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# A GLOBAL DIFFERENTIABILITY RESULT FOR SOLUTIONS OF NONLINEAR ELLIPTIC PROBLEMS WITH CONTROLLED GROWTHS

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*Abstract.* Let $\Omega$ be a bounded open subset of $\mathbb{R}^n$, $n > 2$. In $\Omega$ we deduce the global differentiability result

$$u \in H^2(\Omega, \mathbb{R}^N)$$

for the solutions $u \in H^1(\Omega, \mathbb{R}^n)$ of the Dirichlet problem

$$u - g \in H^1_0(\Omega, \mathbb{R}^N),$$
$$-\sum_i D_i a^i(x, u, Du) = B_0(x, u, Du)$$

with controlled growth and nonlinearity $q = 2$.

The result was obtained by first extending the interior differentiability result near the boundary and then proving the global differentiability result making use of a covering procedure.

*Keywords*: global differentiability of weak solutions, elliptic problems, controlled growth, nonlinearity with $q = 2$

*MSC 2000*: 35J60, 35D10, 58B10

## 1. Introduction

Let $\Omega$ be a bounded open set in $\mathbb{R}^n$, $n > 2$,[1] for instance of class $C^2$ with points $x = (x_1, x_2, \ldots, x_n)$.

We denote by $d_\Omega$ the diameter of $\Omega$. $N$ is an integer $> 1$, $(\cdot|\cdot)_k$ and $\|\cdot\|_k$ are the scalar product and the norm in $\mathbb{R}^k$, respectively. We will drop the subscript $k$ when there is no fear of confusion.

[1] This argumentation is obviously modified if $n = 2$.

If $u\colon \Omega \to \mathbb{R}^N$, we set $Du = (D_1u, \dots, D_nu)$ where, as usual, $D_i = \partial/\partial x_i$; clearly $Du \in \mathbb{R}^{nN}$ and we denote by $p = (p^1, \dots, p^n)$, $p^j \in \mathbb{R}^N$, a typical vector of $\mathbb{R}^{nN}$. $H^k = H^{k,2}$ and $H_0^k = H_0^{k,2}$ are the usual Sobolev spaces ($k$ integer $\geqslant 0$).[2]

Let us consider the variational elliptic nonlinear system

$$-\sum_{i=1}^{n} D_i a^i(x, u, Du) = B_0(x, u, Du). \tag{1.1}$$

We suppose that

(1.2) $a^i(x, u, p)$, $\forall\, i = 1, 2, \dots, n$, are vectors of class $C^1(\overline{\Omega}, \mathbb{R}^N, \mathbb{R}^{nN})$ such that

$$a^i(x, u, 0) = 0 \quad \forall\, x \in \Omega,\ \ \forall u \in \mathbb{R}^N,\ \ \forall\, i = 1, 2, \dots, n,$$

$$\left\| \frac{\partial a^i(x, u, p)}{\partial u_k} \right\| + \left\| \frac{\partial a^i(x, u, p)}{\partial p_k^j} \right\| \leqslant M, \quad \forall\, (x, u, p) \in \Lambda = \Omega \times \mathbb{R}^N \times \mathbb{R}^{nN},$$
$$\forall\, i, j = 1, 2, \dots, n,\ \ \forall\, k = 1, 2, \dots, N$$

where $M$ is a suitable positive constant;

$$\|a^i(x, u, p)\| + \left\| \frac{\partial a^i(x, u, p)}{\partial x_s} \right\| \leqslant f(x) + c\Big\{ \|u\|^\alpha + \sum_j \|p^j\| \Big\} \ \forall\, i, s = 1, 2, \dots, n,$$
$$\forall\, (x, u, p) \in \Lambda \ \text{ with } \alpha \leqslant \frac{n}{n-2} \ \text{ and } \ f \in L^2(\Omega);$$

(1.3) there exist a positive constant $\nu$ such that

$$\sum_{i,j} \sum_{h,k} \frac{\partial a_h^i(x, u, p)}{\partial p_k^j} \xi_h^i \xi_k^j \geqslant \nu \|\xi\|^2 \quad \forall\, (x, u, p) \in \Lambda,\ \ \forall\, \xi \in \mathbb{R}^{nN};$$

(1.4) the vector $B^0(x, u, p)$ defined in $\Lambda$ is measurable in $x$, continuous in $(u, p)$ and satisfies the following condition $\forall\, (x, u, p) \in \Lambda$:

$$\|B^0(x, u, p)\| \leqslant f_0(x) + c(k) \Big\{ \|u\|^\alpha + \sum_j \|p^j\| \Big\}$$

where $f_0 \in L^2(\Omega)$, $\alpha \leqslant n/(n-2)$ and $c$ is a positive constant.

Condition (1.2) is considered only for the sake of simplicity. In fact if $a^i(x, u, 0)$ is different from zero, we can consider the operator $\overline{a}^i = a^i(x, u, p) - a^i(x, u, 0)$ instead of $a^i(x, u, p)$ and we add the term $a^i(x, u, 0)$ to $B_0(x, u, Du)$.

[2] $H^0(\Omega) = L^2(\Omega)$.

From (1.2) it easily follows that $\forall\,(x,u,p)\in\Lambda$, $\forall\,i=1,2,\dots,n$ we have

$$\|a^i(x,u,p)\|\leqslant M\|p\|. \tag{1.5}$$

A solution of system (1.1) in $\Omega$ is a vector $u\in H^1(\Omega,\mathbb{R}^N)$ such that

$$\int_\Omega\sum_i\bigl(a^i(x,u,Du)\big|D_i\varphi\bigr)\,\mathrm{d}x=\int_\Omega(B^0|\varphi)\,\mathrm{d}x\quad\forall\,\varphi\in H^1_0(\Omega,\mathbb{R}^N). \tag{1.6}$$

In this paper at first we recall an interior differentiability result due to S. Campanato (see [2], Theorem 1.I, p. 167) and from it we immediately obtain an interior differentiability result; afterwards we prove a differentiability result near the boundary and then a global differentiability result.

S. Campanato in [4] investigated the problem of differentiability of the solution $u\in H^1(\Omega,\mathbb{R}^N)$ of the basic system

$$\sum_i D_ia^i(Du)=0$$

both in the interior and near the boundary, achieving results of the same type in both cases.

Moreover, the problem to achieve global differentiability results had been investigated by Campanato in [1] for solutions $u\in H^k(\Omega,\mathbb{R}^N)$ of linear elliptic systems. He proved that the solutions of the Dirichlet problem with zero boundary data belong to $H^{k+2}(\Omega,\mathbb{R}^N)$.

