Czechoslovak Mathematical Journal

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Czechoslovak Mathematical Journal, Vol. 58 (2008), No. 2, 469–479

Persistent URL: http://dml.cz/dmlcz/128270

Czechoslovak Mathematical Journal, 58 (133) (2008), 469–479

CARDINALITY OF RETRACTS OF MONOUNARY ALGEBRAS

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(Received March 29, 2006)

Abstract. For an uncountable monounary algebra (A, f) with cardinality κ it is proved that (A, f) has exactly 2^κ retracts. The case when (A, f) is countable is also dealt with.

Keywords: monounary algebra, retract, cardinality

MSC 2000: 08A60

1. Introduction

The notion of retract was investigated in many areas of mathematics, first for topological spaces, later for algebraic structures as groups, lattices, posets etc. The importance of the notion is well known and is commonly appreciated. There are dozens of papers dealing with retracts of algebraic structures, we quote [7], [8], [9], [13], [14]. Retracts of algebras are also closely connected to direct limits, e.g., in [2] it is proved that the class of all retracts of a finite algebra is a direct limit class.

Monounary algebras are algebras with one unary operation. They play a significant role in the study of algebraic and relational structures (cf., e.g., Jónsson [6], Skornjakov [12], Chvalina [1], Novotný [10]).

The aim of this paper is to investigate the cardinality of the set of all retracts of a given monounary algebra (A, f). For an uncountable monounary algebra (A, f) with cardinality κ it is proved that (A, f) has exactly 2^κ retracts. If (A, f) is a countable monounary algebra, necessary and sufficient conditions under which (A, f) has only finitely many retracts (has exactly $\aleph_0$ retracts or has $2^{\aleph_0}$ retracts) are found.

This work was suported by the the Slovak VEGA Grant No. 1/3003/06. This work was supported by Science and Technology Assistance Agency under the contract No. APVT-20-004104.

First we recall some basic notions.

By a monounary algebra we understand a pair (A, f) where A is a nonempty set and $f\colon A \longrightarrow A$ is a mapping.

Let (A, f) be a monounary algebra. A nonempty subset M of A is said to be a *retract* of (A, f) if there is a mapping φ of A onto M such that φ is an endomorphism of (A, f) and $\varphi(x) = x$ for each $x \in M$. The mapping φ is then called a *retraction endomorphism* (or, briefly, retraction) corresponding to the retract M.

A monounary algebra (A, f) is called *connected* if for arbitrary elements $x, y \in A$ there are non-negative integers n, m such that $f^n(x) = f^m(y)$. A maximal connected subalgebra of a monounary algebra is called a *(connected) component.*

An element $x \in A$ is referred to as cyclic if there exists a positive integer n such that $f^n(x) = x$. In this case the set $\{x, f^1(x), f^2(x), \dots, f^{n-1}(x)\}$ is called a cycle.

Let C be a cycle of (A, f). Then C is called minimal if whenever D is a cycle of (A, f) such that $|D|$ divides $|C|$ then $|D| = |C|$.

The notion of the degree $s(x)$ of an element $x \in A$ was introduced in [11] (cf. also [10]) as follows. Let us denote by $A^{(\infty)}$ the set of all elements $x \in A$ such that there exists a sequence $\{x_n\}_{n\in\mathbb{N}\cup\{0\}}$ of elements belonging to A with the property $x_0 = x$ and $f(x_n) = x_{n-1}$ for each $n \in \mathbb{N}$. Further, we put $A^{(0)} = \{x \in A\colon\ f^{-1}(x) = \emptyset\}$. Now we define a set $A^{(\lambda)} \subseteq A$ for each ordinal λ by induction. Assume that we have defined A^{α} for each ordinal $\alpha < \lambda$. Then we put $A^{(\lambda)} = \Big\{x \in A - \bigcup\limits_{\alpha<\lambda} A^{(\alpha)}\colon\ f^{-1}(x) \subseteq \bigcup\limits_{\alpha<\lambda} A^{(\alpha)}\Big\}$.

The sets $A^{(\lambda)}$ are pairwise disjoint. For each $x \in A$, either $x \in A^{(\infty)}$ or there is an ordinal λ with $x \in A^{(\lambda)}$. In the former case we put $s(x) = \infty$, in the latter we set $s(x) = \lambda$. We put $\lambda < \infty$ for each ordinal λ.

