent to:*

(c) *For any $m \in \mathrm{Cl}^M_{\mathscr{X}}(N) \setminus N$ we have $(N : m) = \mathrm{Ann}\, m$.*

(d) *For any $m \in \mathrm{Cl}^M_{\mathscr{X}}(N) \setminus N$ we have $Rm \cap N = 0$.*

Proof. (c) ⇒ (d) ⇒ (a) ⇔ (b). These are obvious.

(b) ⇒ (c). Let $x \in \mathrm{Cl}^M_{\mathscr{X}}(N) \setminus N$, then there exists $I \in \mathscr{X}$ with $Ix \subseteq N$, then $Ix = 0$. Hence $I \subseteq \mathrm{Ann}\, x \subseteq (N : x)$, therefore $(N : x) \in \mathscr{X}$ and $(N : x) = \mathrm{Ann}\, x$. Thus we deduce that $Rx \cap N = 0$. □

Lemma 3.4. *Let $N \subseteq M$ be an $\mathscr{X}$-honest submodule, thus*

$$\mathrm{Cl}^M_{\mathscr{X}}(N) = N \cup T_{\mathscr{X}}(M).$$

Proof. Let $x \in \mathrm{Cl}^M_{\mathscr{X}}(N) \setminus N$, there exists $I \in \mathscr{X}$ such that $0 = Ix \subseteq N$, thus $x \in T_{\mathscr{X}}(M)$. □

Corollary 3.5. *Let $N \subseteq M$ be a submodule such that $T_{\mathscr{X}}(M) \subseteq N$, then $N \subseteq M$ is an $\mathscr{X}$-honest submodule if and only if it is $\mathscr{X}$-closed.*

In particular, if M is $\mathscr{X}$-torsionfree, then a submodule $N \subseteq M$ is $\mathscr{X}$-honest if and only if it is $\mathscr{X}$-closed.

Proof. Indeed, if $N \subseteq M$ is $\mathscr{X}$-honest then $\mathrm{Cl}^M_{\mathscr{X}}(N) = N \cup T_{\mathscr{X}}(M) = N$, and N is $\mathscr{X}$-closed. □

Corollary 3.6. *Let $\mathscr{L}$ be a linear filter, then for any R-module M the torsion submodule $T_{\mathscr{L}}(M) \subseteq M$ is $\mathscr{L}$-honest.*

Proof. Since $\mathscr{L}$ is a linear filter then $T_{\mathscr{L}}(M)$ is a $\mathscr{L}$-closed submodule, hence $\mathscr{L}$-honest. □

Remark 3.7. In the above Corollary we have in fact that $\mathscr{L}$ is a linear filter if and only if $T_{\mathscr{L}}(M) \subseteq M$ is $\mathscr{L}$-closed for any left R-module M if and only if $T_{\mathscr{L}}(M) \subseteq M$ is $\mathscr{L}$-honest for any left R-module M.

Corollary 3.8. *Let $\mathscr{X}$ be a set of ideals weak closed under intersections, then any $\mathscr{X}$-honest submodule $N \subseteq M$ satisfies either $N \subseteq T_{\mathscr{X}}(M)$ or $T_{\mathscr{X}}(M) \subseteq N$.*

Proof. Since N is $\mathscr{X}$-honest in M, $\mathrm{Cl}^M_{\mathscr{X}}(N) = N \cup T_{\mathscr{X}}(M)$. Since $\mathscr{X}$ is weak closed under intersections, $\mathrm{Cl}^M_{\mathscr{X}}(N)$ and $T_{\mathscr{X}}(M)$ are subgroups. Hence either N is included in $T_{\mathscr{X}}(M)$ or $T_{\mathscr{X}}(M)$ is included in N. □

If $\mathscr{L}$ is a linear filter and $N \subseteq T_{\mathscr{L}}(M)$ is $\mathscr{L}$-honest, then, by Lemma 3.1, we have that $N \subseteq M$ is honest.