In this paper we investigate the problem of global differentiability of the solutions $u\in H^1(\Omega,\mathbb{R}^N)$ of the Dirichlet problem with nonzero boundary data for nonlinear elliptic systems with controlled growth and we prove

**Theorem 1.1.** *Let $u\in H^1(\Omega,\mathbb{R}^N)$ be a solution of the Dirichlet problem*

$$u-g\in H^1_0(\Omega,\mathbb{R}^N)\;-\sum_i D_ia^i(x,u,Du)=B^0(x,u,Du).$$

*If $\Omega$ is of class $C^2$ and $g\in H^2(\Omega,\mathbb{R}^N)$ and if $a^i$, $B_0$ satisfy conditions* (1.1)–(1.4) *then*

$$u\in H^2(\Omega,\mathbb{R}^N)$$

*and we have*

$$\int_\Omega\sum_{ij}\|D_{ij}u\|^2\,\mathrm{d}x \leqslant c(\nu,M)\int_\Omega\bigl[1+|f|^2+|f_0|^2+\|u\|^{2^*}+\|Du\|^2+\|Dg\|^2+\|D^2g\|^2\bigr]\,\mathrm{d}x.$$

## 2. Preliminaries, notation

We define

$$B(x^0,\sigma)=\{x\in\mathbb{R}^n:\ \|x-x^0\|<\sigma\};$$

moreover, if $x_n^0=0$,

$$\begin{aligned}B^+(x^0,\sigma)&=\{x\in B(x^0,\sigma):\ x_n>0\},\\ \Gamma(x^0,\sigma)&=\{x\in B(x^0,\sigma):\ x_n=0\}.\end{aligned}$$

We will simply write $B^+(\sigma)$, $\Gamma(\sigma)$ and $\Gamma$ instead of $B^+(0,\sigma)$, $\Gamma(0,\sigma)$ and $\Gamma(0,1)$ respectively.

If $u\in H^k(\Omega,\mathbb{R}^N)$ ($k$ integer $\geqslant 0$), we define[3]

$$\begin{aligned}|u|^2_{k,\Omega}&=\int_\Omega\sum_{|\alpha|=k}\|D^\alpha u\|^2\,\mathrm{d}x,\\ \|u\|_{k,\Omega}&=\left\{\sum_{h=0}^k|u|^2_{h,\Omega}\right\}^{1/2}.\end{aligned}$$

$H^\vartheta(\Omega,\mathbb{R}^N)$, $0<\vartheta<1$ is the space of those vectors $u\in L^2(\Omega,\mathbb{R}^N)$ such that

$$|u|_{\vartheta,\Omega}=\int_\Omega\mathrm{d}x\int_\Omega\frac{\|u(x)-u(y)\|^2}{\|x-y\|^{n+2\vartheta}}\,\mathrm{d}y<+\infty$$

and $H^{1+\vartheta}(\Omega,\mathbb{R}^N)$, $0<\vartheta<1$ is the space of those vectors $u\in L^1(\Omega,\mathbb{R}^N)$ such that

$$D_iu\in H^\vartheta(\Omega,\mathbb{R}^N),\quad i=1,2,\ldots,n.$$

Let us consider $t\in(0,1)$, $h\in\mathbb{R}$ such that $|h|<(1-t)\sigma$ and $x\in B(t\sigma)\subset\Omega$. If $u\colon B(\sigma)\to\mathbb{R}^N$ we define

$$\tau_{s,h}u(x)=u(x+he^s)-u(x),\quad s=1,2,\ldots,n$$

where $\{e^s\}_{s=1,2,\ldots,n}$ is the standard basis of $\mathbb{R}^n$.

Now we recall some well known lemmas that we will utilize in what follows.

[3] $|u|_{0,\Omega}=\|u\|_{0,\Omega}=\left\{\int_\Omega\|u\|^2\,\mathrm{d}x\right\}^{1/2}$.

**Lemma 2.1.** *Let $u \in L^2(B^+(R); \mathbb{R}^N)$, such that*

$$\left| \int_{B^+(R)} (u \big| D_n \varphi) \, \mathrm{d}x \right| \leqslant \mathscr{M} |\varphi|_{0,B^+(R)} \quad \forall \varphi \in C_0^\infty(B^+(R), \mathbb{R}^N)$$

*then there exists the weak derivative $D_n u \in L^2(B^+(R); \mathbb{R}^N)$ and there holds*

$$|D_n u|_{0,B^+(R)} \leqslant \mathscr{M}.$$

Proof. Since $C_0^\infty(B^+(R); \mathbb{R}^N)$ is dense in $L^2(B^+(R); \mathbb{R}^N)$ there exists a unique $F \in L^2(B^+(R); \mathbb{R}^N)^*$, such that

$$\langle F, \varphi \rangle = - \int_{B^+(R)} (u \big| D_n \varphi) \, \mathrm{d}x \quad \forall \varphi \in C_0^\infty(B^+(R), \mathbb{R}^N).$$

By Riesz's representation theorem one gets a unique $v \in L^2(B^+(R); \mathbb{R}^N)$ with

$$\langle F, \varphi \rangle = (v, \varphi)_{L^2(B^+(R); \mathbb{R}^N)} \quad \forall \varphi \in L^2(B^+(R); \mathbb{R}^N).$$

In particular,

$$\int_{B^+(R)} (v \big| \varphi) \, \mathrm{d}x = - \int_{B^+(R)} (u \big| D_n \varphi) \, \mathrm{d}x \quad \forall \varphi \in C_0^\infty(B^+(R), \mathbb{R}^N).$$

By the definition of weak derivative we have $v = D_n u$ and the assertion follows from

$$|D_n u|_{0,B^+(R)} = \|F\|_{L^2(B(R))^*} \leqslant \mathscr{M}.$$

**Lemma 2.2.** *If $u \in H^1(B(\sigma), \mathbb{R}^N)$, $\sigma > 0$, then $\forall t \in (0,1)$ and $\forall h$ such that $|h| < (1-t)\sigma$ we have*

$$\|\tau_{s,h} u\|_{0,B(t\sigma)} \leqslant |h| \|D_s u\|_{0,B(\sigma)}, \quad s = 1, 2, \ldots, n.$$

See, for instance, [2], Chap. 1, Lemma 3.VI.