In the sequel we will use the following notation for some algebras.

We denote by $(\mathbb{Z}, \mathrm{suc})$ and $(\mathbb{N}, \mathrm{suc})$ the algebras such that suc is the operation of the successor.

For $n, k \in \mathbb{N}$ let $Z_{n,k} = \mathbb{Z}_n \cup \{1, 2, \dots, k\}$, where $\mathbb{Z}_n$ for $n \in \mathbb{N}$ is the set of all integers modulo n. We define a monounary algebra $(Z_{n,k}, \mathrm{suc})$ by putting for $x \in Z_{n,k}$

$$\mathrm{suc}(x) = \begin{cases} 0_n & \text{if } x = k,\\ x + 1 & \text{otherwise.} \end{cases}$$

Next we denote $Z_{n,\infty} = \mathbb{Z}_n \cup \mathbb{N}$ and for $x \in Z_{n,\infty}$ we set

$$\operatorname{suc}(x) = \begin{cases} x+1 & \text{if } x \in \mathbb{Z}_n, \\ x-1 & \text{if } x \in \mathbb{N} - \{1\}, \\ 0_n & \text{if } x = 1. \end{cases}$$

Monounary algebras isomorphic to some of the algebras $(\mathbb{Z}, \operatorname{suc})$, $(\mathbb{N}, \operatorname{suc})$, $(Z_{n,k}, \operatorname{suc})$, $(n \in \mathbb{N}, k \in \mathbb{N} \cup \{0\})$, $(Z_{n,\infty}, \operatorname{suc})$, $(n \in \mathbb{N})$ will be called *basic*. A retract M of (A, f) is said to be basic if (M, f) is a basic algebra.

In [3] the following theorem characterizing retracts of connected monounary algebras was proved.

Theorem 2.1. *Let (A, f) be a connected monounary algebra and let (M, f) be a subalgebra of (A, f). Then M is a retract of (A, f) if and only if the following condition is satisfied:*

If $y \in f^{-1}(M)$, then there is $z \in M$ with $f(y) = f(z)$ and $s(y) \leqslant s(z)$.

In what follows we will use some known notions and results of the set theory.

A subset B of a partially ordered set A is cofinal if for every a in A there is b in B such that $a \leqslant b$. Let α be an ordinal number. Cofinality of α is the least ordinal λ such that there is a cofinal subset $X \subseteq \alpha$, X ordered by type λ. We will denote cofinality of α by $\operatorname{cf}(\alpha)$.

An ordinal κ is called a cardinal number if $|\kappa| \neq |\beta|$ for all $\beta < \kappa$. An infinite cardinal κ is *regular* if $\operatorname{cf}(\kappa) = \kappa$. It is *singular* if $\operatorname{cf}(\kappa) < \kappa$. Infinite cardinal numbers are usually denoted by the normal ordinal function $\aleph$.

The following basic facts are known:

Lemma 2.2. $\operatorname{cf}(\operatorname{cf}(\alpha)) = \operatorname{cf}(\alpha)$, *hence* $\operatorname{cf}(\alpha)$ *is always a regular cardinal.*

Lemma 2.3. *An infinite cardinal κ is singular if and only if there exist a cardinal $\lambda < \kappa$ and a family $\{S_\xi \colon \xi < \lambda\}$ of subsets of κ such that $|S_\xi| < \kappa$ for each $\xi < \lambda$ and $\kappa = \bigcup_{\xi<\lambda} S_\xi$. (And the least cardinal λ that satisfies the condition is in fact equal to* $\operatorname{cf}(\kappa)$*.)*

Lemma 2.4 (Axiom of choice)**.** *Every $\aleph_{\alpha+1}$ is a regular cardinal.*

3. Connected case

In this section we will deal with retracts of infinite connected monounary algebras.

Lemma 3.1. *Let κ be regular, $\kappa > \aleph_0$, let (A, f) be a connected monounary algebra, $|A| = \kappa$. Then there exists an element $z \in A$ such that $|f^{-1}(z)| = \kappa$.*

Proof. For $x, y \in A$ let $d(x, y) = \min\{m + n\colon m, n \in \omega; f^m(x) = f^n(y)\}$. Let $y \in A$ be fixed. Denote $Y_0 = \{y\}$; $Y_i = \{x \in A\colon d(x, y) = i\}$ for $i \geqslant 1$.

