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HOMOMORPHIC IMAGES OF FINITE SUBDIRECTLY IRREDUCIBLE UNARY ALGEBRAS

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Abstract. We prove that a finite unary algebra with at least two operation symbols is a homomorphic image of a (finite) subdirectly irreducible algebra if and only if the intersection of all its subalgebras which have at least two elements is nonempty.

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We are concerned with the following natural question: *Which algebras are homomorphic images of subdirectly irreducible algebras?*

A necessary condition, discovered in [3]–[5], for an algebra A with at least one at least binary operation to be a homomorphic image of some subdirectly algebra, is that the intersection of all ideals of A is nonempty. (By an *ideal* of A we mean a nonempty subset I such that $f(a_1, \dots, a_n) \in I$ whenever f is a fundamental operation and a_1, \dots, a_n are elements of A with $a_i \in I$ for at least one i .) It was proved in [2] and independently in [8] that the condition is also sufficient. In fact, it was proved in those two papers that any algebra A with at least one at least binary operation and with a smallest ideal is isomorphic to a factor of a subdirectly irreducible algebra B through its monolith, and the construction is such that if A is finite then also B is finite. The case that remains is that of algebras with only unary operations. (Note that nullary operations play no role in investigation of congruences.)

If there is just one unary operation, the characterization is simple; see e.g. [9] or [2]. For two or more unary operations, the situation is more complicated. In the present

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paper we are going to characterize *finite* algebras with at least two unary operations that are homomorphic images of a subdirectly irreducible algebra. We leave it as an open problem to do the same for infinite algebras. An example found in [2] suggests that it will be probably much harder to characterize such unary algebras that are isomorphic to a subdirectly irreducible algebra through its monolith.

One can ask similar questions in particular varieties of algebras. For instance, the construction from [8] yields for an idempotent algebra an idempotent subdirectly irreducible one. (In fact, it preserves any identity of the form $t(x) \approx x$, where t is a unary term. And if the algebra contains no proper ideal, then the construction preserves any identity in one variable.) However, it seems that no other interesting identities are kept.

Indeed, in many varieties almost no algebras can be represented as homomorphic images of subdirectly irreducibles—for instance in distributive lattices, for an obvious reason. On the other hand, in several familiar varieties the answer is known to be similar to the general case. It is proved in [1] that a semigroup is a homomorphic image of some subdirectly irreducible semigroup if and only if the intersection of all its ideals is nonempty. However, there are finite semigroups (e.g. any right zero band) which are not isomorphic to the factor of a *finite* subdirectly irreducible semigroup over its monolith. Every quasigroup, group and every lattice (indeed, all of them have no non-trivial ideals, in our sense) is also a homomorphic image of a subdirectly irreducible one (quasigroup, group, lattice) and for a finite one the subdirectly irreducibles can be constructed finite. For lattices, an easy construction was found by Ralph Freese (unpublished). For quasigroups and groups, a solution was found by R. McKenzie and D. Stanovský [7].

Throughout this paper let σ be a finite set of unary operation symbols, $|\sigma| \geq 2$. By a unary algebra we mean a σ -algebra. By a word we mean a word over the alphabet σ .

For a word $w = f_1 \dots f_n$ and any $i = 0, \dots, n - 1$ we define a word $w^{[i]} = f_{i+1} \dots f_n f_1 \dots f_i$. This is a word of the same length as w . We put $w^i = ww \dots w$ where w is repeated i times.

Let A be a unary algebra; let $a \in A$ and let $w = f_1 \dots f_n$ be a word. For $i = 0, \dots, n$ we put $w_{\langle i \rangle}(a) = f_i \dots f_2 f_1(a)$ (so that $w_{\langle 0 \rangle}(a) = a$). By an e-pair for A we mean a pair a, w where $a \in A$, w is a word of a positive length n , $w_{\langle n \rangle}(a) = a$ and $A = \{w_{\langle 0 \rangle}(a), \dots, w_{\langle n-1 \rangle}(a)\}$. (Informally, an e-pair is a path through all elements of A using the operations.) An e-pair a, w is said to start at a ; w is called the e-pair's path; the e-pair's length is the length of its path (clearly, the length is at least $|A|$).