**Lemma 2.3.** *If $u \in L^2(B(\sigma), \mathbb{R}^N)$ and there exists $M > 0$ such that*

$$\|\tau_{s,h} u\|_{0,B(t\sigma)} \leqslant |h| M \quad \forall |h| < (1-t)\sigma, \ \forall t \in (0,1), \ s = 1, 2, \ldots n$$

*then $u \in H^1(B(t\sigma), \mathbb{R}^N)$ and*

$$\|D_s u\|_{0,B(t\sigma)} \leqslant M, \quad s = 1, 2, \ldots, n.$$

See [2], Chap. I, Theorem 3.X.

**Lemma 2.4.** *If $u \in H^1(\Omega, \mathbb{R}^N)$ is a solution of the system* (1.1) *and if conditions* (1.2)–(1.4) *hold, then $u \in H^2_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$ and for all $B(\varrho) \subset B(\sigma) \subset \Omega$ we have*

$$|u|^2_{2,B(\varrho)} \leqslant \frac{c(\nu, M)}{(\sigma-\varrho)^2}\left\{\int_{B(\sigma)}\left[1+|f|^2+|f_0|^2+\|u\|^{2^*}+\sum_i\|D_iu\|^2\right]\mathrm{d}x\right\}.$$

For this lemma see for instance, [2], Theorem 1.1, Chap. V.

From Lemma 2.4 we immediately deduce the following interior differentiability result:

**Theorem 2.1.** *Let $u \in H^1(\Omega, \mathbb{R}^N)$ be a solution of the system* (1.1) *where we suppose that conditions* (1.2)–(1.4) *are verified. Then for every open set $\Omega^* \subset\subset \Omega$ we have*

$$|u|^2_{2,\Omega^*} \leqslant c\int_\Omega\left[1+|f|^2+|f_0|^2+\|u\|^{2^*}+\sum_i\|D_iu\|^2\right]\mathrm{d}x \tag{2.1}$$

*where the constant $c$ depends also on $d = \operatorname{dist}(\overline{\Omega}^*, \partial\Omega)$.*

If we extend this theorem to the solution of the system

$$-\sum_{i=1}^n D_ia^i(x, u+g, Du+Dg) = B_0(x, u+g, Du+Dg) \tag{2.2}$$

with $g \in H^2(\Omega, \mathbb{R}^N)$, we obtain immediately

**Theorem 2.2.** *Let $u \in H^1(\Omega, \mathbb{R}^N)$ be a solution of the system* (2.2) *with $g \in H^2(\Omega, \mathbb{R}^N)$, where we suppose that conditions* (1.2)–(1.4) *hold. Then for every open set $\Omega^* \subset\subset \Omega$ we have*

$$|u|^2_{2,\Omega^*} \leqslant c\int_\Omega\left[1+|f|^2+|f_0|^2+\|u\|^{2^*}+\|g\|^{2^*}+\|Du\|^2+\|Dg\|^2+\|D^2g\|^2\right]\mathrm{d}x \tag{2.3}$$

*where the constant $c$ depends also on $d = \operatorname{dist}(\overline{\Omega}^*, \partial\Omega)$.*

## 3. Differentiability near the boundary

In the hemisphere $B^+(1)$ let us consider the problem

$$
(3.1)\qquad \begin{cases} u \in H^1(B^+(1), \mathbb{R}^N), \\ u = 0 \ \text{ on } \ \Gamma, \\ -\sum\limits_{i=1}^{n} D_i a^i(x,u,Du) = B^0(x,u,Du). \end{cases}
$$

The last equality means that

$$
(3.2)\qquad \int_{B^+(1)} \sum_i \big(a^i(x,u,Du)\big|D_i\varphi\big)\,\mathrm{d}x = \int_{B^+(1)} \big(B^0(x,u,Du)\big|\varphi\big)\,\mathrm{d}x
$$

for all $\varphi \in H_0^1(B^+(1), \mathbb{R}^N)$.

Then we want to prove the following differentiability theorem.

**Theorem 3.1.** *If $u \in H^1(B^+(1), \mathbb{R}^N)$ is a solution of the problem* (3.1), *under the conditions* (1.2)–(1.4), *then for every $\sigma < 1$ we have*

$$
(3.3)\qquad u \in H^2(B^+(\sigma), \mathbb{R}^N)
$$

*and*

$$
(3.4)\qquad |Du|_{1,B^+(\sigma)} \leqslant \frac{c(\nu,M)}{(1-\sigma)} \Big\{ \int_{B^+(1)} \big[1 + |f|^2 + |f_0|^2 + \|u\|^{2^*} + \|Du\|^2\big]\,\mathrm{d}x \Big\}^{1/2}.
$$

P r o o f. The proof will be divided into two steps. First let us suppose that

$$
r = 1, 2, \ldots, n-1.
$$

Let us choose

$$
0 < \sigma < 1, \quad 0 < \varrho < 1 - \sigma
$$

and a function $\vartheta \in C_0^\infty(\mathbb{R}^n)$ satisfying

$$
0 \leqslant \vartheta \leqslant 1, \quad \vartheta = 1 \ \text{ in } \ B(\sigma), \quad \vartheta = 0 \ \text{ in } \ \mathbb{R}^n \setminus B(\varrho), \quad |D_i\vartheta| \leqslant \frac{c}{1-\sigma}.
$$

Then, taking into account the fact that $u = 0$ on $\Gamma$, in (3.2) we can use as a test function

$$
\varphi = \tau_{r,-\varrho}(\vartheta^2 \tau_{r,\varrho} u), \quad r = 1, 2, \ldots, n-1
$$

and we obtain

$$\int_{B^+(1)} \sum_i (\tau_{r,\varrho} a^i(x,u,Du) | D_i(\vartheta^2 \tau_{r,\varrho} u))\, dx = \int_{B^+(1)} (B^0 | \tau_{r,-\varrho}(\vartheta^2 \tau_{r,\varrho} u))\, dx, \qquad r = 1,2,\dots,n-1. \tag{3.5}$$

