Czechoslovak Mathematical Journal

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Czechoslovak Mathematical Journal, Vol. 58 (2008), No. 2, 517–540

Persistent URL: http://dml.cz/dmlcz/128275

Czechoslovak Mathematical Journal, 58 (133) (2008), 517–540

ALTERNATIVE CHARACTERISATIONS OF LORENTZ-KARAMATA SPACES

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(Received May 5, 2006)

Abstract. We present new formulae providing equivalent quasi-norms on Lorentz-Karamata spaces. Our results are based on properties of certain averaging operators on the cone of non-negative and non-increasing functions in convenient weighted Lebesgue spaces. We also illustrate connections between our results and mapping properties of such classical operators as the fractional maximal operator and the Riesz potential (and their variants) on the Lorentz-Karamata spaces.

Keywords: Lorentz-Karamata spaces, equivalent quasi-norms, weighted norm inequalities, fractional maximal operators, Riesz potentials

MSC 2000: 46E30, 26D10, 47B38, 47G10

1. Introduction

In [9], [14] and [15] new characterisations of Lorentz spaces were given by means of quasi-norms that were shown to be equivalent to the classical ones. However, Lorentz spaces (and their special cases, the Lebesgue spaces) are but the simplest of a whole scale of spaces of proven usefulness in analysis, such as those of Lorentz-Zygmund and generalised Lorentz-Zygmund type. The Lorentz-Karamata spaces, defined by means of slowly varying functions, contain all these spaces and are currently attracting a good deal of attention, not least because they enable results involving them to be proved with no more difficulty than is needed to give the ad hoc arguments necessary to establish the corresponding results in more specialised spaces. For an account of these spaces, together with illustrations of their usefulness, we refer to [5] and [13]. There is a standard way of defining Lorentz-Karamata spaces by means of quasi-norms; here we provide alternative characterisations by means of equivalent quasi-norms.

To explain our results in a little more detail, let (Ω, μ) be a totally σ-finite measure space with a non-atomic measure μ. A non-negative measurable function b on $(0, \infty)$ is called slowly varying if, given any $\varepsilon > 0$, the functions $t \longmapsto t^{\varepsilon} b(t)$ and $t \longmapsto t^{-\varepsilon} b(t)$ are respectively equivalent to non-decreasing and non-increasing functions on $(0, \infty)$. Suppose that $p, q \in (0, \infty]$ and that b is slowly varying. Then the Lorentz-Karamata space $L_{p,q,b}$ is the set of all μ-measurable functions f on Ω such that

$$\|f\|_{p,q,b} := \|t^{1/p-1/q} b(t) f^*(t)\|_{q,(0,\infty)} < \infty.$$

Here f^* is the non-increasing rearrangement of f and $\|\cdot\|_{q,(0,\infty)}$ is the usual Lebesgue quasi-norm on $(0, \infty)$. The Lorentz, Lorentz-Zygmund and generalised Lorentz-Zygmund spaces are all special cases of these spaces, obtained by making particular choices of the slowly varying function b. We show that if $p, r, s \in (0, \infty]$, $q \in (-\infty, 0)$ and a, b are slowly varying, $0 \not\equiv a \not\equiv \infty$, then

$$f \longmapsto \left\| t^{1/q-1/r} \frac{b(t)}{a(t)} \|\tau^{1/p-1/q-1/s} a(\tau) f^*(\tau)\|_{s,(0,t)} \right\|_{r,(0,\infty)}$$

is a quasi-norm equivalent to $\|\cdot\|_{p,q,b}$. A corresponding statement holds when $q \in (0, \infty)$ or when the function f^* is replaced by its maximal function f^{**} (see Theorems 3.1, 3.2, 3.5 and 3.6 below). These results extend those given in [9] and [14] for Lorentz spaces and, in particular, they generalize and complement the result of A. P. Calderón (cf. Remark 3.7 below).

We also illustrate (see Section 4) connection of our results with the mapping properties of such classical operators as the maximal operator and the Riesz potential, together with their variants, in the context of Lorentz-Karamata spaces. Some ideas that explain our original motivation are mentioned at the end of Section 4. However, we do not follow these ideas in the proofs of Theorems 3.1, 3.2, 3.5 and 3.6; our proofs of these theorems are based on properties of certain averaging operators on the cone of non-negative and non-increasing functions in convenient weighted Lebesgue spaces (cf. Remark 3.3 below).

2. Preliminaries

Given any quasi-Banach spaces X and Y, we write $X \hookrightarrow Y$ if $X \subset Y$ and the natural embedding of X in Y is continuous; $X = Y$ means that $X \hookrightarrow Y$ and $Y \hookrightarrow X$.

We write $A \lesssim B$ (or $A \gtrsim B$) if $A \leqslant cB$ (or $cA \geqslant B$) for some positive constant c independent of appropriate quantities involved in A and B; $A \approx B$ means that $A \lesssim B$ and $B \lesssim A$. Throughout the paper we use the abbreviation LHS($*$) (RHS($*$)) for the left(right)-hand side of the relation (*). By χ_S we shall mean the characteristic function of the set S.

Let (Ω, μ) be a totally σ-finite measure space with a non-atomic measure μ. We denote by $\mathcal{M}(\Omega, \mu)$ the set of all μ-measurable functions on Ω and by $\mathcal{M}^+(\Omega, \mu)$ the subset of this consisting of all non-negative functions; when Ω is an interval $(a, b) \subseteq \mathbb{R}$ and μ is Lebesgue measure on this interval, we shall denote these sets by $\mathcal{M}(a, b)$ and $\mathcal{M}^+(a, b)$, respectively; when $\Omega = \mathbb{R}^n$ and μ is Lebesgue measure dx we shall write $\mathcal{M}(\mathbb{R}^n)$ instead of $\mathcal{M}(\mathbb{R}^n, dx)$. By $\mathcal{M}^+(a, b; \uparrow)$ and $\mathcal{M}^+(a, b; \downarrow)$ we shall mean the subsets of $\mathcal{M}^+(a, b)$ containing all non-decreasing and all non-increasing functions, respectively. Given any $f \in \mathcal{M}(\Omega, \mu)$, the non-increasing rearrangement f^* of f is defined by

$$f^*(t) = \inf\{\lambda > 0\colon\ \mu\{x \in \Omega\colon\ |f(x)| > \lambda\} \leqslant t\},\ t \in (0, \infty),$$

and we write $f^{**}(t) = t^{-1} \int_0^t f^*(s)\,\mathrm{d}s,\ t > 0$. If $q \in (0, \infty]$ and $-\infty \leqslant c < d \leqslant \infty$, $\|\cdot\|_{q,(c,d)}$ will stand for the usual Lebesgue quasi-norm on (c, d) with respect to Lebesgue measure.

A function $b \in \mathcal{M}^+(0, \infty)$ is called *slowly varying* on $(0, \infty)$ if given any $\varepsilon > 0$, there are functions $g_\varepsilon \in \mathcal{M}^+(0, \infty; \uparrow)$ and $g_{-\varepsilon} \in \mathcal{M}^+(0, \infty; \downarrow)$ such that

$$t^\varepsilon b(t) \approx g_\varepsilon(t) \ \text{ and } \ t^{-\varepsilon} b(t) \approx g_{-\varepsilon}(t) \ \text{ for all } t \in (0, \infty).$$

Here we follow the definition given in [11]; for other definitions see, for example, [3] and [5]. The family of all slowly varying functions is denoted by SV; it includes not only powers of iterated logarithms and the broken logarithmic functions of [10], but also such functions as $t \to \exp\left(|\log t|^a\right),\ a \in (0, 1)$. (The last mentioned function has the interesting property that it tends to infinity more quickly than any positive power of the logarithmic function.)

We shall need the following properties of elements of SV, for which we refer to [11], Proposition 2.2 and Remark 2.3(i).

Lemma 2.1. *Let b, b_1 and b_2 belong to SV. Then*

(i) $b_1 b_2 \in SV$ *and* $b^r \in SV$ *for each* $r \in \mathbb{R}$;

(ii) *given positive numbers ε and κ, there are positive constants c_ε and C_ε such that*

$$c_\varepsilon \min\{\kappa^{-\varepsilon}, \kappa^\varepsilon\} b(t) \leqslant b(\kappa t) \leqslant C_\varepsilon \max\{\kappa^{-\varepsilon}, \kappa^\varepsilon\} b(t) \quad \text{for all } t > 0;$$

(iii) *if $\alpha > 0$ and $q \in (0, \infty]$, then for all $t > 0$,*

$$\|\tau^{\alpha - 1/q} b(\tau)\|_{q,(0,t)} \approx t^\alpha b(t) \quad \text{and} \quad \|\tau^{-\alpha - 1/q} b(\tau)\|_{q,(t,\infty)} \approx t^{-\alpha} b(t);$$

(iv) *there exists* $d \in SV \cap C(0,\infty)$ *such that* $b \approx d$;
(v) *if* $h \in \mathcal{M}^+(0,\infty;\uparrow)$ *and* $t/h^\delta(t) \in \mathcal{M}^+(0,\infty;\uparrow)$ *for some* $\delta > 0$, *then* $b \circ h \in SV$.