Then $A = \bigcup\limits_{i\in\omega} Y_i$ and $\kappa = |A| = |\bigcup\limits_{i\in\omega} Y_i|$. Since κ is regular there is $i \in \omega$ such that $|Y_i| = \kappa$. Let i be the least number with $|Y_i| = \kappa$. Then there is $z \in Y_{i-1}$ such that $|f^{-1}(z)| = \kappa$, since $|Y_{i-1}| < \kappa$. □

Lemma 3.2. *Let κ be singular, (A, f) connected, $|A| = \kappa$. Then there is an element $z \in A$ with $|f^{-1}(z)| = \kappa$ or there is a subset $\{z_i\colon i \in I\} \subseteq A$ such that $\{|f^{-1}(z_i)|\colon i \in I\}$ is cofinal in κ.*

Proof. If κ is singular, then κ is limit. By way of contradiction, let us assume that there is a cardinal $\lambda < \kappa$ with $|f^{-1}(x)| \leqslant \lambda$ for all $x \in A$. Then there exists $Y \subseteq A$ such that $|Y| = \lambda^+ < \kappa$, since κ is limit.

λ^+ is regular, the subalgebra $[Y]$ generated by Y has cardinality λ^+ and it is a connected subalgebra of (A, f). According to Lemma 3.1 there is an element $z \in [Y]$ such that $|f^{-1}(z)| = \lambda^+$, which is contradiction. □

Theorem 3.3. *Let κ be a cardinal, $\kappa \geqslant \aleph_0$, let (A, f) be a connected monounary algebra and let $z \in A$ be such that $|f^{-1}(z)| = \kappa$. Then (A, f) contains at least 2^κ retracts.*

Proof. First suppose that there is an element $x \in f^{-1}(z)$ such that $s(x)$ is the maximal degree over all elements in $f^{-1}(z)$.

Let $M = \Bigl(A - \bigcup\limits_{k=1}^{\infty} f^{-k}(z)\Bigr) \cup \bigcup\limits_{k=0}^{\infty} f^{-k}(x)$. Then (M, f) is a subalgebra of (A, f). Let $g\colon \kappa \longmapsto f^{-1}(z) - \{x\}$ be an arbitrary bijection.

We assign a subalgebra M_Y to every subset $Y \subseteq \kappa$, where $M_Y = M \cup \bigcup\limits_{k=0}^{\infty} f^{-k}(g[Y])$. For $y \in f^{-1}(M_Y) - M_Y$ we have $y \in f^{-1}(z)$, thus $f(y) = f(x) = z$ and $s(x) \geqslant s(y)$. This implies that according to Theorem 2.1, M_Y is a retract of (A, f).

Let there be no such element with the maximal degree. Let $\alpha = \sup\{s(x)\colon x \in f^{-1}(z)\}$, $\alpha \in On$. Since $|f^{-1}(z)| = \kappa$, we have $\mathrm{cf}(\alpha) \leqslant \kappa$. There exist $K, Z \subseteq f^{-1}(z)$

with $f^{-1}(z) = K \cup Z$, $K \cap Z = \emptyset$, $\{s(x)\colon x \in K\}$ is cofinal in α and $|Z| = \kappa$. Let $M = \Bigl(A - \bigcup_{k=1}^{\infty} f^{-k}(z)\Bigr) \cup \bigcup_{k=0}^{\infty} f^{-k}(K)$, and let $g\colon \kappa \longmapsto Z$ be any bijection.

Then $M_Y = M \cup \bigcup_{k=0}^{\infty} f^{-k}(g[Y])$ for every $Y \subseteq \kappa$ is a retract, since for every $y \in f^{-1}(z)$ and $y \notin M_Y$ there is $x \in K$ such that $s(x) \geqslant s(y)$ and $f(x) = f(y) = z$. In both the cases we have a one to one mapping from 2^κ into the set of all retracts of (A, f). □

Remark 3.4. Under the assumptions as in Theorem 3.3 these retracts can be chosen such that for each retract B of them we have $A - B \subseteq \bigcup_{n=0}^{\infty} f^{-n}(z)$, i.e., if φ is the corresponding retraction of A onto B and $x \in A$, $\varphi(x) \neq x$, then $x \in \bigcup_{n=0}^{\infty} f^{-n}(z)$.

