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RINGS WHICH HAVE PROJECTIVE COFLAT MODULE

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INTRODUCTION AND PRELIMINARIES

In [1] R. R. Colby has defined a ring R, as left (right) IF if every injective left (right) R-module is flat. In [2] R. F. Damiano has defined FC ring over which left (right) flat and left (right) coflat modules are precisely the same. Induced by these paper we have defined a ring R as left (right) PC if every left (right) projective R-module is coflat. A ring R is quasi-frobenius ring if R as a right R-module is noetherian and injective. In [3] U. Chase characterised a left perfect ring R as a ring over which every left flat R-module is projective.

Through-out this paper R will be understood a ring with unity, and every module over R will be unitary. Every right (left) R-module M is denoted by M_R $({}_RM)$ and dual of M is denoted by ${}_RM^*$ (M_R^*), i.e. $M^* = \mathrm{Hom}_R(M,R)$. S is the R-endomorphisms ring of M, i.e.

$$S = \mathrm{Hom}_R(M,M) = \mathrm{End}_R(M).$$

Trace of an R-module M on M is denoted by $T_M(M)$ and defined as

$$T_M(M) = \left\{\sum \mathrm{Im}\, f \colon f \in \mathrm{Hom}_R(M,M)\right\}.$$

Definition. A ring R is right (left) PC ring if every right (left) projective R-module is coflat. A ring R is PC ring if it is both right as well as left PC ring.

Obvioulsy every FC ring and every QF ring is PC ring.

Theorem 1. *For a ring R following statements are equivalent.*

1. *R is right PC ring.*
2. *Every right free module over R is coflat.*
3. *For every right free R module M, trace of M on M is coflat.*

Proof. $1 \Rightarrow 2$ Let R is right PC ring and M is a free module over R. Then obviousely it is coflat, since every free module is projective and over right PC ring every right projective module is coflat.

$2 \Rightarrow 3$ Let every free right R-module M is coflat. But M is a flat module [$\because$ Free $\Rightarrow$ Projective $\Rightarrow$ flat] hence ${}_sM \bigotimes_R R \cong {}_sM$ is a flat module. Therefore by [4, Lemma 1.3(2)] M generates all kernels of homomorphism $M^n \to M$, $n \in N$, which implies by [4, Lemma 1.3(1)] that the mapping $\emptyset'(M)$: $\mathrm{Hom}_R(M, M) \bigotimes_s M \to T_M(M)$ is an isomorphism or

$$S \bigotimes_s M_R \quad \text{is isomorphhic to} \quad T_M(M)$$

i.e. M_R is isomrphic to $T_M(M)$ and so $T_M(M)$ is coflat.

$3 \Rightarrow 1$ Let for every free right R-module M, $T_M(M)$ is coflat. Since M is also flat, by the same argument as in $2 \Rightarrow 3$ we have $T_M(M)$ is isomorphic to M. Which implies very free right R-module M is coflat. Now let P be any projective right R module. Then P is a direct summand of a free module, say M. But M is coflat so by [2, Theorem 1.8] every direct summand of M is coflat i.e. P is coflat. Hence R is right PC ring. □

This theorem is true for left PC ring R and left free R-module M also.

Example 1. [2, R. F. Damiano, example 2.7]

Let $R = \mathbf{Z} \bigoplus \mathbf{Q}/\mathbf{Z}$ with multiplication defined by $(n_1, q_1) \cdot (n_2 q_2) = (n_1 n_2, n_1 q_2 + n_2 q_1)$, $n_i \in \mathbf{Z}$, $q_i \in \mathbf{Q}/\mathbf{Z}$. Then R is a commutative ring with Jacobson radical

$$J(R) = \{(n, q) \mid n = o\}.$$

More-over it is clear that every finitely generated ideal of R is principal. Hence R is PC ring, but $R/J(R) \cong \mathbf{Z}$ is not PC ring, as the homomorphism $f : n\mathbf{Z} \to \mathbf{Z}$ via $n\nu \to \nu$ can not extend over $\mathbf{Z}$, which implies $\mathbf{Z}$ over $\mathbf{Z}$ is not coflat.