Next we define the Lorentz-Karamata spaces. Given any $p, q \in (0,\infty]$ and any $b \in SV$, the *Lorentz-Karamata* space $L_{p,q,b} = L_{p,q,b}(\Omega,\mu)$ is the set of all $f \in \mathcal{M}(\Omega,\mu)$ such that

$$\|f\|_{p,q,b} := \|t^{1/p-1/q} b(t) f^*(t)\|_{q,(0,\infty)} < \infty. \tag{2.1}$$

Particular choices of b give well-known spaces. Obviously when b is the function identically equal to 1, the corresponding Lorentz-Karamata space coincides with the Lorentz space $L_{p,q}$. Moreover, if $m \in \mathbb{N}$ and

$$b(t) = \prod_{i=1}^m l_i^{\alpha_i}(t) \quad \text{for } t > 0, \quad \text{where } \alpha_1, \ldots, \alpha_m \in \mathbb{R},$$

and, for $t > 0$,

$$l_1(t) = 1 + |\log t|\,, \; l_i(t) = l_1(l_{i-1}(t)) \quad \text{if } i > 1,$$

then the Lorentz-Karamata space $L_{p,q,b}$ is the generalised Lorentz-Zygmund space $L_{p,q,\alpha_1,\ldots,\alpha_m}$ of [7], which in turn becomes the Lorentz-Zygmund space $L^{p,q}(\log L)^{\alpha_1}$ of Bennett and Rudnick [1] when $m = 1$.

Two further lemmas will be needed. For the first we refer to [11], Lemma 2.7.

Lemma 2.2. *Let* $1 \leqslant P \leqslant \infty$, $\nu \in \mathbb{R} \setminus \{0\}$ *and suppose that* $d \in SV$.
(i) *The inequality*

$$\left\| t^{\nu-1/P} d(t) \int_0^t g(\tau)\,\mathrm{d}\tau \right\|_{P,(0,\infty)} \lesssim \|t^{\nu+1-1/P} d(t) g(t)\|_{P,(0,\infty)} \tag{2.2}$$

holds for all $g \in \mathcal{M}^+(0,\infty)$ *if and only if* $\nu < 0$.
(ii) *The inequality*

$$\left\| t^{\nu-1/P} d(t) \int_t^\infty g(\tau)\,\mathrm{d}\tau \right\|_{P,(0,\infty)} \lesssim \|t^{\nu+1-1/P} d(t) g(t)\|_{P,(0,\infty)} \tag{2.3}$$

holds for all $g \in \mathcal{M}^+(0,\infty)$ *if and only if* $\nu > 0$.

The next is Theorem 2.2 of [12].

Lemma 2.3. *Let $0 < P \leqslant Q \leqslant 1$ and suppose that $k\colon (0,\infty)\times(0,\infty)\to[0,\infty]$ is Lebesgue measurable. Then there is a constant $C\in[0,\infty)$ such that the inequality*

$$\left\| w(t)\int_0^\infty k(t,\tau)g(\tau)\,\mathrm{d}\tau \right\|_{Q,(0,\infty)} \leqslant C\,\|v(t)g(t)\|_{P,(0,\infty)} \tag{2.4}$$

holds for all $g\in\mathcal{M}^+(0,\infty;\downarrow)$ if and only if

$$\left\| w(t)\int_0^\varrho k(t,\tau)\,\mathrm{d}\tau \right\|_{Q,(0,\infty)} \leqslant C\,\|v(t)\|_{P,(0,\varrho)} \quad \text{for all } \varrho\in(0,\infty). \tag{2.5}$$

3. Main results

Our first alternative way of characterising the Lorentz-Karamata space $L_{p,q,b}$ is contained in the following.

Theorem 3.1. *Let $p,r,s\in(0,\infty]$, $q\in(-\infty,0)$ and suppose that $a,b\in SV$, $0\not\equiv a\not\equiv\infty$. Then for all $f\in\mathcal{M}(\Omega,\mu)$,*

$$\|f\|_{p,r,b} \approx \left\| t^{1/q-1/r}\frac{b(t)}{a(t)}\|\tau^{1/p-1/q-1/s}a(\tau)f^*(\tau)\|_{s,(0,t)} \right\|_{r,(0,\infty)}. \tag{3.1}$$

Proof. Let $f\in\mathcal{M}(\Omega,\mu)$. Then $f^*\in\mathcal{M}(0,\infty;\downarrow)$ and for all $t>0$,

$$\|\tau^{1/p-1/q-1/s}a(\tau)f^*(\tau)\|_{s,(0,t)} \geqslant f^*(t)\|\tau^{1/p-1/q-1/s}a(\tau)\|_{s,(0,t)}.$$

Moreover, since $1/p-1/q>0$ and $a\in SV$, it follows from Lemma 2.1 (iii) that

$$\|\tau^{1/p-1/q-1/s}a(\tau)\|_{s,(0,t)} \approx t^{1/p-1/q}a(t) \quad \text{for all } t>0. \tag{3.2}$$

Hence

$$\mathrm{RHS}(3.1) \gtrsim \|t^{1/p-1/r}b(t)f^*(t)\|_{r,(0,\infty)} = \mathrm{LHS}(3.1).$$

To prove the reverse estimate, we distinguish several cases.

(i) Let $s=1$. Then

$$\mathrm{RHS}(3.1) = \left\| t^{1/q-1/r}\frac{b(t)}{a(t)}\int_0^t \tau^{1/p-1/q-1}a(\tau)f^*(\tau)\,\mathrm{d}\tau \right\|_{r,(0,\infty)}. \tag{3.3}$$

If $r \in [1,\infty]$, we apply the Hardy-type inequality (2.2), with $P = r$, $\nu = 1/q$, $d = b/a$ and $g(\tau) = \tau^{1/p-1/q-1}a(\tau)f^*(\tau)$, and obtain, for all $f \in \mathcal{M}(\Omega,\mu)$,

$$\text{RHS}(3.1) \lesssim \|t^{1/p-1/r}b(t)f^*(t)\|_{r,(0,\infty)} = \text{LHS}(3.1). \tag{3.4}$$

If $r \in (0,1)$, we put $P = Q = r$ and

$$k(t,\tau) = \chi_{(0,t)}(\tau)\tau^{1/p-1/q-1}a(\tau), w(t) = t^{1/q-1/r}b(t)/a(t), v(t) = t^{1/p-1/r}b(t)$$

for all $\tau, t \in (0,\infty)$. Then the inequality

$$\text{RHS}(3.1) \leqslant C\,\text{LHS}(3.1) \tag{3.5}$$

(where C is a positive constant independent of f) can be rewritten as (2.4). Thus by Lemma 2.3, inequality (3.5) holds if and only if (2.5) is satisfied. Moreover, for all $\varrho \in (0,\infty)$,

$$\begin{aligned} \text{LHS}(2.5) &\lesssim \left\|w(t)\int_0^\varrho k(t,\tau)\,\mathrm{d}\tau\right\|_{Q,(0,\varrho)} + \left\|w(t)\int_0^\varrho k(t,\tau)\,\mathrm{d}\tau\right\|_{Q,(\varrho,\infty)} \\ &:= L_1 + L_2. \end{aligned} \tag{3.6}$$

Then

$$L_1 = \left\|t^{1/q-1/r}\frac{b(t)}{a(t)}\int_0^t \tau^{1/p-1/q-1}a(\tau)\,\mathrm{d}\tau\right\|_{r,(0,\varrho)},$$

and on using (3.2) with $s = 1$ we arrive at

$$L_1 \approx \left\|t^{1/p-1/r}b(t)\right\|_{r,(0,\varrho)} \quad \text{for all } \varrho \in (0,\infty). \tag{3.7}$$

In a similar way we obtain

$$\begin{aligned} L_2 &= \left\|t^{1/q-1/r}\frac{b(t)}{a(t)}\int_0^\varrho \tau^{1/p-1/q-1}a(\tau)\,\mathrm{d}\tau\right\|_{r,(\varrho,\infty)} \\ &\approx \varrho^{1/p-1/q}a(\varrho)\left\|t^{1/q-1/r}\frac{b(t)}{a(t)}\right\|_{r,(\varrho,\infty)}. \end{aligned}$$

Since $q < 0$, Lemma 2.1 (iii) implies that

$$\left\|t^{1/q-1/r}\frac{b(t)}{a(t)}\right\|_{r,(\varrho,\infty)} \approx \varrho^{1/q}b(\varrho)/a(\varrho) \quad \text{for all } \varrho \in (0,\infty).$$

Hence

$$L_2 \approx \varrho^{1/p}b(\varrho) \quad \text{for all } \varrho \in (0,\infty). \tag{3.8}$$

We claim that

$$L_2 \lesssim L_1. \tag{3.9}$$

To justify this, take $\varepsilon \in (0, \infty)$ and observe that for all $\varrho \in (0, \infty)$,

$$\|t^{1/p-1/r} b(t)\|_{r,(0,\varrho)} \geqslant \|t^{1/p+\varepsilon-1/r} t^{-\varepsilon} b(t)\|_{r,(\varrho/2,\varrho)}. \tag{3.10}$$

Since the function $t \longmapsto t^{-\varepsilon} b(t)$ is equivalent to some $g \in \mathcal{M}(0, \infty; \downarrow)$ on $(0, \infty)$, we see that

$$\mathrm{RHS}(3.10) \gtrsim \varrho^{-\varepsilon} b(\varrho) \|t^{1/p+\varepsilon-1/r}\|_{r,(\varrho/2,\varrho)} \approx \varrho^{1/p} b(\varrho) \quad \text{for all } \varrho \in (0, \infty).$$

Together with (3.10), this gives

$$\|t^{1/p-1/r} b(t)\|_{r,(0,\varrho)} \gtrsim \varrho^{1/p} b(\varrho) \quad \text{for all } \varrho \in (0, \infty).$$

Our claim (3.9) now follows on using (3.7) and (3.8).