Corollary 3.5. *Let (A, f) be a connected monounary algebra, $|A| = \kappa \geqslant \aleph_0$. If there is an element $z \in A$ such that $|f^{-1}(z)| = \kappa$, then (A, f) has exactly 2^κ retracts.*

Corollary 3.6. *If κ is an uncountable regular cardinal, (A, f) a connected monounary algebra, then (A, f) has exactly 2^κ retracts.*

Let κ be a singular cardinal, $\{\kappa_i\colon i \in I\}$ a cofinal subset in κ where all κ_i are cardinals and $|I| = \operatorname{cf}(\kappa) < \kappa$. Then $2^\kappa = \Bigl|\prod_{i \in I} 2^{\kappa_i}\Bigr|$.

Theorem 3.7. *Let κ be a singular cardinal, (A, f) a connected monounary algebra, $|A| = \kappa$. There are exactly 2^κ retracts of (A, f).*

P r o o f. If there exists $z \in A$, such that $|f^{-1}(z)| = \kappa$, then there are 2^κ retracts according to Corollary 3.5.

Suppose that such an element does not exist, let $Z = \{z_i\colon i \in I\}$ be a subset of A with the following properties.

1. There is no $i \in I$ such that z_i is cyclic.
2. $\{|f^{-1}(z_i)|\colon i \in I\}$ is cofinal in κ and $|Z| = \operatorname{cf}(\kappa)$.
3. For every $i \in I$, $|f^{-1}(z_i)| > \operatorname{cf}(\kappa)$.

The existence of a subset Z with the above properties follows from the assumptions and from Lemma 3.2. Denote $\kappa_i = |f^{-1}(z_i)|$. Let $Y_i = \Bigl\{y \in f^{-1}(z_i)\colon Z \cap \bigcup_{n=0}^{\infty} f^{-n}(y) = \emptyset\Bigr\}$ for $i \in I$. Then $|Y_i| = \kappa_i$, since $|Z| = \operatorname{cf}(\kappa)$ and $\kappa_i > \operatorname{cf}(\kappa)$.

Let $M = A - \bigcup_{n=0}^{\infty} f^{-n}\Bigl(\bigcup_{i \in I} Y_i\Bigr)$. Then M is a subalgebra of (A, f). Let $A_i = M \cup \bigcup_{n=0}^{\infty} f^{-n}(Y_i)$.

According to Theorem 3.3 and to Remark 3.4 there exist 2^{κ_i} retracts of A_i such that if φ is some of the retractions then $\varphi(x) \neq x$ implies $x \in A_i - M$. Thus we have a bijection g_i of 2^{κ_i} into the system of these retracts. For $h \in \prod_{i\in I} 2^{\kappa_i}$ we denote $M_h = M \bigcup_{i\in I} g_i(h(i))$. It can be verified that M_h is a retract of (A, f). Thus we have an injective mapping from 2^κ to the set of all retracts of (A, f). □

4. General non-countable case

There are, up to isomorphism, countably many connected monounary algebras having no proper retract, namely $(\mathbb{Z}, \mathrm{suc})$, $(\mathbb{N}, \mathrm{suc})$, $(\mathbb{Z}_n, \mathrm{suc})$.

Theorem 4.1. *Let $\kappa > \aleph_0$ be a cardinal, (A, f) a monounary algebra with $|A| = \kappa$. Then (A, f) has exactly 2^κ retracts.*

Proof. If there is a component of cardinality κ, then by Corollary 3.6 and Theorem 3.7 there are 2^κ retracts. Now assume that each connected component of A has less than κ elements.

a) Suppose that there is no connected component of cardinality κ in (A, f), but there are κ components of (A, f).

1. There are κ components of (A, f) having a proper retract. Let A_α for each $\alpha < \kappa$ be such components and let R_α be any fixed proper retract of A_α. Let $B = A - \bigcup_{\alpha<\kappa} A_\alpha$. To every subset $X \subseteq \kappa$ we can assign a retract $M(X) = \bigcup_{\alpha\in X} R_\alpha \cup \bigcup_{\alpha\notin X} A_\alpha \cup B$, thus we have at least 2^κ retracts of (A, f).