Corollary 3.9. *Let $\mathscr{X}$ be a set of ideals weak closed under intersection. If $0 \neq N \subseteq M$ is an $\mathscr{X}$-torsionfree $\mathscr{X}$-honest submodule, then M is $\mathscr{X}$-torsionfree and $N \subseteq M$ is $\mathscr{X}$-closed.*

At this point, by Corollaries 3.5 and 3.8, we have a nice characterization of $\mathscr{X}$-honest submodules containing the subgroup $T_{\mathscr{X}}(M)$: *they are the $\mathscr{X}$-closed submodules.* We deal now with the problem of studying the $\mathscr{X}$-honest submodules which are contained in the $\mathscr{X}$-torsion submodule. Thus we may restrict ourselves to the case in which M is $\mathscr{X}$-torsion.

3.1. Torsion honest submodules.

The following is an interesting result on the annihilator of some torsion submodules.

Lemma 3.10. *Let $\mathscr{X}$ be an inductive set of ideals. Let M be a left R-module then for any $\mathscr{X}$-honest submodule $N \subsetneq T_{\mathscr{X}}(M)$ and any $m \in T_{\mathscr{X}}(M) \setminus N$ we have $0 \neq \mathrm{Ann}\, m \subseteq \mathrm{Ann}\, N \in \mathscr{X}$.*

Proof. We take $m \in T_{\mathscr{X}}(M) \setminus N$, then $\mathrm{Ann}\, m \in \mathscr{X}$. If $\mathrm{Ann}\, m \nsubseteq \mathrm{Ann}\, n$, for some $n \in N$, we have $\mathrm{Ann}\, mn \neq 0$. Thus we have $0 \neq \mathrm{Ann}\, mn = \mathrm{Ann}\, m(m+n) \subseteq N$, hence $m+n \in N$, as N is $\mathscr{X}$-honest. Thus $m \in N$, which is a contradiction. □

Corollary 3.11. *Let $\mathscr{X}$ be an inductive set of ideals. Let $N \subseteq M$ be an $\mathscr{X}$-honest submodule such that $N \subsetneqq T_{\mathscr{X}}(M)$, then*

$$\bigcup\{\operatorname{Ann} m\colon\ m \in T_{\mathscr{X}}(M) \setminus N\} \subseteq \operatorname{Ann} N \neq 0.$$

3.2. Honest submodules which are direct summands.

Theorem 3.12. *Let $\mathscr{X}$ be an inductive set of ideals. Let $N \subseteq M$ be an $\mathscr{X}$-honest submodule such that M/N is cyclic, nonzero and $\mathscr{X}$-torsion, then N is a direct summand of M.*

Proof. We assume $M/N = R(m + N)$, then $m \notin N$ and $(N : m) \in \mathscr{X}$. Since N is $\mathscr{X}$-honest in M and $m \notin N$, $Rm \cap N = 0$ (see Proposition 3.3(d)) and $M = Rm \oplus N$. □

This result can be extended as follows.

Theorem 3.13. *Let $\mathscr{X}$ be an inductive set of ideals. Let $N \subseteq M$ be an $\mathscr{X}$-honest submodule such that M/N is nonzero, $\mathscr{X}$-torsion and a direct sum of cyclic submodules, then N is a direct summand of M.*

Proof. Let us assume that $M/N = \oplus_{\lambda\in\Lambda} R(m_\lambda + N)$ is a direct sum with $m_\lambda + N \neq 0$ for any $\lambda \in \Lambda$, then we have that $Rm_\lambda \cap N = 0$ for any $\lambda \in \Lambda$. Otherwise, if we take $0 \neq m \in \Big(\sum\limits_{\lambda\in\Lambda} Rm_\lambda\Big) \cap N$, we may assume $m = \sum\limits_{i=1}^{t} r_i m_{\lambda_i}$ for some $r_i \in R$ and $r_i m_{\lambda_i} \neq 0$ for $i = 1, \ldots, t$. Thus we have:

$$r_1 m_{\lambda_1} + N = -\sum_{i=2}^{t} r_i m_{\lambda_i} + N \in R(m_{\lambda_1} + N) \cap \sum_{i=2}^{t} R(m_{\lambda_i} + N) = 0.$$