By (3.6), (3.9) and (3.7),

$$\mathrm{LHS}(2.5) \lesssim \|t^{1/p-1/r} b(t)\|_{r,(0,\varrho)} = \mathrm{RHS}(2.5) \quad \text{for all } \varrho \in (0, \infty).$$

Hence (2.5) is satisfied, and so (3.5) also holds. This completes the proof of case (i).

(ii) Assume now that $s = \infty$. Then

$$\mathrm{RHS}(3.1) = \left\| t^{1/q-1/r} \frac{b(t)}{a(t)} \operatorname*{ess\,sup}_{\tau \in (0,t)} \tau^{1/p-1/q} a(\tau) f^*(\tau) \right\|_{r,(0,\infty)}.$$

Using the estimate

$$\begin{aligned}
\operatorname*{ess\,sup}_{\tau \in (0,t)} \tau^{1/p-1/q} a(\tau) f^*(\tau) &\approx \operatorname*{ess\,sup}_{\tau \in (0,t)} \left(\int_0^\tau \sigma^{1/p-1/q-1} a(\sigma) \, \mathrm{d}\sigma \right) f^*(\tau) \\
&\lesssim \operatorname*{ess\,sup}_{\tau \in (0,t)} \int_0^\tau \sigma^{1/p-1/q-1} a(\sigma) f^*(\sigma) \, \mathrm{d}\sigma \\
&= \int_0^t \sigma^{1/p-1/q-1} a(\sigma) f^*(\sigma) \, \mathrm{d}\sigma
\end{aligned}$$

for all $f \in \mathcal{M}(\Omega, \mu)$ and every $t \in (0, \infty)$, we find that

$$\mathrm{RHS}(3.1) \lesssim \left\| t^{1/q-1/r} \frac{b(t)}{a(t)} \int_0^t \sigma^{1/p-1/q-1} a(\sigma) f^*(\sigma) \, \mathrm{d}\sigma \right\|_{r,(0,\infty)} = \mathrm{RHS}(3.3).$$

The result now follows from part (i).

(iii) Suppose finally that $s \in (0,\infty)$. Let $f \in \mathcal{M}(\Omega,\mu)$. Putting $h = |f|^s$, $P = p/s$, $Q = q/s$, $R = r/s$, $\widetilde{b} = b^s$ and $\widetilde{a} = a^s$, we have

$$\text{RHS(3.1)} = \left\| t^{1/Q-1/R} \frac{\widetilde{b}(t)}{\widetilde{a}(t)} \int_0^t \tau^{1/P-1/Q-1} \widetilde{a}(\tau) h^*(\tau)\,\mathrm{d}\tau \right\|_{R,(0,\infty)}^{1/s}. \tag{3.11}$$

Applying the same method as that used to estimate RHS(3.3), we obtain

$$\text{RHS(3.11)} \lesssim \|t^{1/P-1/R}\widetilde{b}(t)h^*(t)\|_{R,(0,\infty)}^{1/s} = \text{LHS(3.1)},$$

which, together with (3.11), implies that RHS(3.1) $\lesssim$ LHS(3.1). □

Now we turn to the situation in which the parameter q is positive.

Theoerem 3.2. *Let $p, r, s \in (0,\infty]$, $q \in (0,\infty)$ and suppose that $a, b \in SV$, $0 \not\equiv a \not\equiv \infty$. If $p = q$ and $r \in (0,1)$, we additionally suppose that a is equivalent to a monotone function on $(0,\infty)$. Then for all $f \in \mathcal{M}(\Omega,\mu)$,*

$$\|f\|_{p,r,b} \approx \left\| t^{1/q-1/r} \frac{b(t)}{a(t)} \|\tau^{1/p-1/q-1/s} a(\tau) f^*(\tau)\|_{s,(t,\infty)} \right\|_{r,(0,\infty)}. \tag{3.12}$$

P r o o f. This follows the general line of the proof of Theorem 3.1 but with additional technical complications. Let $f \in \mathcal{M}(\Omega,\mu)$. Then $f^* \in \mathcal{M}(0,\infty;\downarrow)$ and for all $t \in (0,\infty)$,

$$\begin{aligned} \|\tau^{1/p-1/q-1/s} a(\tau) f^*(\tau)\|_{s,(t,\infty)} &\geqslant \|\tau^{1/p-1/q-1/s} a(\tau) f^*(\tau)\|_{s,(t,2t)} \\ &\geqslant f^*(2t) \|\tau^{1/p-1/q-1/s} a(\tau)\|_{s,(t,2t)}. \end{aligned} \tag{3.13}$$

Furthermore, by Lemma 2.1 (ii) (with $\varepsilon = 1$ and $\kappa = \tau/t$) we have for all $\tau \in (t, 2t)$ that

$$a(\tau) = a\Big(\frac{\tau}{t}t\Big) \leqslant C_1 \max\Big\{\frac{t}{\tau}, \frac{\tau}{t}\Big\} a(t) \leqslant C_1 \max\Big\{\frac{t}{t}, \frac{2t}{t}\Big\} a(t) = 2C_1 a(t)$$

and

$$a(\tau) = a\Big(\frac{\tau}{t}t\Big) \geqslant c_1 \min\Big\{\frac{t}{\tau}, \frac{\tau}{t}\Big\} a(t) \geqslant c_1 \min\Big\{\frac{t}{2t}, \frac{t}{t}\Big\} a(t) = \frac{1}{2} c_1 a(t),$$

which means that

$$a(\tau) \approx a(t) \quad \text{for all } \tau \in (t, 2t) \text{ and every } t \in (0,\infty). \tag{3.14}$$

Thus for all $t \in (0, \infty)$,

$$\|\tau^{1/p-1/q-1/s}a(\tau)\|_{s,(t,2t)} \approx a(t)\|\tau^{1/p-1/q-1/s}\|_{s,(t,2t)} \approx a(t)t^{1/p-1/q}.$$

Together with (3.13), the last estimate implies that

$$\begin{aligned} \text{(3.15)} \qquad \mathrm{RHS}(3.12) &\gtrsim \left\| t^{1/q-1/r}\frac{b(t)}{a(t)}a(t)t^{1/p-1/q}f^*(2t)\right\|_{r,(0,\infty)} \\ &= \|t^{1/p-1/r}b(t)f^*(2t)\|_{r,(0,\infty)} \approx \mathrm{LHS}(3.12), \end{aligned}$$

the final estimate following from a change of variables and use of Lemma 2.1 (ii).

To establish the reverse estimate, we distinguish several cases.

(i) Let $s = 1$. Then

$$\text{(3.16)} \qquad \mathrm{RHS}(3.12) = \left\| t^{1/q-1/r}\frac{b(t)}{a(t)}\int_t^\infty \tau^{1/p-1/q-1}a(\tau)f^*(\tau)\,\mathrm{d}\tau\right\|_{r,(0,\infty)}.$$

If $r \in [1, \infty]$, we apply the Hardy-type inequality (2.3) (with $P = r$, $\nu = 1/q$, $d = b/a$ and $g(\tau) = \tau^{1/p-1/q-1}a(\tau)f^*(\tau)$) to show that for all $f \in \mathcal{M}(\Omega, \mu)$,

$$\text{(3.17)} \qquad \mathrm{RHS}(3.16) \lesssim \|t^{1/p-1/r}b(t)f^*(t)\|_{r,(0,\infty)} = \mathrm{LHS}(3.12).$$

If $r \in (0, 1)$, we put $P = Q = r$ and

$$k(t,\tau) = \chi_{(t,\infty)}(\tau)\tau^{1/p-1/q-1}a(\tau),\ w(t) = t^{1/q-1/r}b(t)/a(t),\ v(t) = t^{1/p-1/r}b(t)$$

for all $t, \tau \in (0, \infty)$. Then the inequality

$$\text{(3.18)} \qquad \mathrm{RHS}(3.12) \leqslant C\ \mathrm{LHS}(3.12)$$

(where C is a positive constant independent of f) can be rewritten as (2.4). Consequently, by Lemma 2.3, inequality (2.4) holds if and only if (2.5) is satisfied. Moreover, for all $\varrho \in (0, \infty)$,

$$\begin{aligned} \text{(3.19)} \quad \mathrm{LHS}(2.5) &\lesssim \left\|w(t)\int_0^\varrho k(t,\tau)\,\mathrm{d}\tau\right\|_{Q,(0,\varrho)} + \left\|w(t)\int_0^\varrho k(t,\tau)\,\mathrm{d}\tau\right\|_{Q,(\varrho,\infty)} \\ &=: L_1 + L_2. \end{aligned}$$