Example 2. A left PC ring which is not ring PC.

(R. F. Damiano [2, Example 2.8]). Let R be an algebra over a field F with basis $\{1, e_0, e_1, e_2, \ldots, x_1, x_2, \ldots\}$ where for all i, j

$$\begin{aligned} e_i e_j &= \delta_{i,j} e_j, \\ x_i e_j &= \delta_{i,j+1} x_i, \\ e_i x_j &= \delta_{i,j} x_j, \\ x_i x_j &= 0. \end{aligned}$$

It is easy to see that R is left coherent and that every R-homomorphism

$$f\colon {}_RI \to {}_RR$$

extends to one over R. Then R is left coflat, and hence R is left PC ring.

However R is not right coflat since the homomorphism $x_iR \to e_oR$.

Via $x_ir \to e_or$ can not be extended over R. Thus every projective right R-module is not coflat. Therefore R is not right PC ring.

Theorem 2. *If R is right PC ring for a finitely generated projective right R-module P following are true.*

1. *$S = \mathrm{Hom}_R(P, P)$ is coflat.*
2. *$P^* = \mathrm{Hom}_R(P, R)$ is coflat.*

Proof. (1) Let $o \to I \to S$ be an exact sequence with I finitely generated right ideal of S. Now P_R is projective so P_R is flat. Hence ${}_sP \bigotimes_R R \cong {}_sP$ is flat. So we have an exact sequence $o \to I \bigotimes_s P \to S \bigotimes_s P$ with $I \bigotimes_s P$ finitely generated. Again P_R is coflat module [$\because$ R is right PC ring]. So consider the following commutative diagram

$$\begin{array}{ccccc}
\mathrm{Hom}_R(S \bigotimes_s P, P) & \longrightarrow & \mathrm{Hom}_R(I \bigotimes_s P, P) & \longrightarrow & 0 \\
\big\downarrow \wr & & \big\downarrow \wr & & \\
\mathrm{Hom}_s(S, \mathrm{Hom}(P, P)) & \longrightarrow & \mathrm{Hom}_s(I, \mathrm{Hom}(P, P)) & \longrightarrow & 0
\end{array}$$

The exactness of bottem row implies that $\mathrm{Hom}_R(P, P)$ is coflat.

(2) Similar to the proof of (1). □

Above theorem is true for the left PC ring also.

Theorem 3. *If R is any right PC ring, the ring of endomorphisms of a finitely generated projective right R-module P is also PC ring.*

Proof. Let S be the ring of endomorphism of P i.e. $S = \mathrm{Hom}_R(P, P)$. Let Q be any left (right) projective module over S. Then by the dual basis lemma choose a generating set $\{x_i\}_{i\in I}$ for Q and $\{f_i\}_{i\in I} \subset \mathrm{Hom}_s(Q, S)$ such that for each $x \in Q$, $f_i(x) = 0$ for all but a finite number of i and $x = \sum_{i\in I} f_i(x)x_i$ $\left(\sum_{i\in I} x_if_i(x)\right)$. For each $i \in I$ define $g_i\colon Q \to Sx_i$ (x_iS) by $g_i(x) = f_i(x)x_i$ $\left(x_if_i(x)\right)$, $x \in Q$. Then g_i is an endomorphism of Q. Therefore $g_i(Q)$ is a direct summand of Q, hence a direct summand of Sx_i (x_iS). Thus $g_i(Q)$ is cyclic.

Then $g_i(Q) \cong Se$ (eS) for any nonzero idempotente of S.

But R is right PC ring so by theorem 2 S is coflat as S module. Hence Se (eS) is coflat ($\because$ Se(eS) is direct summand of S and [2, Theorem 1.8]).

Thus each $g_i(Q)$ is coflat.

But $Q = \sum_{i \in I} g_i(Q)$.