Hence $r_1 m_{\lambda_1} \in N$. Therefore $r_1 m_{\lambda_1} = 0$, which is a contradiction. Thus we obtain $\sum\limits_{\lambda\in\Lambda} Rm_\lambda \cap N = 0$ and $M = N \oplus \Big(\sum\limits_{\lambda\in\Lambda} Rm_\lambda\Big)$. □

As a consequence if M is an Abelian group and $\mathscr{X}$ is the set of all nonzero ideals, then any $\mathscr{X}$-honest submodule N of M, such that M/N is finitely generated and torsion, is a direct summand.

Problem 3.14. Is it true, in general, that if $\mathscr{X}$ is inductive and $N \subseteq M$ is an $\mathscr{X}$-honest submodule such that M/N is nonzero finitely generated and $\mathscr{X}$-torsion, then N is a direct summand of M?

3.3. Direct summands. Let us study the $\mathscr{X}$-honest direct summands of a left R-module M, mainly when $\mathscr{X}$ is an inductive set of ideals.

We deal now with the problem of determining which direct summands are $\mathscr{X}$-honest.

Proposition 3.15. *Let $\mathscr{X}$ be an inductive set of ideals. Let $M = N \oplus K$ be a direct sum such that K is not $\mathscr{X}$-torsionfree, then the following statements are equivalent:*

(a) *$N \subseteq M$ is $\mathscr{X}$-honest.*

(b) $\bigcup\{\operatorname{Ann} k\colon\ 0 \neq k \in T_{\mathscr{X}}(K)\} \subseteq \operatorname{Ann} N \neq 0$.

Moreover, in this case N is $\mathscr{X}$-torsion.

Proof. (a) $\Rightarrow$ (b). For any $0 \neq k \in T_{\mathscr{X}}(K)$, if $Ik = 0$ for some $I \in \mathscr{X}$, then for any $n \in N$ we have $I(n+k) = In \subseteq N$, thus $In = 0$ and we have $I \subseteq \operatorname{Ann} N$.

(b) $\Rightarrow$ (a). Let $m \in M$ and $I \in \mathscr{X}$ be such that $0 \neq Im \subseteq N$. We may assume $m = n + k$, hence $Ik = 0$, and $k \in T_{\mathscr{X}}(M)$. If $k \neq 0$, then $I \subseteq \operatorname{Ann} k \subseteq \operatorname{Ann} N$ and we have $Im = 0$, which is a contradiction. Hence $k = 0$ and $m = n \in N$. □

The case in which K is $\mathscr{X}$-torsionfree is well-known. In fact, it follows directly from Corollary 3.5 that:

Proposition 3.16. *Let $M = N \oplus K$ be a direct sum such that K is $\mathscr{X}$-torsionfree, then the following statements are equivalent:*

(a) *$N \subseteq M$ is $\mathscr{X}$-honest.*

(b) $N = \operatorname{Cl}^M_{\mathscr{X}}(N)$.

Proposition 3.17. *Let $M = N \oplus K$ be a direct sum such that N is nonzero and $\mathscr{X}$-torsionfree, then the following statements are equivalent:*

(a) *$N \subseteq M$ is $\mathscr{X}$-honest.*

(b) *K is $\mathscr{X}$-torsionfree.*

(c) $N = \operatorname{Cl}^M_{\mathscr{X}}(N)$.