In our case we have

$$\text{(3.20)} \qquad L_1 = \left\| t^{1/q-1/r}\frac{b(t)}{a(t)}\int_t^\varrho \tau^{1/p-1/q-1}a(\tau)\,\mathrm{d}\tau\right\|_{r,(0,\varrho)}$$

and

$$L_2 = \left\| t^{1/q-1/r} \frac{b(t)}{a(t)} \int_0^{\varrho} \chi_{(t,\infty)}(\tau) \tau^{1/p-1/q-1} a(\tau)\,\mathrm{d}\tau \right\|_{r,(\varrho,\infty)} = 0.$$

Since RHS(2.5) $= C \left\| t^{1/p-1/r} b(t) \right\|_{r,(0,\varrho)}$, (2.5) can be rewritten as

$$\left\| t^{1/q-1/r} \frac{b(t)}{a(t)} \int_t^{\varrho} \tau^{1/p-1/q-1} a(\tau)\,\mathrm{d}\tau \right\|_{r,(0,\varrho)} \leqslant C \| t^{1/p-1/r} b(t) \|_{r,(0,\varrho)} \tag{3.21}$$

for all $\varrho \in (0,\infty)$. To verify (3.21), we distinguish three cases.
(i-1) Assume that $1/p - 1/q < 0$. Then, by Lemma 2.1 (iii),

$$\int_t^{\varrho} \tau^{1/p-1/q-1} a(\tau)\,\mathrm{d}\tau \leqslant \int_t^{\infty} \tau^{1/p-1/q-1} a(\tau)\,\mathrm{d}\tau \approx t^{1/p-1/q} a(t)$$

for all $t \in (0,\varrho)$. Hence

$$\text{LHS(3.21)} \lesssim \| t^{1/p-1/r} b(t) \|_{r,(0,\varrho)},$$

which means that (3.21) holds.
(i-2) Suppose that $1/p - 1/q > 0$. Then $p < q$ and since $q < \infty$, we see that $p < \infty$. Therefore by Lemma 2.1 (iii),

$$\| t^{1/p-1/r} b(t) \|_{r,(0,\varrho)} \approx \varrho^{1/p} b(\varrho) \text{ for all } \varrho \in (0,\infty). \tag{3.22}$$

Moreover, again by Lemma 2.1 (iii),

$$\int_t^{\varrho} \tau^{1/p-1/q-1} a(\tau)\,\mathrm{d}\tau \leqslant \int_0^{\varrho} \tau^{1/p-1/q-1} a(\tau)\,\mathrm{d}\tau \approx \varrho^{1/p-1/q} a(\varrho) \text{ for all } t \in (0,\varrho),$$

and since $1/q > 0$ this implies that

$$\begin{aligned} \text{LHS(3.21)} &\lesssim \varrho^{1/p-1/q} a(\varrho) \left\| t^{1/q-1/r} \frac{b(t)}{a(t)} \right\|_{r,(0,\varrho)} \\ &\approx \varrho^{1/p-1/q} a(\varrho) \varrho^{1/q} b(\varrho)/a(\varrho) = \varrho^{1/p} b(\varrho) \text{ for all } \varrho \in (0,\infty). \end{aligned} \tag{3.23}$$

Estimates (3.22) and (3.23) show that (3.21) is satisfied.
(i-3) Finally we assume that $p = q$. Then $p < \infty$ and so (3.22) holds. Moreover,

$$\begin{aligned} \text{LHS(3.21)} &\lesssim \left\| t^{1/p-1/r} \frac{b(t)}{a(t)} \int_t^{\varrho} \tau^{-1} a(\tau)\,\mathrm{d}\tau \right\|_{r,(0,\varrho/2)} \\ &\quad + \left\| t^{1/p-1/r} \frac{b(t)}{a(t)} \int_t^{\varrho} \tau^{-1} a(\tau)\,\mathrm{d}\tau \right\|_{r,(\varrho/2,\varrho)} \\ &=: L_{11} + L_{12}. \end{aligned} \tag{3.24}$$

To estimate L_{12}, we deduce from Lemma 2.1 (ii) that

$$a(\tau) \approx a(\varrho) \approx a(t) \quad \text{for all } \tau, t \in (\varrho/2, \varrho). \tag{3.25}$$

Hence, for all $\tau, t \in (\varrho/2, \varrho)$,

$$\int_t^{\varrho} \tau^{-1} a(\tau)\, \mathrm{d}\tau \approx a(t) \int_t^{\varrho} \tau^{-1}\, \mathrm{d}\tau = a(t)(\ln \varrho - \ln t). \tag{3.26}$$

Using the mean value theorem we obtain, for some $t_\theta \in (t, \varrho)$,

$$\ln \varrho - \ln t = (\varrho - t)/t_\theta \leqslant (\varrho - t)/t \leqslant 1 \quad \text{for all } t \in (\varrho/2, \varrho). \tag{3.27}$$

The combination of estimates (3.25)–(3.27) gives

$$L_{12} \lesssim \|t^{1/p-1/r} b(t)\|_{r,(\varrho/2,\varrho)} \approx \varrho^{1/p} b(\varrho). \tag{3.28}$$

To estimate L_{11}, assume first that

$$a \text{ is equivalent to a non-increasing function on } (0, \infty). \tag{3.29}$$

Then

$$\int_t^{\varrho} \tau^{-1} a(\tau)\, \mathrm{d}\tau \lesssim a(t)(\ln \varrho - \ln t) \quad \text{for all } t \in (0, \varrho/2),$$

and so

$$L_{11} \leqslant \|t^{1/p-1/r} b(t)(\ln \varrho - \ln t)\|_{r,(0,\varrho/2)}. \tag{3.30}$$

Furthermore, integration by parts and Lemma 2.1 (iii) give

$$\begin{aligned} (\mathrm{RHS}(3.30))^r &\approx [(t^{1/p} b(t)(\ln \varrho - \ln t))^r]_0^{\varrho/2} \\ &\quad + \int_0^{\varrho/2} (t^{1/p-1/r} b(t))^r (\ln \varrho - \ln t)^{r-1}\, \mathrm{d}t \\ &\approx (\varrho^{1/p} b(\varrho))^r + \int_0^{\varrho/2} (t^{1/p-1/r} b(t))^r (\ln \varrho - \ln t)^{r-1}\, \mathrm{d}t. \end{aligned} \tag{3.31}$$

By the mean value theorem, for some $t_\theta \in (t, \varrho)$,

$$\ln \varrho - \ln t = (\varrho - t)/t_\theta \geqslant (\varrho - t)/\varrho = 1 - t/\varrho \geqslant 1/2 \quad \text{for all } t \in (0, \varrho/2). \tag{3.32}$$

Since $r \in (0, 1)$, the last estimate and (3.31) give

$$(\mathrm{RHS}(3.30))^r \lesssim (\varrho^{1/p} b(\varrho))^r + \int_0^{\varrho/2} (t^{1/p-1/r} b(t))^r dt \approx (\varrho^{1/p} b(\varrho))^r.$$

Thus

$$L_{11} \lesssim \varrho^{1/p} b(\varrho). \tag{3.33}$$

Next, assume that

$$a \text{ is equivalent to a non-decreasing function on } (0,\infty). \tag{3.34}$$

Then using integration by parts and properties of slowly varying functions, we obtain

$$\begin{aligned} L_{11}^r &\approx \left[\left(t^{1/p}\frac{b(t)}{a(t)}\int_t^{\varrho} \tau^{-1}a(\tau)\,\mathrm{d}\tau\right)^r\right]_0^{\varrho/2} \\ &\quad + \int_0^{\varrho/2}\left(t^{1/p}\frac{b(t)}{a(t)}\right)^r \left(\int_t^{\varrho}\tau^{-1}a(\tau)\,\mathrm{d}\tau\right)^{r-1} t^{-1}a(t)\,\mathrm{d}t \\ &=: N_1 + N_2. \end{aligned} \tag{3.35}$$