2. There exists no system of κ components with a proper retract. Suppose that there are κ isomorphic components having only one retract (this is always the case when $\mathrm{cf}(\kappa) > \omega$). Let A_α for each $\alpha < \kappa$ be such components, $B = A - \bigcup_{\alpha<\kappa} A_\alpha$. Then $M(X) = \bigcup_{\alpha\in X} A_\alpha \cup B$, $X \subseteq \kappa, X \neq \emptyset$ is a retract of (A, f), hence there are at least 2^κ retracts of (A, f).

Suppose that there are no κ isomorphic components of (A, f), then necessarily $\mathrm{cf}(\kappa) = \omega$. Let C_n, $n \in \omega$ be the set of components isomorphic to $(\mathbb{Z}_n, \mathrm{suc})$, $C_n = \{Z_n^\alpha \colon \alpha < |C_n|\}$. From the above assumptions it follows that there is $X \subseteq \omega$, a subset of positive integers, such that $\{|C_n| \colon n \in X\}$ is cofinal in κ. To each $h \in \prod_{n\in X} \mathscr{P}(C_n)$ such that $h(n) \neq \emptyset$ for every $n \in X$ we can assign a retract $M(h) = \bigcup_{n\in X} \bigcup_{\alpha\in h(n)} Z_n^\alpha \cup B$, where $B = A - \bigcup_{n\in X} C_n$. We can find an injective mapping from 2^κ to the set of all retracts of (A, f).

b) Finally, suppose that (A, f) has less than κ components. Then κ is singular. Let $\{A_i\colon\ i \in I\}$ be the system of connected components of (A, f) such that $|I| = \mathrm{cf}(\kappa)$, $\{|A_i|\colon\ i \in I\}$ is cofinal in κ and $|A_i| > \aleph_0$ for each $i \in I$. According to Corollary 3.6 and Theorem 3.7 let $R_\alpha^{A_i}, \alpha < 2^{|A_i|}$ be a system of retract of $A_i, i \in I$ and $B = A - \bigcup\limits_{i\in I} A_i$. For every $h \in \prod\limits_{i\in I} 2^{|A_i|}$, $M(h) = \bigcup\limits_{i\in I} R_{h(i)}^{A_i} \cup B$ is a retract of (A, f), thus we have an injective mapping from 2^κ to the set of all retracts of (A, f). □

5. Countable case

From the results of the previous part it follows that there exist countable monounary algebras (A, f) having finitely many retracts (e.g., $(\mathbb{N}, \mathrm{suc})$) or having $2^{\aleph_0}$ retracts (e.g.,(A, f) satisfying the assumption of Theorem 3.3). Now we will deal with countable monounary algebras. The aim is to find necessary and sufficient conditions under which a countable monounary algebra has

(a) finitely many retracts (cf. 5.3),

(b) exactly $\aleph_0$ retracts (cf. 5.10),

(c) exactly $2^{\aleph_0}$ retracts.

Now let (A, f) be a countable monounary algebra.

Lemma 5.1. *Assume that $f^{-1}(x)$ is finite for each $x \in A$. If the number of basic retracts of (A, f) is infinite, then (A, f) has $2^{\aleph_0}$ retracts.*

P r o o f. Suppose that there exist distinct basic retracts B_m for $m \in \omega$. If $Q \subseteq \omega$, then we put $D_Q = \bigcup\limits_{q\in Q} B_Q$. By 2.1 it is easy to see that the union of retracts is again a retract, thus D_Q is a retract of (A, f). Therefore the number of retracts of (A, f) is $2^{\aleph_0}$. □

Lemma 5.2. *Assume that $f^{-1}(x)$ is finite for each $x \in A$ and that $\{B_1, B_2, \ldots, B_m\}$ is the set of all basic retracts of (A, f), where $m \in \mathbb{N}$. If the set $A - (B_1 \cup \ldots \cup B_m)$ is infinite, then (A, f) has $2^{\aleph_0}$ retracts.*