(d) *M is $\mathscr{X}$-torsionfree.*

Proof. (a) $\Rightarrow$ (b). Let $I \in \mathscr{X}$. If $Ik = 0$, then $IN = 0$, which is a contradiction. □

3.4. The honest operator. As in the case of the closure operator we may define the honest operator using Lemma 3.2.

Let $N \subseteq M$ be a submodule, we define a new submodule as follows:

$$\operatorname{H}^M_{\mathscr{X}}(N) = \bigcap\{H \subseteq M\colon\ N \subseteq H \text{ and } H \subseteq M \text{ is } \mathscr{X}\text{-honest}\}.$$

Since the intersection of a family of $\mathscr{X}$-honest submodules is $\mathscr{X}$-honest, see Lemma 3.2, we have:

Lemma 3.18. *If N is a submodule of M then $\mathrm{H}^M_{\mathscr{X}}(N)$ is the smallest $\mathscr{X}$-honest submodule of M containing N.*

In fact $\mathrm{H}^M_{\mathscr{X}}$ defines an operator in the lattice of all R-submodules of M; the properties of this operator are the following:
(1) Expansive. $N \subseteq \mathrm{H}^M_{\mathscr{X}}(N)$ for any $N \subseteq M$.
(2) Monotone. If $N_1 \subseteq N_2$, then $\mathrm{H}^M_{\mathscr{X}}(N_1) \subseteq \mathrm{H}^M_{\mathscr{X}}(N_2)$ for any N_1, $N_2 \subseteq M$.
(3) Idempotent. $\mathrm{H}^M_{\mathscr{X}}\mathrm{H}^M_{\mathscr{X}} = \mathrm{H}^M_{\mathscr{X}}$.
Observe that we do not include the continuity property. The reason is the following:

Lemma 3.19. *The following statements are equivalent:*
(a) *$\mathrm{H}^M_{\mathscr{X}}$ is a closure operator.*
(b) *Any submodule of any left R-module is $\mathscr{X}$-honest.*
(c) *$\mathscr{X} = \{R\}$.*

P r o o f. (a) $\Rightarrow$ (b). Let $N \subseteq M$ be a submodule, we consider the canonical projection $\eta\colon M \to M/N$. Since $\mathrm{H}^M_{\mathscr{X}}$ is a closure operator, $\eta(\mathrm{H}^M_{\mathscr{X}}(N)) \subseteq \mathrm{H}^{M/N}_{\mathscr{X}}(\eta(N)) = \mathrm{H}^{M/N}_{\mathscr{X}}(0) = 0$, hence $\mathrm{H}^M(N) \subseteq \mathrm{Ker}(\eta) = N$.
(b) $\Rightarrow$ (c). Let $0 \neq I \in \mathscr{X}$, then $0 \neq I1 \subseteq I$, hence $1 \in I$ and $I = R$.
(c) $\Rightarrow$ (a). It is clear that any submodule is $\mathscr{X}$-honest, hence the honest operator is the identity operator which is a closure operator. □

As a consequence $\mathrm{H}^M_{\mathscr{X}}$ is a closure operator if and only if it is the identity operator.
(5) Relation with $\mathrm{C}^M_{\mathscr{X}}$.
Since $\mathrm{C}^M_{\mathscr{X}}(N)$ is $\mathscr{X}$-honest in M, $\mathrm{C}^M_{\mathscr{X}}(N) \supseteq \mathrm{H}^M_{\mathscr{X}}(N)$.

4. Examples and applications

4.1. Left Ore rings. Let R be a ring and $\mathscr{C}$ be the set of all nonzero divisors of R. We say R is a *left Ore ring* if

$$\forall a \in R,\ \forall s \in \mathscr{C},\ \exists b \in R \text{ and } t \in \mathscr{C} \text{ such that } bs = ta,$$

i.e., if the set $\mathscr{C}$ is a left Ore set of R. If R is a left Ore ring, then $\mathscr{C}^{-1}R$ is named the *classical left ring of quotients* of R or the *total left ring of quotients* of R, and we always have an injective ring map $R \hookrightarrow \mathscr{C}^{-1}R = Q^l_{cl}(R)$.