Take $\varepsilon \in (0,\infty)$. Then using properties of slowly varying functions, we find that

$$\begin{aligned} N_1 &= \lim_{t\to\varrho/2-}\left(t^{1/p}\frac{b(t)}{a(t)}\int_t^{\varrho}\tau^{-1}a(\tau)\,\mathrm{d}\tau\right)^r - \lim_{t\to 0+}\left(t^{1/p}\frac{b(t)}{a(t)}\int_t^{\varrho}\tau^{-1}a(\tau)\,\mathrm{d}\tau\right)^r \\ &= \lim_{t\to\varrho/2-}\left(t^{1/p}\frac{b(t)}{a(t)}\int_t^{\varrho}\tau^{-\varepsilon}a(\tau)\tau^{\varepsilon-1}\,\mathrm{d}\tau\right)^r \\ &\quad - \lim_{t\to 0+}\left(t^{1/p}\frac{b(t)}{a(t)}\int_t^{\varrho}\tau^{\varepsilon}a(\tau)\tau^{-\varepsilon-1}\,\mathrm{d}\tau\right)^r \\ &\lesssim \lim_{t\to\varrho/2-}\left(t^{1/p}\frac{b(t)}{a(t)}t^{-\varepsilon}a(t)\varrho^{\varepsilon}\right)^r - \lim_{t\to 0+}\left(t^{1/p}\frac{b(t)}{a(t)}t^{\varepsilon}a(t)(t^{-\varepsilon}-\varrho^{-\varepsilon})\right)^r \\ &\approx (\varrho^{1/p}b(\varrho))^r - \lim_{t\to 0+}(t^{1/p}b(t)(1-(t/\varrho)^{\varepsilon}))^r = (\varrho^{1/p}b(\varrho))^r. \end{aligned} \tag{3.36}$$

Since $r \in (0,1)$ and (3.34) holds, we have

$$\begin{aligned} N_2 &\lesssim \int_0^{\varrho/2}\left(t^{1/p-1/r}\frac{b(t)}{a(t)}\right)^r \left(a(t)\int_t^{\varrho}\tau^{-1}\,\mathrm{d}\tau\right)^{r-1} a(t)\,\mathrm{d}t \\ &= \int_0^{\varrho/2}(t^{1/p-1/r}b(t))^r(\ln\varrho - \ln t)^{r-1}dt. \end{aligned}$$

Together with (3.32) this implies that

$$N_2 \lesssim \int_0^{\varrho/2}(t^{1/p-1/r}b(t))^r\,\mathrm{d}t \approx (\varrho^{1/p}b(\varrho))^r. \tag{3.37}$$

The estimates (3.35)–(3.37) show that again (3.33) holds. The desired inequality (3.21) follows from (3.24), (3.28), (3.33) and (3.22).

(ii) Suppose that $s=\infty$. Then

$$\mathrm{RHS}(3.12)=\Big\|t^{1/q-1/r}\frac{b(t)}{a(t)}\operatorname*{ess\,sup}_{\tau\in(t,\infty)}\tau^{1/p-1/q}a(\tau)f^*(\tau)\Big\|_{r,(0,\infty)}.$$

Using the estimate

$$\begin{aligned}\operatorname*{ess\,sup}_{\tau\in(t,\infty)}\tau^{1/p-1/q}a(\tau)f^*(\tau)&\approx\operatorname*{ess\,sup}_{\tau\in(t,\infty)}\bigg(\int_{\tau/2}^{\tau}\sigma^{1/p-1/q-1}\,\mathrm{d}\sigma\bigg)a(\tau)f^*(\tau)\\&\approx\operatorname*{ess\,sup}_{\tau\in(t,\infty)}\bigg(\int_{\tau/2}^{\tau}\sigma^{1/p-1/q-1}a(\sigma)\,\mathrm{d}\sigma\bigg)f^*(\tau)\\&\leqslant\operatorname*{ess\,sup}_{\tau\in(t,\infty)}\int_{\tau/2}^{\infty}\sigma^{1/p-1/q-1}a(\sigma)f^*(\sigma)\,\mathrm{d}\sigma\\&=\int_{t/2}^{\infty}\sigma^{1/p-1/q-1}a(\sigma)f^*(\sigma)\,\mathrm{d}\sigma\end{aligned}$$

for all $f\in\mathcal{M}(\Omega,\mu)$ and every $t\in(0,\infty)$, we find that

$$\mathrm{RHS}(3.12)\lesssim\bigg\|t^{1/q-1/r}\frac{b(t)}{a(t)}\int_{t/2}^{\infty}\sigma^{1/p-1/q-1}a(\sigma)f^*(\sigma)\,\mathrm{d}\sigma\bigg\|_{r,(0,\infty)}\approx\mathrm{RHS}(3.16).$$

The result now follows from case (i).

(iii) Suppose finally that $s\in(0,\infty)$. Let $f\in\mathcal{M}(\Omega,\mu)$. Putting $h=|f|^s$, $P=p/s$, $Q=q/s$, $R=r/s$, $\widetilde{b}=b^s$ and $\widetilde{a}=a^s$, we obtain

$$\mathrm{RHS}(3.12)=\bigg\|t^{1/Q-1/R}\frac{\widetilde{b}(t)}{\widetilde{a}(t)}\int_t^{\infty}\tau^{1/P-1/Q-1}\widetilde{a}(\tau)h^*(\tau)\,\mathrm{d}\tau\bigg\|_{R,(0,\infty)}^{1/s}.\tag{3.38}$$

Application of the same method as that used to estimate RHS(3.16) gives

$$\mathrm{RHS}(3.38)\lesssim\|t^{1/P-1/R}\widetilde{b}(t)h^*(t)\|_{R,(0,\infty)}^{1/s}=\mathrm{LHS}(3.12),$$

which, together with (3.38), implies that RHS(3.12) $\lesssim$ LHS(3.12) and completes the proof of the theorem. □

Remark 3.3. To explain the idea behind (3.1), let $X=L^r(v)$, where the weight v is given by $v(t)=t^{1/p-1/r}b(t)$ $(t\in(0,\infty))$ and $L^r(v)$ is the weighted Lebesgue space defined by

$$L^r(v)=\{g\in\mathcal{M}(0,\infty)\colon\ \|g\|_X<\infty\},$$

where $\|g\|_X := \|gv\|_{r,(0,\infty)}$. Consider the weighted averaging operator T given by

$$(Tg)(t) = \frac{1}{t^{1/p-1/q}a(t)}\|\tau^{1/p-1/q-1/s}a(\tau)g(\tau)\|_{s,(0,t)},\ t \in (0,\infty). \tag{3.39}$$

Then (cf. the proof of Theorem 3.1)

(a) T is bounded on X if $r \in [1,\infty]$;
(b) T is bounded on $X \cap \mathcal{M}^+(0,\infty;\downarrow)$ if $r \in (0,1)$;
(c) T has a bounded inverse on $X \cap \mathcal{M}^+(0,\infty;\downarrow)$.

Consequently

$$\|g\|_X \approx \|Tg\|_X \quad \text{for all } g \in X \cap \mathcal{M}^+(0,\infty;\downarrow). \tag{3.40}$$

Thus if $f \in \mathcal{M}(\Omega,\mu)$, the estimate (3.1) follows from (3.40) on putting $g = f^*$, since

$$\|g\|_X = \|t^{1/p-1/r}b(t)f^*(t)\|_{r,(0,\infty)} = \|f\|_{p,r,b}$$

and

$$\|Tg\|_X = \left\|t^{1/q-1/r}\frac{b(t)}{a(t)}\|\tau^{1/p-1/q-1/s}a(\tau)f^*(\tau)\|_{s,(0,t)}\right\|_{r,(0,\infty)}.$$

Similarly, replacing the operator T from (3.39) by S, where

$$(Sg)(t) = \frac{1}{t^{1/p-1/q}a(t)}\|\tau^{1/p-1/q-1/s}a(\tau)g(\tau)\|_{s,(t,\infty)},\ t \in (0,\infty),$$

one can explain the idea behind (3.12).

Remark 3.4. Expressions similar to RHS(3.1) (or RHS(3.12)), with the limiting value $q = \infty$ (or $q = -\infty$) appeared in [11] in connection with "limiting" real interpolation to define spaces that, in general, differ from Lorentz-Zygmund ones. Our results show that in the non-limiting case (that is, when q is finite) the situation is quite different.

Now define the spaces $L_{(p,r,b)}$ by

$$L_{(p,r,b)} := \{f \in \mathcal{M}(\Omega,\mu)\colon\ \|f\|_{(p,r,b)} < \infty\},$$

where

$$\|f\|_{(p,r,b)} := \|t^{1/p-1/r}b(t)f^{**}(t)\|_{r,(0,\infty)}.$$

Note that $\|f\|_{(p,r,b)}$ is obtained from $\|f\|_{p,r,b}$ by replacing f^* by f^{**}. In the proofs of Theorems 3.1 and 3.2 only two facts concerning f^* were used: f^* is non-increasing and it is right-continuous on $(0,\infty)$. Since f^{**} has both these properties, the following variants of Theorems 3.1 and 3.2 hold.