It is easy to see that a finite unary algebra A has an e-pair if and only if it has no proper subalgebra; and if a, w is an e-pair for A then $w_{\langle i \rangle}(a), w^{[i]}$ is an e-pair for A

for any $i = 0, \dots, n - 1$, where n is the length of w . Consequently, if A has no proper subalgebra then it has an e-pair starting at any of its elements.

An e-pair a, w for A is said to be an h-pair if $|A|$ equals the length of w .

Lemma 1. *Let a, w be an e-pair for a finite unary algebra A . Then there exist a finite unary algebra B and an element $b \in B$ such that A is a homomorphic image of B and b, w is an h-pair for B .*

Proof. Put $w = f_1 \dots f_n$. For $i = 0, \dots, n - 1$ put $a_i = w_{\langle i \rangle}(a)$. For every $c \in A$ denote by p_c the number of occurrences of c in $w_{\langle 0 \rangle}(a), \dots, w_{\langle n-1 \rangle}(a)$. Put $B = \{(a_i, j) : 0 \leq i < n, 0 \leq j < p_{a_i}\}$. For $(a_i, j) \in B$ and $f \in \sigma$ put $f(a_i, j) = (f(a_i), k)$ where $k = |\{m : 0 \leq m < i, f(a_m) = a_m\}|$. Then $(a_i, j) \mapsto a_i$ is a homomorphism of B onto A and $(a, 0), w$ is an h-pair for B . \square

A word $w = f_1 \dots f_n$ is said to be irreducible if $w \neq w^{[i]}$ for all $1 \leq i < n$. It is easy to see that a word is irreducible if and only if it cannot be expressed as u^j for a word u and an integer $j \geq 2$. An e-pair is said to be irreducible if its path is irreducible.

Lemma 2. *Let a, w be an e-pair for a finite unary algebra A . Then there exists a finite unary algebra B such that A is a homomorphic image of B and B has an irreducible h-pair.*

Proof. Let $w = f_1 \dots f_n$ for some $f_i \in \sigma$. Take a symbol $g \in \sigma$ different from f_1 and put $b = f_1 g^{2n}(a)$. Let i be such that $b = w_{\langle i \rangle}(a)$ and put $h_1 = f_{i+1}, h_2 = f_{i+2}, \dots, h_n = f_i, \dots, h_m = f_n$, so that $a = h_m \dots h_1(b)$, $A = \{b, h_1(b), \dots, h_m \dots h_1(b)\}$ and $m < 2n$. Thus $a, g^{2n} f_1 h_1 \dots h_m$ is an irreducible e-pair for A . Now use Lemma 1. \square

Lemma 3. *Let a, w be an irreducible h-pair for a finite unary algebra A . Then A is a homomorphic image of a finite subdirectly irreducible unary algebra. Moreover, A is isomorphic to a factor of a finite subdirectly irreducible unary algebra through its monolith.*

Proof. Put $n = |A|$ and $w = f_1 \dots f_n$. For every $c \in A$ there is a unique number $i \in \{0, \dots, n - 1\}$ with $c = w_{\langle i \rangle}(a)$; denote this i by $I(c)$. Let p be a prime number greater than $2n(n + 1)$. Denote by $+$ ' the addition modulo p . Put $B = A \times \{0, \dots, p - 1\}$. Define σ -operations on B by

$$f(c, i) = \begin{cases} (f(c), i) & \text{if } f = f_{I(c)+1} \\ (f(c), i + ' 1) & \text{otherwise.} \end{cases}$$

Evidently, A is a homomorphic image of B under $(c, i) \mapsto c$. We are going to show that the kernel R of this homomorphism is the monolith of B . We have $((c, i), (d, j)) \in R$ if and only if $c = d$. Let \sim be a nontrivial congruence of B . We need to show that R is contained in \sim .

