Czechoslovak Mathematical Journal

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Czechoslovak Mathematical Journal, Vol. 43 (1993), No. 4, 675–693

Persistent URL: http://dml.cz/dmlcz/128428

Czechoslovak Mathematical Journal, 43 (118) 1993, Praha

ON DIRECTED CONVEX SUBSETS OF PARTIAL MONOUNARY ALGEBRAS

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(Received March 6, 1992)

In the present paper the notions of a directed convex subset and an up-directed convex subset of a partial monounary algebra will be introduced; they are in a certain sense analogous to the same notions in a partially ordered set.

Let (A, f) be a partial monounary algebra. We denote by $\mathrm{DC}(A, f)$ and $\mathrm{DuC}(A, f)$ the system of all directed and of all up-directed convex subsets of (A, f), respectively.

The aim of the present paper is to investigate the following problems:

1. To what extent the partial operation f on A is determined by the system $\mathrm{DuC}(A, f)$ (i.e., we are to describe all partial operations g on A such that $\mathrm{DuC}(A, f) = \mathrm{DuC}(A, g)$, where (A, f) is a given partial monounary algebra).

2. The same for the system $\mathrm{DC}(A, f)$.

The answer to the question 1 is given in Theorem 5.6. Theorem 1.8 and the remark after 1.8 answer the Question 2.

This paper can be considered to be a continuation of [2], [3] and [4]; in these papers analogous problems concerning the system of all convex subsets and the system of all intervals of a partial monounary algebra have been studied.

Similar problems were investigated by *G. Birkhoff* and *M. K. Bennett* [1] (for the case of convex subsets of a partially ordered set) and by *M. Kolibiar* [5] (for the case of directed convex subsets of a down-directed set).

0. PRELIMINARIES

Let $\mathscr{U}$ be the class of all partial monounary algebras. To each $\mathscr{A} = (A, f) \in \mathscr{U}$ there corresponds a directed graph $G(\mathscr{A}) = (A, E)$ without loops and multiple edges which is defined as follows: an ordered pair (a, b) of distinct elements of A belongs to E iff $f(a) = b$.

0.1. Definition. A subset B of A will be called convex (in $\mathscr{A}$), if, whenever a, b_1, b_2 are distinct elements of A having the property that b_1, $b_2 \in B$ and there is a path (in $G(\mathscr{A})$) going from b_1 to b_2, not containing the element b_2 twice and containing the element a, then a belongs to B as well.

0.2. Definition. A subset B of A is said to be directed if, whenever b_1, $b_2 \in B$, then there are paths X_1, X_2, Y_1, Y_2 in $G(\mathscr{A})$ and $u, v \in B$ such that X_i goes from b_i to u and Y_i goes from v to b_i for $i = 1, 2$.

0.3. Definition. A subset B of A will be called up-directed if, whenever b_1, $b_2 \in B$, then there are paths X_1, X_2 in $G(A)$ and $u \in B$ such that X_i goes from b_i to u for $i = 1, 2$.

0.4. Remark. The author wishes to correct an inaccuracy in the definition of convexity in [2]; namely, the assumption "not containing the element b_2 twice" (cf. Definition 0.1 above), was not expressed in [2]. In the whole paper [2] the convexity is to be understood in the sense of the above Definition 0.1.

For $(A, f) \in \mathscr{U}$ let $\mathrm{DC}(A, f)$ and $\mathrm{DuC}(A, f)$ be as in the introduction. Both the systems $\mathrm{DC}(A, f)$ and $\mathrm{DuC}(A, f)$ are considered to be partially ordered by inclusion; the empty set is the least element in both $\mathrm{DC}(A, f)$ and $\mathrm{DuC}(A, f)$.

Let (A, f) and (A', f') belong to $\mathscr{U}$. Instead of the condition

(i) $A = A'$ and $\mathrm{DuC}(A, f')$

we can consider a more general condition

(ii) $\mathrm{DuC}(A, f) \cong \mathrm{DuC}(A', f')$.