Let us observe that

$$\tau_{r,\varrho} a^i(x,u,Du) = \int_0^1 \frac{d}{dt} a^i(x + t\varrho e^r, u(x) + t\tau_{r,\varrho} u(x), Du(x) + t\tau_{r,\varrho} Du(x))\, dt, \tag{3.6}$$

if we set

$$A_r^i(x) = \int_0^1 \frac{\partial a^i(x + t\varrho e^r, u(x) + t\tau_{r,\varrho} u(x), Du(x) + t\tau_{r,\varrho} Du(x))}{\partial x^r}\, dt, \qquad r = 1,2,\dots,n-1; \tag{3.7}$$

$$B_{hk}^i(x) = \int_0^1 \frac{\partial a_h^i}{\partial u_k}(x + t\varrho e^r, u(x) + t\tau_{r,\varrho} u(x), Du(x) + t\tau_{r,\varrho} Du(x))\, dt, \qquad B^i = \{B_{hk}^i\}, \quad h,k = 1,2,\dots,N, \tag{3.8}$$

$$C_{ij}^{hk}(x) = \int_0^1 \frac{\partial a_h^i}{\partial p_k^j}(x + t\varrho e^r, u(x) + t\tau_{r,\varrho} u(x), Du(x) + t\,\tau_{r,\varrho} Du(x))\, dt, \qquad C_{ij} = \{C_{ij}^{hk}\}, \quad h,k = 1,2,\dots,N, \tag{3.9}$$

we have that

$$\tau_{r,\varrho} a^i(x,u,Du) = A_r^i \varrho + B^i \tau_{r,\varrho} u + \sum_{j=1}^n C_{ij} \tau_{r,\varrho} D_j u. \tag{3.10}$$

Then from (3.5) we obtain

$$\begin{aligned} A &= \int_{B^+(1)} \vartheta^2 \sum_{ij} (C_{ij} \tau_{r,\varrho} D_j u | \tau_{r,\varrho} D_i u)\, dx \\ &= -2 \int_{B^+(1)} \sum_{ij} (C_{ij} \tau_{r,\varrho} D_j u | \tau_{r,\varrho} u) \vartheta D_i \vartheta\, dx \\ &\quad - \int_{B^+(1)} \sum_i (B^i \tau_{r,\varrho} u | D_i(\vartheta^2 \tau_{r,\varrho} u))\, dx \\ &\quad - \varrho \int_{B^+(1)} \sum_i (A_r^i | D_i(\vartheta^2 \tau_{r,\varrho} u)\, dx \\ &\quad + \int_{B^+(1)} (B^0(x,u,D,u) | \tau_{r,-\varrho}(\vartheta^2 \tau_{r,\varrho} u))\, dx = B + C + D + E. \end{aligned} \tag{3.11}$$

From (1.2), (3.7), (3.8), (3.9) we deduce

$$\|B_i\| + \|C_{ij}\| \leqslant M, \tag{3.12}$$
$$\|A^i_r\| \leqslant \widetilde{f}(x) + c\{\|u\|^\alpha + \|\tau_{r,\varrho} u\|^\alpha + \|Du\| + \|\tau_{r,\varrho} Du\|\}$$

where $\widetilde{f}(x) = \int_0^1 f(x + t\varrho \mathrm{e}^r)\,\mathrm{d}t$ and then, from (1.3) we get

$$A \geqslant \nu \int_{B^+(1)} \vartheta^2 \|\tau_{r,\varrho} Du\|^2\,\mathrm{d}x. \tag{3.13}$$

In virtue of (3.12) we obtain

$$\begin{aligned} |B| &\leqslant \frac{c(M)}{(1-\sigma)} \int_{B^+(1)} \|\tau_{r,\varrho} u\| \vartheta \|\tau_{r,\varrho} Du\|\,\mathrm{d}x \\ &\leqslant \varepsilon \int_{B^+(1)} \vartheta^2 \|\tau_{r,\varrho} Du\|^2\,\mathrm{d}x + \frac{c(M,\varepsilon)}{(1-\sigma)^2} \int_{B^+(\varrho)} \|\tau_{r,\varrho} u\|^2\,\mathrm{d}x \end{aligned} \tag{3.14}$$

and

$$\begin{aligned} |C| &\leqslant c(M) \int_{B^+(1)} \|\tau_{r,\varrho} u\| \Big\{ \vartheta^2 \|\tau_{r,\varrho} D\,u\| + \frac{\vartheta}{1-\sigma} \|\tau_{r,\varrho} u\| \Big\}\,\mathrm{d}x \\ &\leqslant \varepsilon \int_{B^+(1)} \vartheta^2 \|\tau_{r,\varrho} Du\|^2\,\mathrm{d}x + c(M,\varepsilon) \int_{B^+(\varrho)} \|\tau_{r,\varrho} u\|^2\,\mathrm{d}x \\ &\quad + \frac{c(M)}{(1-\sigma)^2} \int_{B^+(\varrho)} \|\tau_{r,\varrho} u\|^2\,\mathrm{d}x. \end{aligned} \tag{3.15}$$

From (3.12) we obtain

$$\begin{aligned} |D| &\leqslant c \sum_i \int_{B^+(1)} \varrho \|A^i_r\| \Big\{ \vartheta^2 \|\tau_{r,\varrho} Du\| + \frac{\vartheta}{(1-\sigma)} \|\tau_{r,\varrho} u\| \Big\}\,\mathrm{d}x \\ &\leqslant \varepsilon \int_{B^+(1)} \vartheta^2 \|\tau_{r,\varrho} Du\|^2\,\mathrm{d}x + \frac{c(M,\varepsilon)}{(1-\sigma)^2} \varrho^2 \\ &\quad \times \int_{B^+(\varrho)} \{\|\widetilde{f}\|^2 + \|u\|^{2\alpha} + \|\tau_{r,\varrho} u\|^{2\alpha} + \|Du\|^2 + \|\tau_{r,\varrho} Du\|^2\}\,\mathrm{d}x \\ &\quad + c(\varepsilon) \int_{B^+(\varrho)} \|\tau_{r,\varrho} u\|^2\,\mathrm{d}x. \end{aligned} \tag{3.16}$$

Taking into account that

$$\int_{B^+(\varrho)} |\widetilde{f}(x)|^2 \,\mathrm{d}x = \int_0^1 \mathrm{d}t \int_{B^+(\varrho)} |f(x+t\varrho \mathrm{e}^r)|^2 \,\mathrm{d}x \leqslant c \int_{B^+(1)} |f|^2 \,\mathrm{d}x,$$

$$\int_{B^+(\varrho)} \|u\|^{2\alpha} \,\mathrm{d}x \leqslant \int_{B^+(1)} \{1+\|u\|^{2^*}\} \,\mathrm{d}x,$$

$$\int_{B^+(\varrho)} \|\tau_{r,\varrho} u\|^2 \,\mathrm{d}x \leqslant \varrho^2 \int_{B^+(1)} \|Du\|^2 \,\mathrm{d}x \quad \text{by Lemma 2.2,}$$

$$\int_{B^+(\varrho)} \|\tau_{r,\varrho} u\|^{2\alpha} \,\mathrm{d}x = \int_{B^+(\varrho)} \|u(x+\varrho \mathrm{e}^r) - u(x)\|^{2\alpha} \,\mathrm{d}x \leqslant c \int_{B^+(1)} \{1+\|u\|^{2^*}\} \,\mathrm{d}x,$$

