D. J. F. Nonnenmacher

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EVERY M_1-INTEGRABLE FUNCTION IS PFEFFER INTEGRABLE

D. J. F. NONNENMACHER, Ulm

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Dedicated to Erdmute

It is well-known that, in contrary to the one-dimensional case, the *Denjoy-Perron* integral in higher dimensions does not integrate the divergence of any differentiable vector field. To remove this deficiency Mawhin presented in [Maw2] the GP-integral which indeed gives the divergence theorem without any integrability assumptions. Although this integral shows further satisfactory results its main lack is the complete failure of the additivity property, i.e. if an interval is subdivided into two intervals on each of which a function is integrable, the function needs not be integrable on the original interval (see, e.g., ex. 1–2 in [JKS]).

In 1983 the M_1-integral was introduced in [JKS] which not only preserves the good properties of the GP-integral but also shows the additivity. Independently Pfeffer defined an integration process in [Pf] which has the same properties as the M_1-integral but yields a more general divergence theorem. In this note we prove that every M_1-integrable function is *Pfeffer* integrable with the same value. This result completes the known relations between several integrals which are summarized at the end.

By $\mathbf{R}$ and $\mathbf{R}^+$ we denote the set of all real and all positive real numbers, respectively. Throughout this note n is a fixed positive integer and we work in the n-dimensional Euclidean space $\mathbf{R}^n$.

If $E \subseteq \mathbf{R}^n$ we denote by ∂E, $d(E)$ and $|E|_n$ the boundary, diameter and outer Lebesgue measure of E.

An interval I in $\mathbf{R}^n$ is always assumed to be compact and non-degenerate. A finite family of pairs (x_i, I_i) is called a partition of the interval I if the I_i are intervals, having disjoint interiors, $I = \bigcup_i I_i$ and $x_i \in I_i$ for all i. Furthermore, given a function $\delta\colon I \to \mathbf{R}^+$ the partition $\{(x_i, I_i)\}$ is called δ-fine if $d(I_i) < \delta(x_i)$ holds for all i.

If $0 \leqslant k \leqslant n-1$ is an integer we call a k-dimensional linear submanifold H of $\mathbf{R}^n$ which is parallel to k distinct coordinate axes a k-plane or just a plane and we write $\dim H = k$ for the dimension of H. For a subset E of a k-plane ($k \geqslant 1$) we denote by $|E|_k$ its k-dimensional outer Lebesgue measure and if $E \subseteq \mathbf{R}^n$ is finite, then $|E|_0$ denotes the number of elements of E.

Given an interval I and a k-plane H we define the regularity $r(I,H)$ of I with respect to H by $r(I,H) = |H \cap I|_k/d(I)^k$ $(\geqslant |I|_n/d(I)^n)$ if $H \cap I \neq \emptyset$ and by $r(I,H) = |I|_n/d(I)^n$ if $H \cap I = \emptyset$. If $\mathcal{H}$ is a finite family of planes we set $r(I,\mathcal{H}) = \max\{r(I,H) \colon H \in \mathcal{H}\}$ if $\mathcal{H} \neq \emptyset$ and $r(I,\mathcal{H}) = |I|_n/d(I)^n$ else.

We recall the definitions of M_1- and *Pfeffer* integrability, see [JKS] and [Pf].

Given an interval I and a function $f\colon I \to \mathbf{R}$ we call f

- M_1-integrable on I if there is a real number γ with the property that for any $\varepsilon > 0$ and $K > 0$ there exists a $\delta\colon I \to \mathbf{R}^+$ such that $|\gamma - \sum_i f(x_i)|I_i|_n| \leqslant \varepsilon$ holds for every δ- fine partition $\{(x_i, I_i)\}$ of I with $\sum_i d(I_i)|\partial I_i|_{n-1} \leqslant K$. In case of integrability γ is uniquely determined and denoted by ${}^{\mathrm{M}_1}\!\!\int_I f$.
- *Pfeffer* integrable on I if there is a real number γ with the property that for any $\varepsilon > 0$ and any finite family $\mathcal{H}$ of planes there exists a $\delta\colon I \to \mathbf{R}^+$ such that $|\gamma - \sum_i f(x_i)|I_i|_n| \leqslant \varepsilon$ holds for every δ- fine partition $\{(x_i, I_i)\}$ of I with $r(I_i, \mathcal{H}) \geqslant \varepsilon$ for all i. Again γ is, if it exists, uniquely determined and denoted by ${}^{\mathrm{Pf}}\!\!\int_I f$.