Theorem 3.5. *Let $p, r, s \in (0, \infty]$, $q \in (-\infty, 0)$ and suppose that $a, b \in SV$, $0 \not\equiv a \not\equiv \infty$. Then for all $f \in \mathcal{M}(\Omega, \mu)$,*

$$\|f\|_{(p,r,b)} \approx \Big\| t^{1/q-1/r} \frac{b(t)}{a(t)} \|\tau^{1/p-1/q-1/s} a(\tau) f^{**}(\tau)\|_{s,(0,t)} \Big\|_{r,(0,\infty)}. \tag{3.41}$$

Theorem 3.6. *Let $p, r, s \in (0, \infty]$, $q \in (0, \infty)$ and suppose that $a, b \in SV$, $0 \not\equiv a \not\equiv \infty$. If $p = q$ and $r \in (0, 1)$, we additionally suppose that a is equivalent to a monotone function on $(0, \infty)$. Then for all $f \in \mathcal{M}(\Omega, \mu)$,*

$$\|f\|_{(p,r,b)} \approx \Big\| t^{1/q-1/r} \frac{b(t)}{a(t)} \|\tau^{1/p-1/q-1/s} a(\tau) f^{**}(\tau)\|_{s,(t,\infty)} \Big\|_{r,(0,\infty)}. \tag{3.42}$$

In the following remark we present some particular cases of Theorems 3.1 and 3.2. The result mentioned in part (i) of this remark will be used in the next section.

Remark 3.7. (i) Let $1 < p \leqslant \infty$, $0 < r \leqslant \infty$ and $b \in SV$. Then, by Theorem 3.1 (with $1/q = -1 + 1/p$, $a \equiv 1$ and $s = 1$),

$$\|f\|_{p,r,b} \approx \|t^{1/p-1/r} b(t) f^{**}(t)\|_{r,(0,\infty)} = \|f\|_{(p,r,b)} \tag{3.43}$$

for all $f \in \mathcal{M}(\Omega, \mu)$. (Note that (3.43) with $b \equiv 1$ corresponds to Theorem 6 of [4].)

(ii) Let $0 < p < \infty$, $0 < r \leqslant \infty$ and $b \in SV$. Then, by Theorem 3.2 (with $q = p$, $a \equiv 1$ and $s = 1$),

$$\|f\|_{p,r,b} \approx \Big\| t^{1/p-1/r} b(t) \int_t^\infty \tau^{-1} f^*(\tau)\, \mathrm{d}\tau \Big\|_{r,(0,\infty)} \tag{3.44}$$

for all $f \in \mathcal{M}(\Omega, \mu)$ (which is a "dual result" to (3.43)).

(iii) Let $0 < p < 1$, $0 < r \leqslant \infty$ and $b \in SV$. Then, by Theorem 3.2 (with $1/q = -1 + 1/p$, $a \equiv 1$ and $s = 1$),

$$\|f\|_{p,r,b} \approx \Big\| t^{1/p-1/r} b(t) \Big(\frac{1}{t} \int_t^\infty f^*(\tau)\, \mathrm{d}\tau \Big) \Big\|_{r,(0,\infty)} \tag{3.45}$$

for all $f \in \mathcal{M}(\Omega, \mu)$ (which is a counterpart of (3.43)).

(iv) Let $0 < p \leqslant \infty$, $0 < r < \infty$ and $b \in SV$, with b equivalent to a monotone function on $(0, \infty)$ if $p = r \in (0, 1)$. Then, by Theorem 3.2 (with $q = r$, $a = b$ and $s = \infty$),

$$\|f\|_{p,r,b} \approx \| \operatorname*{ess\,sup}_{\tau \in (t,\infty)} \tau^{1/p-1/r} b(\tau) f^*(\tau)\|_{r,(0,\infty)} \tag{3.46}$$

for all $f \in \mathcal{M}(\Omega, \mu)$.

(v) Let $0 < p \leqslant \infty$, $0 < r < \infty$ and $b \in SV$. Then, by Theorem 3.2 (with $q = r$, $a \equiv 1$ and $s = \infty$),

$$\|f\|_{p,r,b} \approx \|b(t) \operatorname*{ess\,sup}_{\tau \in (t,\infty)} \tau^{1/p-1/r} f^*(\tau)\|_{r,(0,\infty)} \tag{3.47}$$

for all $f \in \mathcal{M}(\Omega, \mu)$.

(vi) Let $0 < p < \infty$, $0 < r \leqslant \infty$ and $b \in SV$, with b equivalent to a monotone function on $(0, \infty)$ if $r \in (0, 1)$. Then, by Theorem 3.2 (with $q = p$, $a = b$ and $s = \infty$),

$$\|f\|_{p,r,b} \approx \|t^{1/p-1/r} \operatorname*{ess\,sup}_{\tau \in (t,\infty)} b(\tau) f^*(\tau)\|_{r,(0,\infty)} \tag{3.48}$$

for all $f \in \mathcal{M}(\Omega, \mu)$.

4. Maximal operators and Riesz potentials

Here we present interesting connections between Theorem 3.6 and the actions of these classical operators, and some of their generalisations, on the Lorentz-Karamata spaces we have been considering. Throughout this section the measure space (Ω, μ) will be taken to be $(\mathbb{R}^n, dx)$, and we shall assume that $\gamma \in [0, n)$ and $a \in SV$, $0 \not\equiv a \not\equiv \infty$, satisfy

$$\text{either } \gamma \in (0, n), \text{ or } \gamma = 0 \text{ and } a \approx d \in \mathcal{M}(0, \infty; \downarrow). \tag{4.1}$$

The *fractional maximal operator* $\mathcal{M}_{\gamma,a}$ is defined by

$$(\mathcal{M}_{\gamma,a} f)(x) = \sup_{Q \ni x} \frac{1}{|Q|^{1-\gamma/n} a(|Q|)} \int_Q |f(y)|\, \mathrm{d}y,\ f \in \mathcal{M}(\mathbb{R}^n),\ x \in \mathbb{R}^n, \tag{4.2}$$

where the supremum is taken over all cubes Q in $\mathbb{R}^n$ with sides parallel to the coordinate axes. When $a = 1$, this is just the usual fractional maximal operator, which becomes the classical Hardy-Littlewood maximal operator when $\gamma = 0$; if a is a power of the logarithm, then the operator becomes one of those fractional maximal operators studied in [8] and [16].

Theorem 4.1. *Suppose that $\gamma \in [0, n)$ and $a \in SV$, $0 \not\equiv a \not\equiv \infty$, satisfy (4.1). Then there is a positive constant C, depending only on n, γ and a such that, for all $f \in \mathcal{M}(\mathbb{R}^n)$ and every $t \in (0, \infty)$,*

$$(\mathcal{M}_{\gamma,a} f)^*(t) \leqslant C \sup_{t<\tau<\infty} \tau^{\gamma/n} \{a(\tau)\}^{-1} f^{**}(\tau). \tag{4.3}$$

Inequality (4.3) is sharp in the sense that for every $\varphi \in \mathcal{M}(0, \infty; \downarrow)$ there exists a function $f \in \mathcal{M}(\mathbb{R}^n)$ such that $f^* = \varphi$ a.e. on $(0, \infty)$ and, for all $t \in (0, \infty)$,

$$(\mathcal{M}_{\gamma,a} f)^*(t) \geqslant c \sup_{t<\tau<\infty} \tau^{\gamma/n} \{a(\tau)\}^{-1} f^{**}(\tau), \tag{4.4}$$

where c is positive constant which again depends only on n, γ and a.

The proof of this theorem can be carried out in a way similar to that of Theorem 3.1 in [8]; we omit the details.

Remark 4.2. Let the assumptions of Theorem 4.1 hold. Then the mappings

$$\mathcal{M}_{\gamma,a}\colon\ L^1(\mathbb{R}^n) \to L_{n/(n-\gamma),\infty,a}(\mathbb{R}^n) \tag{4.5}$$

and

$$\mathcal{M}_{\gamma,a}\colon\ L_{n/\gamma,\infty,1/a}(\mathbb{R}^n) \to L^\infty(\mathbb{R}^n) \tag{4.6}$$

are bounded. We refer to [8], Lemma 3.6 for a similar statement.

Now we turn to *operators of Riesz potential type.* Given

$$\gamma \in (0, n) \quad \text{and} \quad a \in SV,\ 0 \not\equiv a \not\equiv \infty, \tag{4.7}$$

let

$$\begin{aligned} (I_{\gamma,a} f)(x) &= (g_{\gamma,a} * f)(x) \\ &= \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\gamma}\, a(|x-y|)}\, \mathrm{d}y,\ \ f \in \mathcal{M}(\mathbb{R}^n),\ \ x \in \mathbb{R}^n, \end{aligned} \tag{4.8}$$

where

$$g_{\gamma,a}(x) = |x|^{\gamma-n} / a(|x|). \tag{4.9}$$

When a is the function identically equal to 1, $I_{\gamma,a}$ is just the classical Riesz potential I_γ. A routine computation shows that

$$g^*_{\gamma,a}(t) \approx t^{\gamma/n-1} / a(t^{1/n}). \tag{4.10}$$

By O'Neil's convolution inequality,

$$(I_{\gamma,a}f)^{**}(t) \leqslant t g^{**}_{\gamma,a}(t) f^{**}(t) + \int_t^\infty g^*_{\gamma,a}(\tau) f^*(\tau)\,\mathrm{d}\tau \quad \text{for all } t > 0.$$