$$\int_{B^+(\varrho)} \|\tau_{r,\varrho} Du\|^2 \,\mathrm{d}x = \int_{B^+(\varrho)} \|Du(x+\varrho \mathrm{e}^r) - Du(x)\|^2 \,\mathrm{d}x \leqslant c \int_{B^+(1)} \|Du\|^2 \,\mathrm{d}x$$

we conclude from (3.14), (3.15) and (3.16)

$$|B| \leqslant \varepsilon \int_{B^+(1)} \vartheta^2 \|\tau_{r,\varrho} Du\|^2 \,\mathrm{d}x + \frac{c(M,\varepsilon)}{(1-\sigma)^2} \varrho^2 \int_{B^+(1)} \|Du\|^2 \,\mathrm{d}x, \tag{3.17}$$

$$|C| \leqslant \varepsilon \int_{B^+(1)} \vartheta^2 \|\tau_{r,\varrho} Du\|^2 \,\mathrm{d}x + \frac{c(M,\varepsilon)}{(1-\sigma)^2} \varrho^2 \int_{B^+(1)} \|Du\|^2 \,\mathrm{d}x, \tag{3.18}$$

$$\begin{aligned} |D| &\leqslant \varepsilon \int_{B^+(1)} \vartheta^2 \|\tau_{r,\varrho} Du\|^2 \,\mathrm{d}x + \frac{c(M,\varepsilon)}{(1-\sigma)^2} \varrho^2 \\ &\quad \times \int_{B^+(1)} \{1+|f|^2+\|u\|^{2^*}+\|Du\|^2\} \,\mathrm{d}x. \end{aligned} \tag{3.19}$$

Moreover, keeping in mind (1.4) and Lemma 2.2 we obtain

$$\begin{aligned} |E| &\leqslant \left\{ \int_{B^+(1)} \|B^0(x,u,Du)\|^2 \,\mathrm{d}x \right\}^{1/2} |\vartheta^2 \varrho \tau_{r,\varrho} u|_{1,B^+(1)} \\ &\leqslant \left\{ \varrho \int_{B^+(1)} \vartheta \{|f_0|^2 + \|u\|^{2\alpha} + \|Du\|^2\} \,\mathrm{d}x \right\}^{1/2} \\ &\quad \times \int_{B^+(1)} \left[ \vartheta^2 \|\tau_{r,\varrho} Du\|^2 + \frac{c}{(1-\sigma)^2} \vartheta \|\tau_{r,\varrho} u\|^2 \,\mathrm{d}x \right]^{1/2} \\ &\leqslant \varepsilon \int_{B^+(1)} \vartheta^2 \|\tau_{r,\varrho} Du\|^2 \,\mathrm{d}x + \frac{c(\varepsilon)\varrho^2}{(1-\sigma)^2} \\ &\quad \times \int_{B^+(\varrho)} \{|f_0|^2 + \|u\|^{2\alpha} + \|Du\|^2\} \,\mathrm{d}x \\ &\leqslant \varepsilon \int_{B^+(1)} \vartheta^2 \|\tau_{r,\varrho} Du\|^2 \,\mathrm{d}x + \frac{c(\varepsilon)\varrho^2}{(1-\sigma)^2} \\ &\quad \times \int_{B^+(1)} \{1+|f_0|^2 + \|u\|^{2^*} + \|Du\|^2\} \,\mathrm{d}x. \end{aligned} \tag{3.20}$$

From (3.11), (3.13), (3,17)–(3.20) with $\varepsilon = \frac{1}{8}\nu$ we have

$$\int_{B^+(1)} \vartheta^2 \|\tau_{r,\varrho} Du\|^2 \,\mathrm{d}x \leqslant \frac{c(\nu, M)}{(1-\sigma)^2} \varrho^2 \int_{B^+(1)} \{1 + |f|^2 + |f_0|^2 + \|u\|^{2^*} + \|Du\|^2\} \,\mathrm{d}x$$

and

$$\int_{B^+(\sigma)} \|\tau_{r,\varrho} Du\|^2 \,\mathrm{d}x \leqslant \frac{c(\nu, M)}{(1-\sigma)^2} \varrho^2 \int_{B^+(1)} \{1 + |f|^2 + |f_0|^2 + \|u\|^{2^*} + \|Du\|^2\} \,\mathrm{d}x$$

and then, from Lemma 2.3 we conclude that there exists $D_r Du \in L^2(B^+(\sigma), \mathbb{R}^N)$, $r = 1, 2, \ldots, n-1$, and

$$\begin{aligned} \text{(3.21)} \qquad \sum_{r=1}^{n-1} \int_{B^+(\sigma)} \|D_r Du\|^2 \,\mathrm{d}x & \\ \leqslant \frac{c(\nu, M)}{(1-\sigma)^2} \int_{B^+(1)} \{1 + |f|^2 + |f_0|^2 + \|u\|^{2^*} + \|Du\|^2\} \,\mathrm{d}x. & \end{aligned}$$

In the case $r = n$ we argue as follows.

Fix $0 < \sigma < R < 1$ and $0 < \varrho < \frac{1}{2}(1 - R)$. We want to estimate the integral

$$\text{(3.22)} \qquad \int_{B^+(R)} (D_n u | D_n \varphi) \,\mathrm{d}x, \quad \varphi \in C_0^\infty(B^+(R), \mathbb{R}^N).$$

We observe that $\forall\, x \in B^+(R) + \varrho \mathrm{e}^n$

$$\begin{aligned} \text{(3.23)} \qquad \tau_{n,-\varrho} a^n(x, u, Du) = C_{n,n}[\tau_{n,-\varrho} D_n u] + \sum_{j=1}^{n-1} C_{nj}[\tau_{n,-\varrho} D_j u] & \\ + B^n[\tau_{n,-\varrho} u] + (-\varrho) A_n^n & \end{aligned}$$

where $C_{ij}$, $B^n$ and $A_n^n$ are defined as in (3.7), (3.8), (3.9), $\varrho$ being replaced by $(-\varrho)$. Now $C_{nn}$ is a nonsingular matrix; in fact if $\xi \equiv (0, 0, \ldots, \xi_n)$ we deduce from (1.2)

$$(C_{nn}(x)\xi^n | \xi^n) \geqslant \nu \|\xi^n\|^2 \quad \forall\, \xi^n \in \mathbb{R}^N, \ \forall\, x \in B^+(1)$$

so that[4]

$$\det C_{nn} \neq 0 \quad \text{and} \quad \|C_{nn}^{-1}(x)\| \leqslant \frac{\sqrt{N}}{\nu} \quad \forall\, x \in B^+(1).$$

[4] If $A = \{A^{hk}\}$ then $\|A\| = \Big\{\sum_{hk} |A^{hk}|^2\Big\}^{1/2}$.