Hence by [2, Theorem 1.8] Q is coflat and $S0$ S is a PC ring. □

Corollary 1. *If R is a right PC ring then for a non-zero idempotent $e \in R$, eRe is a PC ring.*

Proof. For a nonzero idempotent $e \in R$, eR is a finitely generated projective right R-module. Hence $\mathrm{End}_R(eR)$ is a PC ring by theorem 3. But

$$eRe \cong \mathrm{End}_R(eR).$$

So that eRe is a PC ring. □

Corollary 2. *Let R be a right PC ring and M is an right R module having a PC endomorphisms ring. Then every direct summand of M also has a PC endomorhism ring.*

Proof. Let $S = \mathrm{Hom}_R(M, M)$ and N a diret summand of M. Let e is the projection onto N. Then e is an idempotent of S.

Now $\mathrm{Hom}_R(N, N) = eSe$, and by corollary 1 eSe is a PC ring. Hence $\mathrm{Hom}_R(N, N)$ is a PC ring. □

Corollary 3. *If R is a ring and $n > o$ is an integer, let $M_n(R)$ denote the ring of $n \times n$ matrices over R. Then R is a right PC ring iff $M_n(R)$ is PC ring.*

Proof. Let R is right PC ring. Now R^n is a finitely generated projective module, hence $\mathrm{Hom}_R(R^n, R^n)$ is PC ring by theorem 3. But $M_n(R) \cong \mathrm{Hom}_R(R^n, R^n)$, so $M_n(R)$ is PC ring.

Conversselly let $M_n(R)$ is PC ring. Then $\mathrm{Hom}_R(R^n, R^n) \cong M_n(R)$ is PC ring. Since R_R is a direct summand of R^n_R, by corollary 2 $\mathrm{Hom}_R(R, R)$ is PC ring. Which implies that R is right PC ring. □

Theorem 4. *For a left perfect ring R followings are equivalent.*

1. *R is left PC ring.*
2. *R is left FC ring.*
3. *Every left coflat module is projective.*

Proof. $1 \Rightarrow 2$ Let R is left PC ring, then every projective left R-module is coflat. But R is left perfect ring where every left flat module is projective. Hence every left flat module is coflat, and so R is left FC ring [2, Theorem 2.4].

$2 \Rightarrow 3$ Let R is left FC ring. Then every left coflat module is flat. But R is left perfect, so every left flat module is projective. Hence every coflat module is projective.

$3 \Rightarrow 1$ Let every left coflat module projective.

Then every left coflat module is flat.

$\Rightarrow$ R is left FC ring [2, Theorem 2.4].

$\Rightarrow$ R is left PC ring [$\because$ Every left FC ring is left PC ring]. □

Example. A PC ring which is not FC ring. In [5] Rosenberg and Zelinsky gave an example of a quasi-frobenius, hence FC ring R and a nonzero idempotent $e \in R$ such that eRe is not QF. Thus, since eRe is artinian, it is not FC ring. But eRe is PC ring, (since, R is a PC ring and eR is a finitely generated projective module which implies that $eRe \cong \mathrm{End}_R(eR)$ is a PC ring by theorem 3).

Theorem 5. *If R is right noetherian ring following statements are equivalent.*

1. *R is right PC ring.*
2. *R is QF ring.*
3. *R is right IF ring.*
4. *R is FC ring.*

Proof. $1 \Rightarrow 2$ Let R is right PC ring. Then every right projective module is coflat. Hence R_R is coflat. But R is right noetherian hence by [2, Corollary 1.10] R_R is injective. Thus R_R is noetherian and coflat so that R is QF ring.

$2 \Rightarrow 3$ Let R is QF ring. Then every right injective R-module is projective or every right injective R-module is flat. Hence R is right IF ring.

$3 \Rightarrow 4$ Let R is right IF ring. Then every right injective R-module is flat. As R is right noetherian every right coflat module is right injective, so every coflat module is flat. Hence R is FC ring [2, Theorem 2.4].

$4 \Rightarrow 1$ Let R is FC ring, then every right flat module is coflat. Hence every right projective module is right coflat and so R is right PC ring. □

Example. A PC ring which is not QF ring.

Let K is a field and for any set A

$$R = K^A = \prod_{i \in A} K_i, \quad K_i = K \ \forall i \in A.$$

Then R is a commutative regular ring.

Or R is a PC ring. But R is not QF ring. Since $\bigoplus_{i \in A} R_{ij} R_i = R$ is projective but not injective as R is not noetherian ring.

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