Moreover, by (4.10),

$$g^{**}_{\gamma,a}(t) \approx g^*_{\gamma,a}(t) \approx t^{\gamma/n-1}/a(t^{1/n}) \quad \text{if } \gamma \in (0,n). \tag{4.11}$$

Hence

$$\begin{aligned}(I_{\gamma,a}f)^*(t) &\leqslant (I_{\gamma,a}f)^{**}(t) \lesssim \frac{t^{\gamma/n}}{a(t^{1/n})} f^{**}(t) + \int_t^\infty \frac{\tau^{\gamma/n-1}}{a(\tau^{1/n})} f^*(\tau)\,\mathrm{d}\tau \\ &\approx \int_t^\infty \frac{\tau^{\gamma/n-1}}{a(\tau^{1/n})} f^{**}(\tau)\,\mathrm{d}\tau \approx \int_t^\infty g^*_{\gamma,a}(\tau) f^{**}(\tau)\,\mathrm{d}\tau.\end{aligned} \tag{4.12}$$

To justify the last line, note that by Fubini's theorem,

$$\begin{aligned}\int_t^\infty g^*_{\gamma,a}(\tau) f^{**}(\tau)\,\mathrm{d}\tau &= \int_t^\infty \frac{g^*_{\gamma,a}(\tau)}{\tau}\left(\int_0^\tau f^*(\sigma)\,\mathrm{d}\sigma\right)\mathrm{d}\tau \\ &= \int_0^t f^*(\sigma)\left(\int_t^\infty \frac{g^*_{\gamma,a}(\tau)}{\tau}\,\mathrm{d}\tau\right)\mathrm{d}\sigma \\ &\quad + \int_t^\infty f^*(\sigma)\left(\int_\sigma^\infty \frac{g^*_{\gamma,a}(\tau)}{\tau}\,\mathrm{d}\tau\right) d\sigma \\ &\approx \frac{t^{\gamma/n-1}}{a(t^{1/n})}\int_0^t f^*(\sigma)\,\mathrm{d}\sigma + \int_t^\infty f^*(\sigma)\frac{\sigma^{\gamma/n-1}}{a(\sigma^{1/n})}\,\mathrm{d}\sigma,\end{aligned}$$

since

$$\int_s^\infty \frac{g^*_{\gamma,a}(\tau)}{\tau}\,\mathrm{d}\tau = \int_s^\infty \frac{\tau^{\gamma/n-2}}{a(\tau^{1/n})}\,\mathrm{d}\tau \approx \frac{s^{\gamma/n-1}}{a(s^{1/n})} \quad \text{for all } s > 0.$$

Thus by (4.12),

$$(I_{\gamma,a}f)^*(t) \lesssim \int_t^\infty \frac{\tau^{\gamma/n-1}}{a(\tau^{1/n})} f^{**}(\tau)\,\mathrm{d}\tau.$$

The arguments above lead to

Theorem 4.3. *Suppose that γ and a satisfy (4.7). Then there is a positive constant C, depending only on n, γ and a, such that, for all $f \in \mathcal{M}(\mathbb{R}^n)$ and every $t \in (0,\infty)$,*

$$\begin{aligned}(I_{\gamma,a}f)^*(t) &\leqslant C\int_t^\infty \frac{\tau^{\gamma/n-1}}{a(\tau^{1/n})} f^{**}(\tau)\,\mathrm{d}\tau \\ &\left(\approx \frac{t^{\gamma/n-1}}{a(t^{1/n})}\int_0^t f^*(\sigma)\,\mathrm{d}\sigma + \int_t^\infty f^*(\sigma)\frac{\sigma^{\gamma/n-1}}{a(\sigma^{1/n})}\,\mathrm{d}\sigma\right).\end{aligned} \tag{4.13}$$

This inequality is sharp in the sense that for every $\varphi \in \mathcal{M}^+(0,\infty;\downarrow)$ there is a function $f \in \mathcal{M}(\mathbb{R}^n)$ such that $f^* = \varphi$ a.e. on $(0,\infty)$ and, for all $t \in (0,\infty)$,

$$(I_{\gamma,a}f)^*(t) \geqslant c\int_t^\infty \frac{\tau^{\gamma/n-1}}{a(\tau^{1/n})} f^{**}(\tau)\,\mathrm{d}\tau, \tag{4.14}$$

where c is a positive constant which again depends only on n, γ and a.

P r o o f. That (4.13) holds follows from the arguments leading up to the theorem. The sharpness assertion (4.14) may be proved by arguments similar to those in the proof of Lemma 3.4 in [6]. □

Remark 4.4. It is interesting to compare the estimates (4.3) and (4.13) for $(\mathcal{M}_{\gamma,a}f)^*(t)$ and $(I_{\gamma,a}f)^*(t)$, respectively. The right-hand sides of both are of the form

$$\|\tau^{\gamma/n-1/s}d(\tau)f^{**}(\tau)\|_{s,(t,\infty)}$$

for some $s \in (0,\infty]$ and $d \in SV$: for (4.3), $s=\infty$ and $d = 1/a$, while for (4.13), $s = 1$ and $d(\tau) = 1/a(\tau^{1/n})$ $(\tau \in (0,\infty))$.

Remark 4.5. Assume that (4.7) holds. Then the mappings

$$I_{\gamma,a}\colon\ L^1(\mathbb{R}^n) \to L_{n/(n-\gamma),\infty,a(t^{1/n})}(\mathbb{R}^n) \tag{4.15}$$

and

$$I_{\gamma,a}\colon\ L_{n/\gamma,1,1/a(t^{1/n})}(\mathbb{R}^n) \to L^\infty(\mathbb{R}^n) \tag{4.16}$$

are bounded. (Compare (4.5) and (4.6) with (4.15) and (4.16).) Indeed, by (4.10),

$$\sup_{0<t<\infty} t^{1-\gamma/n}a(t^{1/n})g^*_{\gamma,a}(t) < \infty,$$

so that

$$g_{\gamma,a} \in L_{n/(n-\gamma),\infty,a(t^{1/n})}(\mathbb{R}^n) := X.$$

Hence

$$|(I_{\gamma,a}f)(x)| = \left|\int_{\mathbb{R}^n} g_{\gamma,a}(x-y)f(y)\,\mathrm{d}y\right| \leqslant \|g_{\gamma,a}\|_X\,\|f\|_{X'}$$

for all $f \in X' = L_{n/\gamma,1,1/a(t^{1/n})}(\mathbb{R}^n)$, and (4.16) follows. Since $I'_{\gamma,a} = I_{\gamma,a}$, we see that

$$I_{\gamma,a}\colon\ (L^\infty(\mathbb{R}^n))' \to \left(L_{n/\gamma,1,1/a(t^{1/n})}(\mathbb{R}^n)\right)',$$

which coincides with (4.15).

Next we turn to the mapping properties of our version of the Riesz potential.

Theorem 4.6. *Let $\gamma \in (0,n)$, $1 < p < n/\gamma$, $1/q = 1/p - \gamma/n$, $1 \leqslant r \leqslant \infty$ and $a, B \in SV$, $0 \not\equiv a \not\equiv \infty$. Then*

$$I_{\gamma,a}\colon\ L_{p,r,B(t)/a(t^{1/n})}(\mathbb{R}^n) \to L_{q,r,B}(\mathbb{R}^n). \tag{4.17}$$

Proof. For all $f \in L_{p,r,B(t)/a(t^{1/n})}(\mathbb{R}^n)$,

$$\|I_{\gamma,a}f\|_{q,r,B} = \|t^{1/q-1/r}B(t)\,(I_{\gamma,a}f)^*(t)\,\|_{r,(0,\infty)} \leqslant N_1 + N_2, \tag{4.18}$$

where

$$N_1 := \left\| t^{1/q-1/r}B(t)\frac{t^{\gamma/n-1}}{a(t^{1/n})}\int_0^t f^*(\sigma)\,\mathrm{d}\sigma \right\|_{r,(0,\infty)} \tag{4.19}$$

and

$$N_2 := \left\| t^{1/q-1/r}B(t)\int_t^\infty f^*(\sigma)\frac{\sigma^{\gamma/n-1}}{a(\sigma^{1/n})}\,\mathrm{d}\sigma \right\|_{r,(0,\infty)}. \tag{4.20}$$

Using Lemma 2.2 (i) together with the fact that $1/q = 1/p - \gamma/n$, we obtain

$$\begin{aligned} N_1 &= \left\| t^{1/p-1-1/r}\frac{B(t)}{a(t^{1/n})}\int_0^t f^*(\sigma)\,\mathrm{d}\sigma \right\|_{r,(0,\infty)} \\ &\lesssim \left\| t^{1/p-1/r}\frac{B(t)}{a(t^{1/n})}f^*(t) \right\|_{r,(0,\infty)} = \|f\|_{p,r,B(t)/a(t^{1/n})}. \end{aligned} \tag{4.21}$$

Similarly, applying Lemma 2.2 (ii), we arrive at

$$\begin{aligned} N_2 &\lesssim \left\| t^{1/q+1-1/r}B(t)f^*(t)\frac{t^{\gamma/n-1}}{a(t^{1/n})} \right\|_{r,(0,\infty)} \\ &= \left\| t^{1/p-1/r}\frac{B(t)}{a(t^{1/n})}f^*(t) \right\|_{r,(0,\infty)} = \|f\|_{p,r,B(t)/a(t^{1/n})}. \end{aligned} \tag{4.22}$$

The result now follows from (4.18), (4.21) and (4.22). $\square$

Corresponding to this we have the following for the fractional maximal operator.