In conclusion, from (3.23) we get

$$\tau_{n,-\varrho} D_n u = C_{nn}^{-1}[\tau_{n,-\varrho} a^n(x,u,Du) + G(Du) + F(u) + \varrho A_n^n] \tag{3.24}$$

where $G(Du) = -\sum_{j=1}^{n-1} C_{nj}[\tau_{n,-\varrho} D_j u]$ and $F(u) = -B^n[\tau_{n,-\varrho} u]$.

On the other hand, from (3.2) and by (1.2), (1.4), (3.21) it follows that $D_n a^n(x,u, Du)$ exists and belongs to $L^2(B^+(R))$, $\forall\, R < 1$; moreover, taking into account that

$$\begin{aligned} D_n a^n(x,u,Du) = & -\sum_{i=1}^{n-1} D_i a^i(x,u,Du) + B_0(x,u,Du) \\ = & -\sum_{i=1}^{n-1} \frac{\partial a^i(x,u,Du)}{\partial x_i} - \sum_{i=1}^{n-1}\sum_{j=1}^{n} \frac{\partial a^i(x,u,Du)}{\partial u_j} D_i u^j \\ & -\sum_{i=1}^{n-1}\sum_{j=1}^{n} \frac{\partial a^i(x,u,Du)}{\partial p^j} D_{ij} u + B_0(x,u,Du) \end{aligned}$$

we have

$$\begin{aligned} \int_{B^+(R)} \|D_n a^n(x,u,Du)\|^2 \,\mathrm{d}x \\ \leqslant \frac{c(\nu, M)}{(1-R)^2} \int_{B^+(1)} \left[1 + |f|^2 + |f_0|^2 + \|u\|^{2^*} + \|Du\|^2\right] \mathrm{d}x. \end{aligned} \tag{3.25}$$

Finally, integral (3.22) can be estimated as follows.

If we choose $0 < \varrho < \operatorname{dist}(\operatorname{supp}(\varphi), \partial B^+(R))$. For every $\varphi \in C_0^\infty(B^+(R))$ we have that $\tau_{n,\varrho}\varphi \in C_0^\infty(B^+(R), \mathbb{R}^N)$ and then

$$\int_{B^+(R)} (D_n u \big| \tau_{n,\varrho}\varphi)\,\mathrm{d}x = \int_{B^+(R)} (\tau_{n,-\varrho} D_n u \big| \varphi)\,\mathrm{d}x.$$

Then, taking into account (3.21), (3.23), (3.24), (3.25), if $\varrho$ is small enough, we get[5]

$$\begin{aligned} &\left| \int_{B^+(R)} (D_n u \big| \tau_{n,\varrho}\varphi)\,\mathrm{d}x \right| \\ &\quad = \left| \int_{B^+(R)} [(\tau_{n,-\varrho} a^n(x,u,Du) + G(Du) + F(u) + \varrho A_n^n) \big| (C_{nn}^{-1})^* \varphi]\,\mathrm{d}x \right| \end{aligned}$$

[5] $(C_{nn}^{-1})^*$ is the adjoint of the matrix $C_{nn}^{-1}$.

$$\leqslant c(\nu,M)|\varphi|_{0,B^+(R)}\Big\{\int_{B^+(R)}[\|\tau_{n,-\varrho}a^n(x,u,Du)\|^2 + \sum_{j=1}^{n-1}\|\tau_{n,-\varrho}D_ju\|^2+\|\tau_{n,-\varrho}u\|^2+\|A_n^n\varrho\|^2]\,\mathrm{d}x\Big\}^{1/2}$$

$$\leqslant c(\nu,M)|\varphi|_{0,B^+(R)}\varrho\Big\{\int_{B^+(1)}\|D_na^n(x,u,Du)\|^2 + \sum_{j=1}^{n-1}\|D_{nj}u\|^2+\|D_nu\|^2+\|A_n^n\|^2\,\mathrm{d}x\Big\}^{1/2}$$

$$\leqslant \frac{c(\nu,M)}{(1-R)}|\varphi|_{0,B^+(R)}\varrho\Big\{\int_{B^+(1)}[1+c|f|^2+|f_0|^2+\|u\|^{2^*}+\|Du\|^2]\,\mathrm{d}x\Big\}^{1/2}.$$

Consequently, by dividing all sides by $\varrho$ and taking the limit for $\varrho\to 0$ we obtain that for every $\varphi\in C_0^\infty(B^+(R),\mathbb{R}^N)$

$$\Big|\int_{B^+(R)}(D_nu\big|D_n\varphi)\,\mathrm{d}x\Big| \leqslant \frac{c(\nu,M)}{(1-R)}|\varphi|_{0,B^+(R)}\cdot\Big\{\int_{B^+(1)}[1+|f(x)|^2+|f_0(x)|^2+\|u\|^{2^*}+\|Du\|^2]\,\mathrm{d}x\Big\}^{1/2}. \tag{3.26}$$

Applying now Lemma 2.1, from (3.26) we obtain that $D_nu\in H^1(B^+(R),\mathbb{R}^N)$, and

$$\int_{B^+(\sigma)}\|D_{nn}u\|^2\,\mathrm{d}x \leqslant \frac{c(\nu,M)}{(1-R)^2}\int_{B^+(1)}[1+|f(x)|^2+|f_0(x)|^2+\|u\|^{2^*}+\|Du\|^2]\,\mathrm{d}x. \tag{3.27}$$

Theorem 3.1 follows from (3.21) and (3.27). □

Now let us consider for $g\in H^1(B^+(1),\mathbb{R}^N)$ the problem

$$u=0 \quad \text{on } \Gamma, \qquad -\sum_{i=1}^n D_ia^i(x,u+g,Du+Dg)=B^0(x,u+g,Du+Dg) \tag{3.28}$$

and let us assume that conditions (1.2)–(1.4) are satisfied with $\Omega$ replaced by $B^+(1)$. Then we prove the following result with a proof analogous to that of Theorem 3.1.