Let us remark that the assumption dealt with in [1] and [5] are analogous to (ii) and not to (i). The following result shows that the distinction between (i) and (ii) is not essential.

0.5. Proposition. *Let $(A, f), (A', f') \in \mathscr{U}$ and $\mathrm{DuC}(A, f) \cong \mathrm{DuC}(A', f')$. Then there is a bijection $h\colon A \to A'$ such that the mapping H defined by the formula*

$$\text{if } B \in \mathrm{DuC}(A, f), \quad \text{then} \quad H(B) = \{h(b)\colon b \in B\}$$

is an isomorphism from $\mathrm{DuC}(A, f)$ onto $\mathrm{DuC}(A', f')$.

Proof. Let the assumption be valid. There is an isomorphism $e\colon \mathrm{DuC}(A, f) \to \mathrm{DuC}(A', f')$. If $a \in A$, then $\{a\}$ is a minimal element of $\mathrm{DuC}(A, f)$, thus $e(\{a\})$ is a minimal element of $\mathrm{DuC}(A', f')$, and therefore there is $a' \in A'$ with $e(\{a\}) = \{a'\}$. Put $h(a) = a'$. It is obvious that h is a bijection. Now let $B \in \mathrm{DuC}(A, f)$, $H(B) = \{h(b)\colon b \in B\}$. We will show that $H(B) = e(B)$. Let $x \in H(B)$, i.e., $x = h(b)$ for some $b \in B$. Since $\{b\}$, $B \in \mathrm{DuC}(A, f)$, $\{b\} \subseteq B$, we obtain that $e(\{b\}) \subseteq e(B)$ (e is an isomorphism). Then $\{x\} = \{h(b)\} = e(\{b\}) \subseteq e(B)$, thus

$x \in e(B)$ and $H(B) \subseteq e(B)$. Conversely, let $y \in e(B)$. Then $y \in A'$ and there is $z \in A$ with $h(z) = y$. We have $e(\{z\}) = \{y\} \subseteq e(B)$. Since e is an isomorphism, this implies that $\{z\} \subseteq B$, $z \in B$. Therefore $y = h(z) \in \{h(b): b \in B\} = H(B)$ and $e(B) \subseteq H(B)$. □

Under the assumption and notation as in 0.5, the relation (ii) holds. Now if we identify the elements a and $h(a)$ for each $a \in A$, then we obtain that (i) is valid.

Let us remark that the result analogous to 0.5 is valid also if we take directed subsets instead of up-directed, i.e. DC instead of DuC.

0.6. Remark. Let $(A, f) \in \mathscr{U}$, $x, y \in A$, $n \in \mathbb{N}$. If we write $y = f^n(x)$, then we suppose that $x \in \operatorname{dom} f$, $f(x) \in \operatorname{dom} f$, ..., $f^{n-1}(x) \in \operatorname{dom} f$ and the elements y and $f^n(x)$ are equal. If we write $y \neq f^n(x)$, then either

(i) $x \in \operatorname{dom} f$, $f(x) \in \operatorname{dom} f$, ..., $f^{n-1}(x) \in \operatorname{dom} f$

and then y and $f^n(x)$ are distinct, or (i) fails to hold.

1. Directed convex subsets of partial monounary algebras

In this section we shall study pairs of partial monounary algebras (A, f) and (A, g) such that $\mathrm{DC}(A, f) = \mathrm{DC}(A, g)$.

1.1. Lemma. *Let $(A, f) \in \mathscr{U}$, $a.b \in A$. Assume that there is $n \in \mathbb{N}$ such that $b = f^n(a)$ and $f^k(a) \neq b$ for each $k \in \mathbb{N} \cup \{0\}$, $k < n$. If*

(1) there is a cycle C of (A, f) with more than one element such that $\{f^{n-1}(a), f^n(a)\} \subseteq C$,

then the least convex subset of (A, f) containing a and b is $\{a, f(a), \ldots, f^n(a)\} \cup C$. If (1) does not hold, then the least convex subset of (A, f) containing a and b is $\{a, f(a), \ldots, f^n(a)\}$.