Proposition. *Let I be an interval and assume the function $f\colon I \to \mathbf{R}$ to be M_1-integrable on I. Then f is Pfeffer integrable on I and* ${}^{\mathrm{Pf}}\!\!\int_I f = {}^{\mathrm{M}_1}\!\!\int_I f$.

Proof. Suppose without loss of generality $\frac{1}{2} \geqslant \varepsilon > 0$ and let a finite family $\mathcal{H}$ of planes be given. Set $K = \frac{2n}{\varepsilon}\left(|I|_n + 2^n \sum_{H \in \mathcal{H}} |H \cap I|_{\dim H}\right)$ and choose a $\delta\colon I \to \mathbf{R}^+$ such that

$$\left| {}^{\mathrm{M}_1}\!\!\int_I f - \sum_i f(x_i)|I_i|_n \right| \leqslant \varepsilon$$

holds for every δ-fine partition $\{(x_i, I_i)\}$ of I with $\sum_i d(I_i)|\partial I_i|_{n-1} \leqslant K$ by the M_1-integrability of f. We may assume $\delta \leqslant 1$.

Now take a δ-fine partition $\{(x_i, I_i)\}$ of I with $r(I_i, \mathcal{H}) \geqslant \varepsilon$ for all i. We will show $\sum_i d(I_i)|\partial I_i|_{n-1} \leqslant K$ and thereby prove our proposition.

Note that for each $H \in \mathcal{H}$ the inequality $\sum_i |H \cap I_i|_{\dim H} \leqslant 2^n |H \cap I|_{\dim H}$ holds true. For, assuming first $\dim H = k \geqslant 1$ and denoting by χ_i the characteristic function of I_i and by ${}^{\mathrm{L}k}\!\int$ the k-dimensional Lebesgue integral we conclude

$$\sum_i |H \cap I_i|_k = \sum_i {}^{\mathrm{L}k}\!\!\int_{H\cap I} \chi_i = {}^{\mathrm{L}k}\!\!\int_{H\cap I} \sum_i \chi_i \leqslant 2^n |H \cap I|_k$$

since each $x \in \mathbf{R}^n$ can be contained in at most 2^n of the intervals I_i and this also applies to the case $\dim H = 0$.

If $\mathcal{H} \neq \emptyset$ take a pair (x_i, I_i) of the partition and choose a plane $H \in \mathcal{H}$ with $r(I_i, H) = r(I_i, \mathcal{H}) \geqslant \varepsilon$. Remembering that $|\partial I_i|_{n-1} \leqslant 2nd(I_i)^{n-1}$ and $\delta \leqslant 1$ we consider two cases:

(i) if $H \cap I_i = \emptyset$ then $|I_i|_n \geqslant \varepsilon\, d(I_i)^n$ which implies

$$d(I_i)|\partial I_i|_{n-1} \leqslant \frac{2n}{\varepsilon}|I_i|_n,$$

(ii) if $H \cap I_i \neq \emptyset$ then $|H \cap I_i|_{\dim H} \geqslant \varepsilon\, d(I_i)^{\dim H}$ which gives

$$\begin{aligned} d(I_i)|\partial I_i|_{n-1} &\leqslant 2nd(I_i)^{\dim H} \leqslant \frac{2n}{\varepsilon}|H \cap I_i|_{\dim H} \\ &\leqslant \frac{2n}{\varepsilon} \sum_{H\in\mathcal{H}} |H \cap I_i|_{\dim H}. \end{aligned}$$

Thus, recalling the above remark we obtain indeed

$$\begin{aligned} \sum_i d(I_i)|\partial I_i|_{n-1} &\leqslant \frac{2n}{\varepsilon}|I|_n + \frac{2n}{\varepsilon}\sum_i \sum_{H\in\mathcal{H}} |H \cap I_i|_{\dim H} \\ &\leqslant \frac{2n}{\varepsilon}|I|_n + \frac{2n}{\varepsilon}2^n \sum_{H\in\mathcal{H}} |H \cap I|_{\dim H} = K. \end{aligned}$$

In case of $\mathcal{H} = \emptyset$ the inequality follows from (i) alone, since then $r(I_i, \mathcal{H}) = |I_i|_n / d(I_i)^n \geqslant \varepsilon$ for all i. □