**Theorem 3.2.** *Let $u \in H^1(B^+(1), \mathbb{R}^N)$ be a solution of the problem* (3.28) *under the conditions* (1.2)–(1.4). *Let us assume that $g \in H^2(B^+(1), \mathbb{R}^N)$. Then for every $0 < R < 1$, $Du$ belongs to $H^1(B^+(R), \mathbb{R}^N)$ and*

$$\begin{aligned} &|Du|_{1,B^+(1)} \\ &\leqslant \frac{c(\nu, M)}{(1-R)} \left\{ \int_{B^+(1)} [1 + |f|^2 + |f_0|^2 + \|u\|^{2^*} + \|Du\|^2 + \|Dg\|^2 + \|D^2 g\|^2] \, \mathrm{d}x \right\}^{1/2}. \end{aligned}$$

## 4. A global differentiability result

Let $u \in H^1(\Omega, \mathbb{R}^N)$ be the solution of the Dirichlet problem

$$\begin{gathered} u - g \in H^1_0(\Omega, \mathbb{R}^N), \\ -\sum_i D_i a^i(x, u, Du) = B^0(x, u, Du) \quad \text{in } \Omega \end{gathered} \tag{4.1}$$

where $g \in H^2(\Omega, \mathbb{R}^N)$; the open set $\Omega$ is of class $C^2$, the vector mappings $a^i(x, u, p)$, $i = 1, \ldots, n$ belong to $C^1(\overline{\Omega} \times \mathbb{R}^N \times \mathbb{R}^{nN})$ and satisfy the conditions (1.2)–(1.3); the vector $B^0(x, u, p)$ defined in $\Lambda = \Omega \times \mathbb{R}^N \times \mathbb{R}^{nN}$ is measurable in $x$, continuous in $(u, p)$ and satisfies condition (1.4).

If we assume $w = u - g$, problem (4.1) can be written in the equivalent form

$$\begin{gathered} w \in H^1_0(\Omega, \mathbb{R}^N), \\ -\sum_i D_i a^i(x, w + g, Dw + Dg) = B^0(x, w + g, Dw + Dg) \quad \text{in } \Omega. \end{gathered} \tag{4.2}$$

As $\Omega$ is of class $C^2$, if $x^0 \in \partial\Omega$, about $x^0$ there exists an open neighborhood $\mathfrak{B}$ such that $\overline{\mathfrak{B}}$ is mapped, by a mapping $y = \mathfrak{I}(x)$ of class $C^2$ together with its inverse, onto the ball $\overline{B(0,1)}$ and, in particular, $\Omega \cap \mathfrak{B}$ is sent to $B^+(1)$ and $\partial\Omega \cap \mathfrak{B}$ to $\Gamma$.

Let us set

$$\begin{gathered} \frac{\partial \mathfrak{I}(x)}{\partial x} = \left\{ \frac{\partial \mathfrak{I}_i(x)}{\partial x_j} \right\}, \\ J(x) = \left| \det \frac{\partial \mathfrak{I}(x)}{\partial x} \right|; \end{gathered} \tag{4.3}$$

moreover, for all $y \in B(0,1)$, $u \in \mathbb{R}^N$ and $p \in \mathbb{R}^{nN}$ we define

$$
\begin{aligned}
\alpha_{ij}(y) &= \frac{\partial \mathfrak{J}_i}{\partial x_j}(\mathfrak{J}^{-1}(y)), \\
\beta_{ij}(y) &= \Big(\frac{\partial \mathfrak{J}_i}{\partial x_j}\frac{1}{J}\Big)(\mathfrak{J}^{-1}(y)), \\
q^j(y,p) &= \sum_{r=1}^{n} \alpha_{rj}(y) p^r, \\
q(y,p) &\equiv (q^1, \dots, q^n), \\
\mathfrak{A}^s(y,u,p) &= \sum_{i=1}^{n} \beta_{si}(y) a^i(\mathfrak{J}^{-1}(y), u, q(y,p)), \\
\overline{B}_0(y,u,p) &= B_0(\mathfrak{J}^{-1}(y), u, q(y,p)) \cdot \frac{1}{J(\mathfrak{J}^{-1}(y))}.
\end{aligned} \tag{4.4}
$$

Clearly $q^j$ is a vector of $\mathbb{R}^N$ defined in $B(0,1) \times \mathbb{R}^{nN}$; $\mathfrak{A}^s$ $(s = 1, \dots, n)$ and $\overline{B}_0$ are vectors of $\mathbb{R}^N$ defined in $B(0,1) \times \mathbb{R}^N \times \mathbb{R}^{nN}$; moreover, $\alpha_{ij}$, $\beta_{i,j}$ are functions of class $C^1(\overline{B(0,1)})$. We can easily prove that the vectors $\mathfrak{A}^h(y,u,p)$, $\overline{B}_0(y,u,p)$, by virtue of assumptions (1.2)–(1.4) verify the same conditions as $a^i(x,u,p)$, $B_0(x,u,p)$ in which constants and coefficients are multiplied by a suitable positive constant $c(\mathfrak{J}) = \overline{c}$ and $f$, $f_0$ are replaced by $F$, $F_0$.

If $y \in B^+(1)$ and $u$ is a vector function defined in $\mathfrak{B} \cap \Omega$, then we get

$$
\begin{aligned}
U(y) &= u(\mathfrak{J}^{-1}(y)) \quad \text{and so} \quad u(x) = U(\mathfrak{J}(x)), \\
W(y) &= u(\mathfrak{J}^{-1}(y)) \quad \text{and so} \quad w(x) = W(\mathfrak{J}(x)), \\
G(y) &= g(\mathfrak{J}^{-1}(y)) \quad \text{and so} \quad g(x) = G(\mathfrak{J}(x)), \\
\Phi(y) &= \varphi(\mathfrak{J}^{-1}(y)) \quad \text{and so} \quad \varphi(x) = \Phi(\mathfrak{J}(x)).
\end{aligned} \tag{4.5}
$$