Proof. First assume that (1) does not hold. Then a, $f(a)$, ..., $f^n(a)$ are distinct elements and either none of them belongs to a cycle or only $f^n(a)$ belongs to a cycle. Consider a path X going from a to $b = f^n(a)$ and not containing b twice. Then X consists of the elements a, $f(a)$, ..., $f^n(a)$, hence $\{a, f(a), \ldots, f^n(a)\}$ is a subset of the least convex subset of (A, f) containing a and b. It is obvious that $\{a, f(a), \ldots, f^n(a)\}$ is convex, thus the least convex subset of (A, f) containing a and b is $\{a, f(a), \ldots, f^n(a)\}$.

Now let (1) be valid. Analogously as above, $\{a, f(a), \ldots, f^n(a\}$ is a subset of the least convex subset of (A, f) containing a and b. Put $b_2 = f^{n-1}(a)$ and let Y be a path going from b to b_2 and containing b_2 only once. Then Y consists of the elements b, $f(b)$, ..., b_2, i.e., of the elements of the set C. Therefore

$$\{a, f(a), \ldots, f^n(a)\} \cup C$$

is a subset of the least convex subset of (A, f) containing a and b. This set is convex, hence it coincides with the least convex subset of (A, f) containing a and b. $\square$

1.2. Definition. Let $(A, f) \in \mathscr{U}$, $a, b \in A$. Assume that there is $n \in \mathbb{N} \cup \{0\}$ such that $b = f^n(a)$. The least convex subset of (A, f) containing a and b will be denoted by $[a, b]_f$ and called an interval in (A, f). We denote by $\mathrm{I}(A, f)$ the system of all intervals in (A, f)including the empty set. (This notion was introduced in [4].)

1.3. Lemma. *If $(A, f) \in \mathscr{U}$, then $\mathrm{I}(A, f) \subseteq \mathrm{DC}(A, f)$.*

Proof. Assume that $(A, f) \in \mathscr{U}$ and B is a nonempty interval in (A, f). There are $a, b \in A$ and $n \in \mathbb{N} \cup \{0\}$ with $b = f^n(a)$ such that $B = [a, b]_f$. According to 1.2, B is convex. If $n = 0$, then $[a, b]_f = \{a\} \in \mathrm{DC}(A, f)$. Let $n \in \mathbb{N}$ and suppose that $f^k(a) \neq b$ for each $k \in \mathbb{N} \cup \{0\}$, $k < n$. Consider the condition (1) from 1.1. If (1) does not hold, then 1.1 yields that $[a, b]_f = \{a, f(a), \dots, f^n(a)\}$; if (1) is valid, then $[a, b]_f = \{a, f(a), \dots, f^n(a)\} \cup C$. Therefore there is $k \in \mathbb{N}$ such that

$$[a, b]_f = \{a, f(a), \dots, f^k(a)\},$$

where $f^i(a) \neq f^j(a)$ for each $0 \leqslant i < j \leqslant k$. Let b_1, $b_2 \in B$. Without loss of generality we can suppose that $b_1 = f^i(a)$, $b_2 = f^j(a)$ for some $0 \leqslant i \leqslant j \leqslant k$. Put $u = b_2$, $v = b_1$. There exist paths $X_1 = Y_2 = f^i(a)f^{i+1}(a)\dots f^j(a), X_2 = b_2$ and $Y_1 = b_1$ such that X_i goes from b_i to u and Y_i goes from v to b_i for $i = 1, 2$. Hence B is directed, thus $B \in \mathrm{DC}(A, f)$. $\square$

1.4. Lemma. *Let $(A, f) \in \mathscr{U}$, $a, b \in A$. If there is $B \in \mathrm{DC}(A, f)$ such that $\{a, b\} \subseteq B$, then there is $n \in \mathbb{N} \cup \{0\}$ such that either $a = f^n(b)$ or $b = f^n(a)$.*