At the end let us list the relations between several integration processes which directly follow from the definitions of the corresponding integrals and our proposition. For relations between other integrals see, e.g., [Ost]. Fix an n-dimensional interval I and denote by $\mathrm{L}(I)$, $\mathrm{DP}(I)$, $\mathrm{M}_i(I)$ ($i = 1, 2$), $\mathrm{Pf}(I)$, $\mathrm{GP}(I)$ and $R\mathrm{P}(I)$ the classes of all *Lebesgue*, *Denjoy-Perron*, M_i- ([JKS]), *Pfeffer*, GP- and RP- ([Maw1]) integrable real-valued functions on I. Then for $n = 1$ we have $\mathrm{L}(I) \subset \mathrm{M}_2(I) \subset \mathrm{DP}(I) = \mathrm{M}_1(I) = \mathrm{Pf}(I) = \mathrm{GP}(I) = \mathrm{RP}(I)$ where the first strict inclusion reflects the fact that every derivative is M_2-integrable and the second one can be seen by taking the one-dimensional form of ex. 3 in [JKS]. In case of $n \geqslant 2$ we have $\mathrm{L}(I) \subset \mathrm{DP}(I) \subset \mathrm{M}_1(I) \subseteq \mathrm{Pf}(I) \subset \mathrm{GP}(I) \subset \mathrm{RP}(I)$ and $\mathrm{L}(I) \subset \mathrm{M}_2(I) \subset \mathrm{M}_1(I)$ where the strict inclusions are given by ex. 1–4 in [JKS] and ex. 7.3 in [Pf]. Those examples also prove that $\mathrm{M}_2(I) \not\subseteq \mathrm{DP}(I)$ and $\mathrm{DP}(I) \not\subseteq \mathrm{M}_2(I)$.

R e m a r k s . **(i)** The proof of the proposition reflects the fact that every partition fulfilling the 'local' restriction as used in [Pf] also satisfies the 'global' restriction used in [JKS] (with changed parameters). The converse is not true since if I is an n-dimensional interval ($n \geqslant 2$) and if $K \geqslant 1 + c(n)|I|_n$, $0 < r < 1$, $\delta\colon I \to \mathbf{R}^+$ and a finite family of planes $\mathcal{H}$ are given arbitrarily then it is easy to see that there exists a δ-fine partition $\{(x_i, I_i)\}$ of I with $\sum\limits_i d(I_i)|\partial I_i|_{n-1} \leqslant K$ but $r(I_i, \mathcal{H}) < r$ for at least one i ($c(n)$ being a positive absolute constant). Thus the question whether the inclusion $\mathrm{M}_1(I) \subseteq \mathrm{Pf}(I)$ is strict remains open.

(ii) In order to establish the additivity of the GP-integral a modified version was also proposed in [Mkh]. Unfortunately, the proof of the crucial Lemma which guarantees the existence of certain partitions defining the integral contains a serious gap. Meanwhile the problem seems to have been solved in the plane in [Bcz].

(iii) In [Ju-Kn] the *weak Denjoy-Perron* integral which depends on a parameter $0 < r < 1$ of regularity is investigated and characterized. Denoting for a fixed r by $\mathrm{DP}_r(I)$ the class of all *weakly Denjoy-Perron* integrable real-valued functions on I one can show that $\mathrm{GP}(I) = \bigcap\limits_{0<r<1} \mathrm{DP}_r(I)$.

References

[Bcz] *Buczolich Z.*: A general Riemann complete integral in the plane, to appear.

[JKS] *Jarník J., Kurzweil J., Schwabik Š.*: On Mawhin's approach to multiple nonabsolutely convergent integral, Časopis Pěst. Mat. *108* (1983), 356–380.

[Ju-Kn] *Jurkat W.B., Knizia R.W.*: A characterization of multi-dimensional Perron integrals and the fundamental theorem, Can. J. Math. *43 (3)* (1991), 526–539.

[Maw1] *Mawhin J.*: Generalized Riemann integrals and the divergence theorem for differentiable vector fields, Proceedings of the Int. Christoffel Symposium, Birkhäuser, Basel, 1981, 704–714.

[Maw2] *Mawhin J.*: Generalized multiple Perron integrals and the Green-Goursat theorem for differentiable vector fields, Czech. Math. J. *31 (106)* (1981), 614–632.

[Mkh] *Mkhalfi A.*: On a generalized multiple integral and the divergence theorem, Bull. Soc. Math. Belg. *40* (1988), no. 1, ser. B, 111–130.

[Ost] *Ostaszewski K.M.*: Henstock integration in the plane, Memoirs of the American Math. Soc., Providence *63*, no. 353.

[Pf] *Pfeffer W.F.*: The divergence theorem, Trans American Math. Soc. *295* (1986), 665–685.

Author's address: Abteilung Mathematik II, Universität Ulm, Helmholtzstr. 18, 7900 Ulm, Germany.