Let c, d be two elements of A . By a path from c to d we mean a word $g_1 \dots g_k$ such that $d = g_k \dots g_1(c)$. Such a path is said to be canonical if $g_i = f_{I(g_{i-1} \dots g_1(c))+1}$ for every $i = 1, \dots, k$. Now for every c, d there exists a canonical path from c to d of length at most n , and also a non-canonical path from c to d of length at most n . For $i = 0, \dots, p-1$ we have $g_k \dots g_1(c, i) = (d, i + j)$ where $j = 0$ if the path is canonical and $1 \leq j \leq k$ otherwise.

Let us first prove that if $(c, i) \sim (c, j)$ for some c and some $i \neq j$ then R is contained in \sim . By taking a canonical path from c to a we get $(a, i) \sim (a, j)$. Let $g \in \sigma$ be different from f_1 . We have $g(a, i) \sim g(a, j)$, i.e., $(g(a), i+1) \sim (g(a), j+1)$. Applying a canonical path from $g(a)$ to a we get $(a, i+1) \sim (a, j+1)$. Since p is a prime number, it follows that $(a, k) \sim (a, l)$ for all k and l . If d is any element of A then applying a canonical path from a to d we get $(d, k) \sim (d, l)$.

Now let $(c, i) \sim (d, j)$ where $c \neq d$. Without loss of generality, $j = i + l$ where $0 \leq l < \frac{1}{2}p$. The canonical path from c to c of length n is a non-canonical path from d to some element e_1 , because w is an irreducible word. Applying this path we get $(c, i) \sim (e_1, i+l+k_1)$ for some e_1, k_1 with $1 \leq k_1 \leq n$. If $e_1 = c$ then $(c, i) \sim (c, i+l+k_1)$ where $i \neq i+l+k_1$ and we are done by the above. Otherwise, apply the same path to obtain $(c, i) \sim (e_2, i+l+k_1+k_2)$ for some e_2, k_2 with $1 \leq k_2 \leq n$. Continue in this way $n+1$ times. If it did not happen that $e_m = c$ for some m , in which case we would be finished by the above, we get $(c, i) \sim (e_m, i+(l+k_1+\dots+k_m))$ for all $m = 1, \dots, n+1$. Since p was so large, we have $m \leq l+k_1+\dots+k_m < \frac{1}{2}p+mn < p$ and since all $k_m \neq 0$, the numbers $i+(l+k_1+\dots+k_m)$ are pairwise distinct. However, the elements e_1, \dots, e_{n+1} cannot be pairwise distinct. So, we get into the previous case. \square

Lemma 4. *Let a, w be an irreducible h -pair for a finite unary algebra A . Then there exist a finite subdirectly irreducible unary algebra C and a homomorphism H of C onto A such that for every element $c \in A$ there exists an element $c' \in C$ with $H(c') = c$ and $c' \neq f(d)$ for all $f \in \sigma$ and all $d \in C$.*

Proof. Define B in the same way as in Lemma 3. Put $C = A \times \{0, \dots, p\}$ and define operations on C in such a way that B is a subalgebra of C and $f(c, p) = (f(c), I(c))$ (we have $I(c) < n < p$). Clearly, A is a homomorphic image of C under $(c, i) \mapsto c$. Define a congruence S of C by $((c, i), (d, j)) \in S$ if and only if either $(c, i) = (d, j)$ or $c = d$ and $i, j < p$. In order to prove that S is the monolith of C , it is sufficient (by the proof of Lemma 3) to show that if $(c, i) \sim (d, p)$ where

$(c, i) \neq (d, p)$ then $(c', j) \sim (d', k)$ for some $(c', j) \neq (d', k)$ with $j, k < p$. If $i = p$ then $c \neq d$, $I(c) \neq I(d)$ and so it is sufficient to apply any operation to $(c, p) \sim (d, p)$. Let $i \neq p$. If $i \neq I(d)$, apply $f_{I(c)+1}$; and if $i = I(d)$, apply any operation different from $f_{I(c)+1}$. \square