P r o o f. Let $A - (B_1 \cup \ldots \cup B_m)$ be infinite and suppose that (A, f) has less than $2^{\aleph_0}$ retracts. If $x \in A - (B_1 \cup \ldots \cup B_m)$, then $s(x) \neq \infty$, because in the opposite case x belongs to some basic retract isomorphic to $(\mathbb{Z}, \mathrm{suc})$ or $(\mathbb{Z}_{n,\infty}, \mathrm{suc})$. Then 3.3 implies

$$(1) \qquad x \in A - (B_1 \cup \ldots \cup B_m) \rightarrow \bigcup_{l=0}^{\infty} f^{-l}(x) \text{ is finite.}$$

Denote $X = f^{-1}(B_1 \cup \ldots \cup B_m) - (B_1 \cup \ldots \cup B_m)$. In view of (1), the assumption implies that X is infinite, $X = \{x_i \colon i \in \omega\}$. For each $Q \subseteq \omega$ we set $M_Q = B_1 \cup \ldots \cup B_m \cup \bigcup_{q \in Q} \bigcup_{l=0}^{\infty} f^{-l}(x_q)$. Then M_q is a retract of (A, f). Indeed, let $Q \subseteq \omega$, $y \in f^{-1}(M_Q) - M_Q$, $x = f(y)$. Then $x \in B_1 \cup \ldots \cup B_m$, i.e., $x \in B_i$ for some $i \in \{1, \ldots, m\}$. Since B_i is a retract of (A, f), by 2.1 there is $z \in B_i$ with $s(y) \leqslant s(z)$. Therefore (A, f) has $2^{\aleph_0}$ retracts. □

Theorem 5.3. *A countable monounary algebra (A, f) has only finitely many retracts if and only if the following conditions are satisfied:*
(i) *$f^{-1}(x)$ is finite for each $x \in A$,*
(ii) *the set of basic retracts of (A, f) is $\{B_1, \ldots, B_m\}$, $m \in \mathbb{N}$,*
(iii) *$A - (B_1 \cup \ldots \cup B_m)$ is finite.*

P r o o f. If (i) fails to hold, then 3.3 yields that there are $2^{\aleph_0}$ retracts of (A, f). If (i) is valid and (ii) fails to hold, then again there are $2^{\aleph_0}$ retracts of (A, f) in view of 5.1. If (i),(ii) hold and (iii) does not hold, then we get the same by 5.2.

Conversely, assume that conditions (i)–(iii) are satisfied. Let M be a retract of (A, f). Then M consists of a join of some basic retracts (there are only finitely many possibilities of choosing them) and of some elements of the set $A - (B_1 \cup \ldots \cup B_m)$ (again, finitely many possibilities of choosing them). Thus (A, f) has only finitely many retracts. □

Corollary 5.4. *If (A, f) contains a basic retract, then either (A, f) contains only finitely many retracts or the number of retracts of (A, f) is $2^{\aleph_0}$.*

In 5.5–5.8 we will suppose that (A, f) is a countable monounary algebra such that the conditions (i) of 5.3 and
(iv) (A, f) *contains no basic retract*
are satisfied. Consequently, $s(x) \neq \infty$ for each $x \in A$. Further, by [4],1.7 the algebra (A, f) is unbounded, i.e., for each $a \in A$ we have

$(\forall m \in \omega)\ (\exists n \in \omega)\ (f^{-(m+n)}(f^n(a)) \neq \emptyset)$.

We will denote by $n_m^{(a)}$ the smallest n with the above property, $x_m^{(a)} = f^{n_m^{(a)}}(a)$. Then this implies that $s(x_m^{(a)}) \geqslant m$, $s(f(x_m^{(a)})) \geqslant m + 1$ for each $m \in \omega$.

Lemma 5.5. *For each $a \in A$, the set $\{n_m^{(a)} \colon m \in \omega\}$ is unbounded.*

P r o o f. Suppose that $k = \max\{n_m^{(a)} \colon m \in \omega\}$ exists. Denote $b = f^k(a)$ and let $m \in \mathbb{N}$. There is $c \in A$ with $f^{n_m^{(c)}+m}(c) = f^{n_m^{(a)}}(a)$, which yields $f^{k-n_m^{(a)}}(f^{n_m^{(a)}+m}(c)) = f^{k-n_m^{(a)}}(f^{n_m^{(a)}}(a))$, i.e., $f^{k+m}(c) = f^k(a) = b$. Hence $f^{-(k+m)}(b) \neq \emptyset$ for each $m \in \omega$, which is a contradiction with (i) of 5.3. □