Theorem 4.7. *Let $\gamma \in (0, n)$, $1 < p < n/\gamma$, $1/q = 1/p - \gamma/n$, $1 \leqslant r \leqslant \infty$ and $a, B \in SV$, $0 \not\equiv a \not\equiv \infty$, Then*

$$\mathcal{M}_{\gamma,a}\colon\ L_{p,r,B/a}(\mathbb{R}^n) \to L_{q,r,B}(\mathbb{R}^n). \tag{4.23}$$

Proof. For all $f \in L_{p,r,B/a}(\mathbb{R}^n)$,

$$\begin{aligned}\|\mathcal{M}_{\gamma,a} f\|_{q,r,B} &= \|t^{1/q-1/r} B(t) (\mathcal{M}_{\gamma,a} f)^*(t)\|_{r,(0,\infty)} \\ &\lesssim \|t^{1/q-1/r} B(t) \sup_{t<\tau<\infty} \tau^{\gamma/n} f^{**}(\tau)/a(\tau)\|_{r,(0,\infty)}.\end{aligned}$$

Since

$$\begin{aligned}\sup_{t<\tau<\infty} \tau^{\gamma/n} f^{**}(\tau)/a(\tau) &= \sup_{t<\tau<\infty} \frac{\tau^{\gamma/n-1}}{a(\tau)} \Big\{ \int_0^t f^*(\sigma)\,\mathrm{d}\sigma + \int_t^\tau f^*(\sigma)\,\mathrm{d}\sigma \Big\} \\ &\lesssim \frac{t^{\gamma/n-1}}{a(t)} \int_0^t f^*(\sigma)\,\mathrm{d}\sigma + \sup_{t<\tau<\infty} \int_t^\tau \frac{\sigma^{\gamma/n-1}}{a(\sigma)} f^*(\sigma)\,\mathrm{d}\sigma \\ &= \frac{t^{\gamma/n-1}}{a(t)} \int_0^t f^*(\sigma)\,\mathrm{d}\sigma + \int_t^\infty \frac{\sigma^{\gamma/n-1}}{a(\sigma)} f^*(\sigma)\,\mathrm{d}\sigma,\end{aligned}$$

we obtain

$$\|\mathcal{M}_{\gamma,a} f\|_{q,r,B} \lesssim N_1 + N_2, \tag{4.24}$$

where

$$N_1 := \left\| t^{1/q-1/r} B(t) \frac{t^{\gamma/n-1}}{a(t)} \int_0^t f^*(\sigma)\,\mathrm{d}\sigma \right\|_{r,(0,\infty)}$$

and

$$N_2 := \left\| t^{1/q-1/r} B(t) \int_t^\infty \frac{\sigma^{\gamma/n-1}}{a(\sigma)} f^*(\sigma)\,\mathrm{d}\sigma \right\|_{r,(0,\infty)}$$

(cf. (4.19) and (4.20)). Using the same arguments as those deployed in the proof of Theorem 4.6 to estimate the quantities (4.19) and (4.20), we find that

$$N_1 \lesssim \|f\|_{p,r,B/a} \quad \text{and} \quad N_2 \lesssim \|f\|_{p,r,B/a}. \tag{4.25}$$

The result now follows from (4.24) and (4.25). □

We conclude by mentioning some interesting connections between Theorem 3.6 and mapping properties of some classical operators and their variants. To explain these, assume that $(\Omega, \mu) = (\mathbb{R}^n, dx)$, $0 < p < q < \infty$ and $1/p < 1 + 1/q$. Put $\gamma = n(1/p - 1/q)$; then $\gamma \in (0, n)$. Let $a \in SV$, $0 \not\equiv a \not\equiv \infty$.

(i) Consider the maximal operator $\mathcal{M}_{\gamma,a}$. By Theorem 4.1,

$$\left(\mathcal{M}_{\gamma,1/a}f\right)^*(t) \lesssim \sup_{t<\tau<\infty} \tau^{\gamma/n}a(\tau)f^{**}(\tau)$$
$$(= \|\tau^{1/p-1/q-1/s}a(\tau)f^{**}(\tau)\|_{s,(t,\infty)} \text{ with } s=\infty).$$

Given the space $L_{q,r,b/a}(\mathbb{R}^n) =: Y$, where $b \in SV$ and $r \in [1,\infty]$, put $\overline{Y} = L_{q,r,b/a}((0,\infty))$. Then it can be shown that the space X,

$$X := \{f \in \mathcal{M}(\mathbb{R}^n)\colon \|f\|_X < \infty\}, \tag{4.26}$$

where

$$\|f\|_X := \|\sup_{t<\tau<\infty} \tau^{\gamma/n}a(\tau)f^{**}(\tau)\|_{\overline{Y}} \ (= \mathrm{RHS}(3.42) \text{ with } s=\infty)$$

is the largest rearrangement-invariant Banach function space which is mapped by $\mathcal{M}_{\gamma,1/a}$ into Y. On the other hand, by Theorem 4.7,

$$\mathcal{M}_{\gamma,1/a}\colon L_{(p,r,b)}(\mathbb{R}^n) \to L_{(q,r,b/a)}(\mathbb{R}^n) = Y.$$

(Note that, by (3.43), $L_{p,r,b}(\mathbb{R}^n) = L_{(p,r,b)}(\mathbb{R}^n)$ and $L_{q,r,b/a}(\mathbb{R}^n) = L_{(q,r,b/a)}(\mathbb{R}^n)$ when $p>1$ and $q>1$.) Hence $L_{(p,r,b)}(\mathbb{R}^n) \subset X$ and so (by [2], Chapter 1, Theorem 1.8), $L_{(p,r,b)}(\mathbb{R}^n) \hookrightarrow X$. However, by Theorem 3.6, $L_{(p,r,b)}(\mathbb{R}^n) = X$.

(ii) Put $\widetilde{a}(t) = 1/a(t^n)$ $(t \in (0,\infty))$ and consider the Riesz-type operator $I_{\gamma,\widetilde{a}}$. By Theorem 4.3,

$$\left(I_{\gamma,\widetilde{a}}f\right)^*(t) \lesssim \int_t^\infty \sigma^{\gamma/n-1}a(\sigma)f^{**}(\sigma)\,\mathrm{d}\sigma (= \|\sigma^{1/p-1/q-1}a(\sigma)f^{**}(\sigma)\|_{1,(t,\infty)}).$$

Given the space $L_{q,r,b/a}(\mathbb{R}^n) =: Y$, where $b \in SV$ and $r \in [1,\infty]$, put $\overline{Y} = L_{q,r,b/a}((0,\infty))$. Then it can be shown that the space X from (4.26), where now

$$\|f\|_X := \left\| \int_t^\infty \sigma^{1/p-1/q-1}a(\sigma)f^{**}(\sigma)\,\mathrm{d}\sigma \right\|_{\overline{Y}} \ (= \mathrm{RHS}(3.42) \text{ with } s=1),$$

is the largest rearrangement-invariant Banach function space which is mapped by $I_{\gamma,\widetilde{a}}$ into Y. However, by Theorem 4.6,

$$I_{\gamma,\widetilde{a}}\colon L_{(p,r,b)}(\mathbb{R}^n) \to L_{(q,r,b/a)}(\mathbb{R}^n) = Y,$$

and so $L_{(p,r,b)}(\mathbb{R}^n) \subset X$. By Theorem 3.6, $L_{(p,r,b)}(\mathbb{R}^n) = X$.

(iii) Here we take $(\Omega, \mu) = (\mathbb{R}, dx)$. Since (cf. Theorem 4.7 and Proposition 4.10 in Chapter 3 of [2])

$$(Hf)^*(t) \lesssim \int_t^\infty \sigma^{-1} f^{**}(\sigma)\,\mathrm{d}\sigma = t^{-1}\int_0^t f^*(\sigma)\,\mathrm{d}\sigma + \int_t^\infty \sigma^{-1} f^{**}(\sigma)\,\mathrm{d}\sigma$$

for all $t \in (0, \infty)$, where H is the Hilbert transform, defined by

$$(Hf)(x) = p.v.\int_{\mathbb{R}} \frac{f(y)}{x-y}\,\mathrm{d}y,\;\; x \in \mathbb{R},$$

one can similarly explain the connection between Theorem 3.6 with $q = p$, $s = 1$ and a identically equal to 1, and the mapping properties of the Hilbert transform.

Acknowledgements. This research was supported by The School of Mathematics at Cardiff University, by grant no. 201/05/2033 of the Grant Agency of the Czech Republic and by Institutional Research Plan no. AV0Z10190503 of the Academy of Sciences of the Czech Republic.

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