Now, because from (4.2) we get, in particular,

$$
\begin{aligned}
&\int_{\Omega\cap\mathfrak{B}} \sum_i (a^i(x, w+g, Dw+Dg) \big| D_i\varphi)\,\mathrm{d}x \\
&\quad = \int_{\Omega\cap\mathfrak{B}} (B_0(x, w+g, Dw+Dg) \big| \varphi)\,\mathrm{d}x \quad \forall\, \varphi \in H_0^1(\Omega\cap\mathfrak{B}, \mathbb{R}^N)
\end{aligned} \tag{4.6}
$$

then making use of the transformation of coordinates $y = \mathfrak{J}(x)$ and taking into account that

$$
\begin{cases}
D_i w(x) = \sum_{h=1}^{n} D_h W(\mathfrak{J}(x)) \cdot D_i \mathfrak{J}_h(x), \\
D_i \varphi(x) = \sum_{h=1}^{n} D_h \Phi(\mathfrak{J}(x)) \cdot D_i \mathfrak{J}_h(x)
\end{cases} \tag{4.7}
$$

we have

$$
\begin{aligned}
(4.8)\quad \sum_{i=1}^{n}\int_{B^+(1)}\Big(a^i(\mathfrak{J}^{-1}(y)), W(y)+G(y), \sum_{j=1}^{n}\alpha_{j1}(y)(D_jW(y)+D_jG(y)),\dots,\\
\sum_{j=1}^{n}\alpha_{jn}(y)(D_jW(y)+D_jG(y))\Big|\sum_{h=1}^{n}\beta_{hi}(y)D_h\Phi(y)\Big)\,\mathrm{d}y\\
=\int_{B^+(1)}B_0\Big(\mathfrak{J}^{-1}(y), W(y)+G(y), \sum_{j=1}^{n}\alpha_{j1}(y)\big(D_jW(y)+D_jG(y)\big),\dots,\\
\sum_{j=1}^{n}\alpha_{jn}(y)(D_jW(y)+D_jG(y))\big|\Phi(y)\Big)\frac{1}{J(\mathfrak{J}^{-1}(y))}\,\mathrm{d}y.
\end{aligned}
$$

Then $W$ is a solution of the problem

$$
(4.9)\qquad
\begin{gathered}
W(y)\in H^1(B^+(1),\mathbb{R}^N),\\
W=0\quad\text{on }\Gamma,\\
-\sum_h D_h\mathfrak{A}^h(y,W+G,DW+DG)=\overline{B}_0(y,W+G,DW+DG)\quad\text{in }B^+(1).
\end{gathered}
$$

Since $\mathfrak{J}$ is of class $C^2$ and $g\in H^2(\Omega\cap\mathfrak{B},\mathbb{R}^N)$, $u\in H^1(\Omega\cap\mathfrak{B},\mathbb{R}^N)$, $f$ and $f_0\in L^2(\Omega\cap\mathfrak{B})$, also $G\in H^2(B^+(1),\mathbb{R}^N)$, $U\in H^1(B^+(1),\mathbb{R}^N)$, $F$ and $F_0\in L^2(B^+(1))$ (see [2], Theorem V, p. 375) and we get

$$
(4.10)\qquad
\begin{cases}
\|G\|^2_{H^k(B^+(1),\mathbb{R}^N)}\leqslant c(\mathfrak{J})\|g\|^2_{H^k(\Omega\cap\mathfrak{B},\mathbb{R}^N)}, \quad k=0,1,2,\\
\|U\|^2_{L^{2^*}(B^+(1),\mathbb{R}^N)}\leqslant c(\mathfrak{J})\|u\|^2_{L^{2^*}(\Omega\cap\mathfrak{B},\mathbb{R}^N)},\\
\|G\|^2_{L^{2^*}(B^+(1),\mathbb{R}^N)}\leqslant c(\mathfrak{J})\|g\|^2_{L^{2^*}(\Omega\cap\mathfrak{B},\mathbb{R}^N)},\\
\|U\|^2_{H^1(B^+(1),\mathbb{R}^N)}\leqslant c(\mathfrak{J})\|u\|^2_{H^1(\Omega\cap\mathfrak{B},\mathbb{R}^N)},\\
\|F\|^2_{L^2(B^+(1),\mathbb{R}^N)}\leqslant c(\mathfrak{J})\|f\|^2_{L^2(\Omega\cap\mathfrak{B})},\\
\|F_0\|^2_{L^2(B^+(1),\mathbb{R}^N)}\leqslant c(\mathfrak{J})\|f_0\|^2_{L^2(\Omega\cap\mathfrak{B})}.
\end{cases}
$$

Then from Theorem 3.2 and Sobolev's theorems we get for all $R\in(0,1)$

$$
\begin{aligned}
(4.11)\quad \|DW\|^2_{H^1(B^+(1),\mathbb{R}^N)}\leqslant\frac{c(\nu,M)}{(1-R)^2}\int_{B^+(1)}[1+|F|^2+|F_0|^2+\|W\|^{2^*}+\|G\|^{2^*}\\
+\|DW\|^2+\|DG\|^2+\|D^2G\|^2]\,\mathrm{d}y.
\end{aligned}
$$

Consequently, since $W=U-G$, we have

$$
\begin{aligned}
(4.12)\quad \|DU\|^2_{H^1(B^+(1),\mathbb{R}^N)}\leqslant\frac{c(\nu,M)}{(1-R)^2}\int_{B^+(1)}[1+|F|^2+|F_0|^2+\|U\|^{2^*}+\|G\|^{2^*}\\
+\|DU\|^2+\|DG\|^2+\|D^2G\|^2]\,\mathrm{d}y.
\end{aligned}
$$

If we denote by $\mathfrak{B}(R)$ the inverse image of $B(0,R)$, since the mapping $\mathfrak{J}$ of class $C^2$ preserves the properties ([1], Theorem V, p. 375), from (4.12) we have

$$\|Du\|^2_{H^1(\Omega\cap\mathfrak{B}(R),\mathbb{R}^N)} \leqslant \frac{c(\nu,M,\overline{c})}{(1-R)^2}\int_\Omega [1+\|f\|^2+\|f_0\|^2 + \|u\|^{2^*}+\|g\|^{2^*}+\|Du\|^2+\|Dg\|^2+\|D^2g\|^2]\,\mathrm{d}x. \tag{4.13}$$

Using this local differentiability result near the boundary together with Theorem 2.2, we can prove by the usual covering argument the global differentiability result which follows.

**Theorem 4.1.** *Let* $u\in H^1(\Omega,\mathbb{R}^N)$ *be the solution of the Dirichlet problem* (4.1) *and suppose that*

$$\Omega \text{ is of class } C^2, \tag{4.14}$$

$$g\in H^2(\Omega,\mathbb{R}^N) \tag{4.15}$$

*and* $a^i$, $B_0$ *satisfy conditions* (1.1)–(1.4). *Then*

$$u\in H^2(\Omega,\mathbb{R}^N) \tag{4.16}$$

*and we have*

$$\int_\Omega\sum_{ij}\|D_{ij}u\|^2\,\mathrm{d}x \leqslant c(\nu,M,\overline{c})\int_\Omega [1+|f|^2+|f_0|^2+\|u\|^{2^*}+\|Du\|^2 + \|Dg\|^2+\|D^2g\|^2]\,\mathrm{d}x. \tag{4.17}$$

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