Lemma 5.6. *For each $a \in A$ and $m \in \omega$ there exists $y_m^{(a)} \in f^{-1}(x_m^{(a)}) - \{f^{n_m^{(a)}-1}(a)\}$ such that if $b \in f^{-1}(x_m^{(a)})$, then $s(b) \leqslant s(y_m^{(a)})$.*

P r o o f. Let $a \in A$, $m \in \omega$. The set $f^{-1}(x_m^{(a)}) - \{f^{n_m^{(a)}-1}(a)\}$ is finite by (i), take an element $y_m^{(a)}$ of this set having the maximal degree. We need to show that $s(f^{n_m^{(a)}-1}(a)) \leqslant s(y_m^{(a)})$. Denote $v = f^{n_m^{(a)}-1}(a)$. By way of contradiction, let $s(v) > s(y_m^{(a)})$. We have $s(x_m^{(a)}) \geqslant n_m^{(a)} + m$, hence $s(y_m^{(a)}) \geqslant n_m^{(a)} + m - 1$. Thus $s(v) > n_m^{(a)} + m - 1$ and $\emptyset \neq f^{-n_m^{(a)}+m-1}(v) = f^{-[(n_m^{(a)}-1)+m]}(f^{n_m^{(a)}-1}(a))$. Since $n_m^{(a)}$ is the smallest element with the above property, we have arrived at a contradiction. $\square$

Notation 5.7. *For $a \in A$ we put $K^{(a)} = \{m \in \mathbb{N} \colon\ s(y_m^{(a)}) = s(f^{n_m^{(a)}-1}(a))\}$.*

Lemma 5.8. *Suppose that $a \in A$ and that $K^{(a)}$ is infinite. Then (A, f) has $2^{\aleph_0}$ retracts.*

P r o o f. For each $Q \subseteq K^{(a)}$ let $k = \min Q$, and put

$$M_Q = \bigcup_{q \in Q \cup \{j \in \mathbb{N} - Q \colon\, j > k\}} \{b \in A \colon\ (\exists l \in \omega)(b = f^l(y_q^{(a)}) \text{ or } y_q^{(a)} = f^l(b))\}.$$

Then (M_Q, f) is a subalgebra of (A, f). To prove that M_Q is a retract of (A, f) let $y \in f^{-1}(M_Q) - M_Q$. Then $f(y) = f^j(a)$ for some $j \in \omega$. If $j = n_m^{(a)}$ for some $m \in \omega$, then $f(y) = x_m^{(a)} \in M_Q$, $y \in f^{-1}(x_m^{(a)})$ and $s(y) \leqslant s(y_m^{(a)})$. If now $y_m^{(a)} \notin M_Q$, then $s(y_m^{(a)}) = s(f^{n_m^{(a)}-1}(a))$, hence $s(y) \leqslant s(f^{n_m^{(a)}-1}(a))$, $f^{n_m^{(a)}-1}(a) \in M_Q$. Suppose that $j \neq n_m^{(a)}$ for each $m \in \mathbb{N}$. By 5.5 there is the smallest $m \in \mathbb{N}$ such that $j < n_m^{(a)}$. Since $f(y) \in M_Q$, we obtain that $m > 1$ and $n_{m-1}^{(a)} < j < n_m^{(a)}$. Hence $m - 1 \leqslant s(x_{n_{m-1}^{(a)}})$, $m \leqslant s(f(x_{n_{m-1}^{(a)}})) \leqslant s(f^j(a)) < s(x_{n_m^{(a)}}) \leqslant m$, which is a contradiction. Thus $(A.f)$ has $2^{\aleph_0}$ retracts. $\square$

Theorem 5.9. *A countable connected monounary algebra (A, f) has exactly $\aleph_0$ retracts if and only if the following conditions are satisfied:*
(i) *$f^{-1}(x)$ is finite for each $x \in A$,*
(iv) *(A, f) contains no basic retract,*
(v) *for each $a \in A$, the set $K^{(a)}$ is finite.*