A ring S is a *ring of quotients* if any element which is not a zero divisor is invertible. If S is a ring of quotients and $R \subseteq S$ is a subring such that $S = Q^l_{cl}(R)$, then we say R is a *left order* in S.

Let $\mathscr{C}$ be a subset of R, such that $0 \notin \mathscr{C}$, and consider the following set of left ideals:

$$\mathscr{L}\mathscr{C} = \{I \subseteq R \colon\ I \cap \mathscr{C} \neq \varnothing\}.$$

It is easy to prove that $\mathscr{L}\mathscr{C}$ is an inductive set of left ideals. Hence we may consider the honest operator $H_{\mathscr{L}\mathscr{C}}$. First we obtain:

Theorem 4.1. *Let $\mathscr{C}$ be the set of not containing zero divisors of R, then the following statements are equivalent:*

(a) *The $\mathscr{C}$-honest operator $H_{\mathscr{L}\mathscr{C}}$ is a closure operator.*
(b) *R is a ring of quotients.*
(c) *Each submodule is $\mathscr{L}\mathscr{C}$-honest.*
(d) *The $\mathscr{C}$-honest operator $H_{\mathscr{L}\mathscr{C}}$ is the identity operator.*

Proof. We only need to prove the relationship between statements (a) and (b) as we know the equivalence between (a), (c) and (d).

(a) $\Rightarrow$ (b). From Lemma 3.19 we have that $\mathscr{L}\mathscr{C} = \{R\}$, hence for any $c \in \mathscr{C}$ the left ideal Rc satisfies $Rc = R$, so there exists $d \in R$ such that $dc = 1$, hence $cdc = c$ and, since c is not a zero divisor, we obtain that c is invertible.

(b) $\Rightarrow$ (a). This is trivial as each element of $\mathscr{C}$ is invertible. □

This is an extension of the results given by Fay and Joubert in [3].

Now the following natural question arises: *Could rings of quotients be characterized by the operators $C^M_{\mathscr{L}\mathscr{C}}$ and $\mathrm{Cl}^M_{\mathscr{L}\mathscr{C}}$?*

When we deal with the operator $\mathrm{Cl}^M_{\mathscr{L}\mathscr{C}}$ we arrive at left Ore sets in the following way.

Proposition 4.2. *Let $\mathscr{C}$ be a multiplicative subset of a ring R, then the following statements are equivalent:*

(a) *$\mathscr{C}$ is a left Ore set.*
(b) *$\mathrm{Cl}^M_{\mathscr{L}\mathscr{C}}$ is an idempotent closure operator.*
(c) *$\mathscr{L}\mathscr{C}$ is a linear filter.*
(d) *$\mathscr{L}\mathscr{C}$ is a topological filter.*
(e) *$\mathrm{Cl}^M_{\mathscr{L}\mathscr{C}}(N)$ is a submodule of M for any submodule $N \subseteq M$.*

Proof. (a) $\Rightarrow$ (b). Let $m_1,\ m_2 \in \mathrm{Cl}^M_{\mathscr{L}\mathscr{C}}(N)$, there exist $s_1,\ s_2 \in \mathscr{C}$ such that $Rs_im_i \subseteq N$. Since $\mathscr{C}$ is a left Ore set, there exist $b \in R$ and $t \in \mathscr{C}$ such that $bs_1 = ts_2$, thus $Rts_2(m_1 + m_2) \subseteq N$. On the other hand, if $m \in \mathrm{Cl}^M_{\mathscr{L}\mathscr{C}}(N)$ and

$r \in R$, there exists $s \in \mathscr{C}$ such that $Rsm \subseteq N$, and there exist $b \in R$ and $t \in \mathscr{C}$ such that $bs = tr$, thus $Rtrm = Rbsm \subseteq N$. Finally, if $m \in \mathrm{Cl}^M_{\mathscr{L}\mathscr{C}} \mathrm{Cl}^M_{\mathscr{L}\mathscr{C}}(N)$, then there exists $s \in \mathscr{C}$ such that $sm \in \mathrm{Cl}^M_{\mathscr{L}\mathscr{C}}(N)$, and there exists $t \in \mathscr{C}$ such that $tsm \in N$, thus $m \in \mathrm{Cl}^M_{\mathscr{L}\mathscr{C}}(N)$.