Lemma 5. *Let a, w be an irreducible h -pair for a finite unary algebra A and let N be a positive integer. Then there exist a finite subdirectly irreducible unary algebra C and a homomorphism H of C onto A such that for every element $c \in A$ there exist at least N distinct elements $c' \in C$ with $H(c') = c$ and $c' \neq f(d)$ for all $f \in \sigma$ and all $d \in C$.*

Proof. a, w^N is an e -pair for A . With respect to this e -pair, apply Lemma 1 and Lemma 2 to obtain an algebra A' without proper subalgebras and a homomorphism of A' onto A such that for every element $c \in A$ there are N elements of A' mapped onto c . Then apply Lemma 4 to this algebra A' . \square

Lemma 6. *Let A be a finite unary algebra such that one of the following two cases takes place:*

- (1) *A has a smallest subalgebra U ;*
- (2) *A has two disjoint subalgebras $U, \{a\}$ such that the partial algebra $A - \{a\}$ has a smallest subalgebra U .*

Then A is a homomorphic image of a finite subdirectly irreducible algebra.

Proof. In both cases put $A_0 = U$, so that A_0 has no proper subalgebras. By Lemma 5 there exist, for every positive integer N , a subdirectly irreducible algebra B_0 and a homomorphism H_0 of B_0 onto A_0 such that for every element $c \in A_0$ there are at least N distinct elements $c' \in B_0$ with $H_0(c') = c$ and $c' \neq f(d)$ for all $f \in \sigma$ and all $d \in B_0$.

Define subsets A_0, A_1, \dots of A by induction as follows: A_{i+1} is the set of all elements $b \in A$ not belonging to $A_0 \cup \dots \cup A_i$ such that $f(b) \in A_i$ for some $f \in \sigma$. Let K be the largest integer with A_K nonempty. We have $A = A_0 \cup \dots \cup A_K$ in case (1) and $A = A_0 \cup \dots \cup A_K \cup \{a\}$ in case (2). If $K = 0$, we can use Lemma 3; assume $K > 0$.

Put $B_K = A_K$ and let H_K be the identity on B_K . For every $i = K - 1, \dots, 1$ define a set B_i and a mapping H_i of B_i into A_i as follows: B_i is the disjoint union of A_i with the set of ordered pairs (b, f) such that $b \in B_{i+1}$, $f \in \sigma$ and $f(H_{i+1}(b)) \in A_i$; put $H_i(b, f) = f(H_{i+1}(b))$, and let the restriction of H_i to A_i be the identity. Take N so large that for every $f \in \sigma$ there exists an injective mapping G_f of B_1 into B_0 with the following property: if $b \in B_1$ and $f(H_1(b)) \in A_0$ then $H_0 G_f(b) = f(H_1(b))$ and $G_f(b)$ is not in the range of any fundamental operation of B_0 . Put $B^* =$

$B_K \cup \dots \cup B_1 \cup B_0$ and $H^* = H_K \cup \dots \cup H_1 \cup H_0$; in case (1) put $B = B^*$ and $H = H^*$, while in case (2) put $B = B^* \cup \{a\}$ and let H be the extension of H^* by $H(a) = a$. We are going to define operations on B . In case (2) put $f(a) = a$ for all $f \in \sigma$. Let $f \in \sigma$ and $b \in B_i$. If $i = 0$, define $f(b)$ in B in the same way as in B_0 (so that B_0 is a subalgebra of B). If $i \geq 2$ and $f(H(b)) \in A_{i-1}$, put $f(b) = (b, f)$ (this is an element of B_{i-1}). If $i \geq 2$ and $f(H(b)) \notin A_{i-1}$ (so that either $f(H(b)) \in A_j$ for some $j \geq i$ or (2) takes place and $f(H(b)) = a$), put $f(b) = f(H(b))$. If $i = 1$ and $f(H(b)) \in A_0$, put $f(b) = G_f(b)$. If $i = 1$ and $f(H(b)) \notin A_0$, put $f(b) = f(H(b))$.