P r o o f. The necessity follows from 3.3, 5.4 and 5.8. Suppose that the conditions (i), (iv) and (v) are satisfied. Let $a \in A$. Consider a retract M of (A, f). There is the smallest $m \in \mathbb{N}$ such that $x_m^{(a)} \in M$. The set $\bigcup_{l=0}^{\infty} f^{-l}(x_m^{(a)})$ is finite, hence there are only finitely many possibilities of choosing elements of this set into M. If $s(y_k^{(a)}) >$

$s(f^{n_k^{(a)}-1}(a))$, $k \in \omega$, then $y_k^{(a)} \in M$. If not, it can belong to M, but there are only finitely many of such elements. Further, the set $A - \Big(\bigcup\limits_{l=0}^{\infty} f^{-l}(x_m^{(a)}) \cup \{y_k^{(a)} \colon k \in \omega\} \Big)$ of the remaining elements is also finite. Therefore there are $\aleph_0$ possibilities of choosing a retract M of (A, f). $\square$

Theorem 5.10. *A countable monounary algebra (A, f) has exactly $\aleph_0$ retracts if and only if the following conditions are satisfied:*

(vi) *each connected component of (A, f) has at most $\aleph_0$ retracts,*

(vii) *there exists a connected component having exactly $\aleph_0$ retracts,*

(viii) *there exists no infinite system of connected components having a proper retract,*

(ix) *there exists no infinite system of isomorphic connected components,*

(x) *the number of non-minimal cycles of (A, f) is finite.*

Proof. Let (A, f) be a countable monounary algebra.

a) Assume that (A, f) has exactly $\aleph_0$ retracts. Then (vi) and (vii) hold. If any of the conditions (viii), (ix), (x) fails to hold, then there is a system $B_i, i \in \omega$ of distinct connected components such that one of the following conditions is satisfied:

(1) for each $i \in \omega$, M_i is a proper retract of B_i,

(2) for each $i \in \omega$, B_i contains a cycle C_i which is non-minimal,

(3) for each $i, j \in \omega$, $B_i \cong B_j$.

Let (1) be valid. To each $Q \subseteq \omega$ we assign a set $M_Q = \Big(A - \bigcup\limits_{i \in Q} B_i\Big) \cup \bigcup\limits_{q \in Q} M_i$. It is easy to verify that M_Q is a retract of (A, f), thus the number of retracts of (A, f) is $2^{\aleph_0}$, a contradiction.

Let (2) hold. To $Q \subseteq \omega$ we assign $V_Q = A - \bigcup\limits_{q \in Q} B_q$. Since for $q \in Q$ the cycle B_q is non-minimal, the connected component B_q can be homomorphically mapped into $A - \bigcup\limits_{i \in Q} B_i$, which implies that V_Q is a retract of (A, f). Again, this is a contradiction to the number of retracts of (A, f).

If (3) is valid, then analogously as for the condition (2), the system V_Q, $Q \subseteq \omega$, is an uncountable system of retracts of (A, f), which is a contradiction.

b) Now suppose that (A, f) satisfies the conditions (vi)–(x). Let X consist of the join of the connected components which have a proper retract. Then $A - X$ consists of connected components which are of types $(\mathbb{Z}, \mathrm{suc})$, $(\mathbb{N}, \mathrm{suc})$ or $(\mathbb{Z}_n, \mathrm{suc})$, $n \in \mathbb{N}$, each type occurring only finitely many times by (viii). Consider a retract M of (A, f). According to (viii), (vi) and (vii), there are exactly $\aleph_0$ possibilities of choosing $M \cap X$. Denote $M_1 = M \cap (A - X)$. If $(A - X)$ has only finitely many connected components, then for M_1 we have only finitely many possibilities and (A, f) has exactly $\aleph_0$ retracts.

Let $(A - X)$ consist of infinitely many connected components. Then by (ix) and (x), there is $Y \subseteq A - X$ such that Y consists of cycles $C_j, j \in \omega$ which are minimal and that $(A-X)-Y$ has only finitely many connected components. For each $j\in\omega$, we have $C_j\subseteq M$, i.e., $M\cap Y=Y$. For $M\cap((A-X)-Y)$ there are only finitely many possibilities (in view of (ix) and (x)). Therefore there are exactly $\aleph_0$ retracts of (A,f). □

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