(b) $\Leftrightarrow$ (c) is a consequence of Proposition 2.7. (d) $\Leftrightarrow$ (e) is a consequence of Lemma 2.6. And (c) $\Rightarrow$ (d) is obvious.

(e) $\Rightarrow$ (a). Let $a \in R$ and $s \in \mathscr{C}$, then $T_{\mathscr{L}\mathscr{C}}(R/Rs) \subseteq R/Rs$; since $1 + Rs \in T_{\mathscr{L}\mathscr{C}}(R/Rs)$, then $T_{\mathscr{L}\mathscr{C}}(R/Rs) = R/Rs$ as it is a submodule, hence there exists $t \in \mathscr{C}$ such that $t(a + Rs) = 0$, therefore there exists $b \in R$ such that $ta = bs$ and $\mathscr{C}$ is a left Ore set. □

As a consequence $\mathrm{Cl}^M_{\mathscr{L}\mathscr{C}}$ defines a submodule operator if and only if $\mathscr{C}$ is a left Ore set. In particular, if $\mathscr{C}$ is the set of all elements of R which are not zero divisors of R, then R is a ring of quotients if and only if $\mathrm{Cl}^M_{\mathscr{L}\mathscr{C}}$ is the identity operator.

Thus if $\mathscr{C}$ is the set of all elements of R which are not zero divisors, we have that the following statements are equivalent:

(a) R is a ring of quotients.
(b) $\mathrm{H}^M_{\mathscr{L}\mathscr{C}}$ is a closure operator.
(c) $\mathrm{H}^M_{\mathscr{L}\mathscr{C}} = \mathrm{Cl}^M_{\mathscr{L}\mathscr{C}}$.
(d) $\mathrm{Cl}^M_{\mathscr{L}\mathscr{C}}$ is the identity operator.
(e) $\mathrm{H}^M_{\mathscr{L}\mathscr{C}}$ is the identity operator.

And in the case in which we take a multiplicative subset $\mathscr{C}$ we also have that the following statements are equivalent:

(a) $\mathrm{C}^M_{\mathscr{L}\mathscr{C}} = \mathrm{Cl}^M_{\mathscr{L}\mathscr{C}}$.
(b) $\mathrm{Cl}^M_{\mathscr{L}\mathscr{C}}(N)$ is a submodule of M for any submodule $N \subseteq M$.
(c) $\mathscr{C}$ is a left Ore set.

4.2. Some concrete examples.

Example 4.3. Let us present an example of an honest submodule when we take $\mathscr{C} = \mathbb{Z} \setminus \{0\}$. If we consider $H = \mathbb{Z}(1,1) \subseteq \mathbb{Z} \times \mathbb{Z} = M$, then H is $\mathscr{C}$-honest in M. As a matter of fact, if $0 \neq z(m_1, m_2) \in H$, there exists $a \in \mathbb{Z}$ such that $z(m_1, m_2) = a(1,1)$, hence $zm_1 = a \neq 0$ and $zm_2 = a \neq 0$. Thus $m_1 = m_2$ and $m \in H$.

Example 4.4. $H = \mathbb{Z}_p(1,1) \subseteq \mathbb{Z}_p \times \mathbb{Z}_p = M$. In this case $H \subseteq M$ is honest and is not closed.

Example 4.5. $p\mathbb{Z}_{p^2} \subseteq \mathbb{Z}_{p^2}$ is not honest.