One can easily check that H is a homomorphism of B onto A . We are going to prove that the union R of the monolith of B_0 with id_B is the monolith of B . Let b, c be two distinct elements of B and let \sim be a congruence with $b \sim c$. It is sufficient to prove that there exist two distinct \sim -related elements in B_0 .

Consider first the case $c = a$. We have $b \sim a$, so that $f(b) \sim f(a) = a$ for all $f \in \sigma$. Since $g_k \dots g_1(b) \in B_0$ for some $g_1, \dots, g_k \in \sigma$, we get $d \sim a$ for some $d \in B_0$. Then also $f(d) \sim a$ for all $f \in \sigma$, so that $d \sim f(d)$ where both d and $f(d)$ belong to B_0 ; for at least one f , $d \neq f(d)$.

It remains to consider the case when $b \in B_i$ and $c \in B_j$ for some i, j . Without loss of generality, $i \leq j$. If $i = j = 0$, we are done.

If $i = 0$ and $j = 1$ then there exists an $f \in \sigma$ with $f(c) \in B_0$; we have $f(b) \sim f(c)$ and these two elements are distinct, since $f(c)$ is not in the range of f restricted to B_0 .

If $i = 0$ and $j \geq 2$ then there exists an $f \in \sigma$ with $f(c) \in B_{j-1}$, so j can be reduced by 1 and (after several such steps) this takes us in the previous case.

If $i = j = 1$ then there are $f, g \in \sigma$ with $f(b) \in B_0$ and $g(c) \in B_0$; if $f(c) \in B_0$, take $g = f$. If $f = g$ then applying f to $b \sim c$ yields a pair of \sim -related elements in B_0 ; these two elements are distinct, since G_f is injective. If $f \neq g$ then applying f to $b \sim c$ yields a pair of \sim -related elements, one of which belongs to B_0 while the other does not and the previous case applies.

If $i = 1$ and $j \geq 2$, then an application of a suitable f to $b \sim c$ yields a pair of \sim -related elements, one of which belongs to B_0 while the other does not.

Finally, if $i, j \geq 2$ then take an $f \in \sigma$ with $f(b) \in B_{i-1}$. We have $f(b) \sim f(c)$ where $f(b) \in B_{i-1}$, $f(c) \in B_k$ for some $k \geq 1$ (or $f(c) = a$) and $f(b) \neq f(c)$. \square

Theorem 7. *The following assertions are equivalent for a finite unary algebra A :*

- (1) A is a homomorphic image of a subdirectly irreducible algebra;
- (2) A is a homomorphic image of a finite subdirectly irreducible algebra;
- (3) Either A has a smallest subalgebra or A has two disjoint subalgebras $U, \{a\}$ such that the partial algebra $A - \{a\}$ has a smallest subalgebra U .

- (4) A has a nonempty intersection of the set of all subalgebras B of A such that $|B| \geq 2$.

Proof. (3) implies (2) by Lemma 6, (2) implies (1) clearly. Next we prove that (1) implies (4). Let h be a homomorphism of a subdirectly irreducible algebra B onto A . Suppose that A has a subalgebra U , $|U| \geq 2$. Then $U' = h^{-1}(U)$ is a subalgebra of B , $|U'| \geq 2$ and $U'^2 \cup \text{id}_B$ is a nontrivial congruence of B , denoted by $\Theta_{U'}$. If all such subalgebras U of A with at least two elements had empty intersection, then the corresponding inverse images U' would have empty intersection. However, their corresponding congruences $\Theta_{U'}$ would all be nontrivial, and would have trivial intersection, which contradicts the assumption that B is subdirectly irreducible.

Assume now (4) and denote by V the intersection of all subalgebras of A with at least two elements. Assume that the intersection of all subalgebras of A is empty. It means that the intersection of V and all one-element subalgebras of A is empty. Since A cannot have three pairwise disjoint subalgebras (by (4)), it has at most two one-element subalgebras, at most one in V and exactly one outside of V —denote it by $\{a\}$. Hence the intersection of all subalgebras of V is non-empty (it is either V itself or the one-element subalgebra of V) and so is that of $A - \{a\}$. \square

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