4.3. Semiartinian modules. We shall study as a particular example honest submodules in semiartinian modules over a *commutative ring* A. Remember that an A-module M is semiartinian if the union of the socle series $\{\mathrm{Soc}_\alpha(M)\}_\alpha$ is equal to M, or equivalently, any nonzero homomorphic image of M contains a simple submodule.

Let $\mathscr{P}$ be a nonempty set of maximal ideals and define $\mathscr{L} = \{I \subseteq A\colon$ there exists $P_1, \ldots, P_t \in \mathscr{P}$ such that $P_1 \ldots P_t \subseteq I\}$, then $\mathscr{L}$ is a linear filter and each $\mathscr{L}$-torsion A-module is semiartinian. If, in particular, $\mathscr{P}$ is the set of all maximal ideals, then an A-module M is semiartinian if and only if it is $\mathscr{L}$-torsion.

Proposition 4.6. *Let A be a commutative ring, let $\mathscr{L}$ be defined as before by a set $\mathscr{P}$ of maximal ideals and let $H \subsetneqq T$ be a nonzero $\mathscr{L}$-honest submodule of an $\mathscr{L}$-torsion A-module T. Then there exists an ideal $P \in \mathscr{P}$ such that $\mathrm{Ann}\, H = P$ and for any $0 \neq x \in T$ there exists a positive integer e such that $P^e \subseteq \mathrm{Ann}\, x \subseteq P$.*

Proof. Let $x \in T \setminus H$ with $\mathrm{Ann}\, x \supseteq P_1^{e_1} \ldots P_t^{e_t}$, for some positive integers e_i, $i = 1, \ldots, t$, then $P_1^{e_1} \ldots P_t^{e_t} \subseteq \mathrm{Ann}\, x \subseteq \mathrm{Ann}\, H$. There is a multiple y_i of x such that $\mathrm{Ann}\, y_i \supseteq P_i$ for any index $i = 1, \ldots, t$; since $0 \neq y_i \notin H$ we have $P_i \subseteq \mathrm{Ann}\, H$ for any index $i = 1, \ldots, t$, which is a contradiction if $t \geqslant 2$. As a consequence $\mathrm{Ann}\, x \supseteq P_1^{e_1}$. The other inclusion is a consequence of $\mathrm{Ann}\, H = P_1$. □

Proposition 4.7. *Let A be a commutative ring, let $\mathscr{L}$ be defined as before by a set $\mathscr{P}$ of maximal ideals and let $H \subsetneqq T$ be a nonzero $\mathscr{L}$-honest submodule of an $\mathscr{L}$-torsion A-module T. Then there exists an ideal $P \in \mathscr{P}$ such that H is a direct sum of copies of A/P.*

Proof. We assume $\mathrm{Ann}\, H = P \in \mathscr{P}$, then for any nonzero element $h \in H$ we have $\mathrm{Ann}\, h = P$. Thus H is a (direct) sum of copies of A/P. □

Proposition 4.8. *Let A be a commutative ring, let $\mathscr{L}$ be defined as before by a set $\mathscr{P}$ of maximal ideals and let $H \subseteq T$ be a nonzero $\mathscr{L}$-honest submodule of an $\mathscr{L}$-torsion A-module T. Then H is a direct summand of T.*

Proof. We assume $\mathrm{Ann}\, H = P$. First we point out that $H \cap PT = 0$. Indeed, $PH = 0$ and if for some $t \in T \setminus H$ we have $0 \neq Pt \cap H \subseteq At \cap H$, then $(H\colon t)t \neq 0$, hence $t \in H$, which is a contradiction. Let $L \subseteq T$ be maximal satisfying $L \cap H = 0$ and $PT \subseteq L$. For any $x \in T \setminus (L \oplus H)$ we have $Px \subseteq PT \subseteq L$. On the other hand, by the maximality of L, we have: $(L + Ax) \cap H \neq 0$, hence there are $l \in L$ and $a \in A$ such that $0 \neq l + ax \in (L + Ax) \cap H$, therefore $ax \in L + H$; in particular $a \notin P$, if $a \in P$, then $0 \neq l + ax \in L \cap H = 0$. Thus we have $ax \in L + H$ and $Px \subseteq L \subseteq L + H$, hence $x \in Ax = (Aa + P)x \subseteq L + H$, which is a contradiction. □

4.4. Dedekind domains. We shall study as a particular example honest submodules in torsion modules on a Dedekind domain D; in particular since torsion D-modules are semiartinian modules then we may apply the previous results. Thus we have:

Proposition 4.9. *Let D be a Dedekind domain and $\mathscr{C} = D \setminus \{0\}$. Let $H \subseteq T$ be a nonzero $\mathscr{C}$-honest proper submodule of a $\mathscr{C}$-torsion D-module T. Then there exists a prime ideal P such that T is P-primary.*

Proposition 4.10. *Let D be a Dedekind domain and $\mathscr{C} = D \setminus \{0\}$. Let $H \subseteq T$ be a nonzero $\mathscr{C}$-honest proper submodule of a $\mathscr{C}$-torsion D-module T. Then there exists a prime ideal P such that H is a direct sum of copies of D/P.*

Proposition 4.11. *Let D be a Dedekind domain and $\mathscr{C} = D \setminus \{0\}$. Let $H \subseteq T$ be a nonzero $\mathscr{C}$-honest proper submodule of a $\mathscr{C}$-torsion D-module T. Then H is a direct summand of T.*

4.5. The enveloping algebra of a finite dimensional solvable Lie algebra. Let R be the enveloping algebra of a finite dimensional solvable Lie algebra over the field of complex numbers, then each cofinite prime ideal has codimension one, hence it is maximal as a left and a right ideal. The same result holds when we consider $R = \mathscr{O}_q(SL_n(\mathbb{C}))$, the quantum coordinate algebra of $SL_n(\mathbb{C})$, q not being a root of unity, see [7, Corollary 3.3].

Let us consider the linear filter $\mathscr{L}$ generated by the set of all cofinite prime ideals, then each ideal in $\mathscr{L}$ contains a product $P_1 \dots P_t$, where each P_i is a cofinite prime ideal. Let T be an $\mathscr{L}$-torsion left R-module, and assume $H \subsetneqq T$ is a nonzero $\mathscr{L}$-honest submodule, then we have:

Proposition 4.12. *With the above notation there exists a cofinite prime ideal P_1 such that $\operatorname{Ann} H = P_1$, and for any $x \in T \setminus H$ there exist cofinite prime ideals $Q_2, \dots, Q_s$ such that $P_1 Q_2 \dots Q_s \subseteq \operatorname{Ann} x$.*

P r o o f. Let $x \in T \setminus H$ with $\operatorname{Ann} x \supseteq P_1 \dots P_t$, then $P_1 \dots P_t = \operatorname{Ann} x \subseteq \operatorname{Ann} H$. There is a multiple y of x such that $\operatorname{Ann} y \supseteq P_1$; since $0 \neq y \notin H$ we have $P_1 \subseteq \operatorname{Ann} H$. Now, as P_1 is maximal, we have $\operatorname{Ann} H = P_1$. As a consequence, for any $x \in T \setminus H$ there exist cofinite prime ideals $Q_2, \dots, Q_s$ such that $P_1 Q_2 \dots Q_s \subseteq \operatorname{Ann} x$. □

Proposition 4.13. *With the above notation there exists a prime ideal $P \in \mathscr{P}$ such that H is a direct sum of copies of simple R/P-modules.*

Proof. We have that R/P is a semisimple Artinian algebra, as it is finite dimensional and prime, and H is a left R/P-module. □

Proposition 4.14. *With the above notation H is a direct summand of T.*

Proof. The same proof as in Proposition 4.8 works. □

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