Proof. Suppose that $\{a, b\} \subseteq B \in \mathrm{DC}(A, f)$. In view of the definition there is $v \in A$ and paths Y and Z, Y going from v to a and Z going from v to b. Then $a = f^i(v)$ and $b = f^j(v)$ for some $i, j \in \mathbb{N} \cup \{0\}$. We can assume that $i \leqslant j$. Therefore

$$b = f^j(v) = f^{j-i}(f^i(v)) = f^{j-i}(a).$$

$\square$

1.5. Lemma. *Let $(A, f) \in \mathscr{U}$. If $B \in \mathrm{DC}(A, f)$ and B is finite, then $B \in \mathrm{I}(A, f)$.*

Proof. Let B be a nonempty finite set, $B \in \mathrm{DC}(A, f)$. If $\operatorname{card} B = 1$, then $B \in \mathrm{I}(A, f)$. Let $\operatorname{card} B > 1$. By 1.4, B is a subset of one connected component of (A, f). If B contains only elements of some cycle, then the fact that B is convex implies that B contains all elements of this cycle and then B—a cycle—is an interval in (A,f). Suppose that there is $x \in B$ such that x does not belong to any cycle. Then the assumption that B is finite implies that there is $a \in B$ such that the set

$$\{y \in B\colon a = f^n(y) \text{ for some } n \in \mathbb{N}\}$$

is empty. Let $z \in B$. According to 1.4, there is $k \in \mathbb{N}$ such that $z = f^k(a)$. Hence

$$B \subseteq \{a\} \cup \{f^k(a)\colon k \in \mathbb{N}, f^{k-1}(a) \in \operatorname{dom} f\}.$$

The set B is finite, thus there are $b \in B$, $n \in \mathbb{N}$ with $b = f^n(a)$ and such that either

(1) $$f^k(a) \notin B \text{ for any } k \in \mathbb{N}, \quad k > n, \qquad \text{if} f^{k-1}(a) \in \operatorname{dom} f,$$

or

(2) $$f^n(a) \in \operatorname{dom} f, \quad f^{n+1}(a) \in \{f(a), \dots, f^n(a)\}.$$

We have $\{a, b\} \subseteq B$, B is convex, hence we obtain that

$$\{a, f(a), \dots, f^n(a) = b\} \subseteq B.$$

Therefore $B = \{a, f(a), \dots, f^n(a)\} = [a, b]_f \in \mathrm{I}(A, f)$. □

1.6. Lemma. *Let* (A, f), $(A, g) \in \mathcal{U}$. *If* $\mathrm{DC}(A, f) = \mathrm{DC}(A, g)$, *then* $\mathrm{I}(A, f) = \mathrm{I}(A, g)$.

Proof. Assume that $\mathrm{DC}(A, f) = \mathrm{DC}(A, g)$ and $B \in \mathrm{I}(A, f)$. According to 1.3, $B \in \mathrm{DC}(A, f)$, thus $B \in \mathrm{DC}(A, g)$. Since each interval in (A, f) is finite, in view of 1.5 this yields that $B \in \mathrm{I}(A, g)$. Thus $\mathrm{I}(A, f) \subseteq \mathrm{I}(A, g)$. The convergence inclusion is analogous, therefore $\mathrm{I}(A, f) = \mathrm{I}(A, g)$. □

1.7. Lemma. *Let* $(A, f), (A, g) \in \mathcal{U}$. *If* $\mathrm{I}(A, f) = \mathrm{I}(A, g)$, *then* $\mathrm{DC}(A, f) = \mathrm{DC}(A, g)$.

Proof. Suppose that $\mathrm{I}(A, f) = \mathrm{I}(A, g)$. By virtue of [4], Theorem 3.8, (A, f) and (A, g) must have the same partition into connected components. If A' is a connected component of (A, f), then instead of $f|A'$ or $g|A'$ we write f and g, respectively. Now it suffices to verify that for each connected component A' of (A, f) the relation

(1) $$\mathrm{DC}(A', f) = \mathrm{DC}(A', g)$$

is valid.

Take (A', f) fixed and consider the possibilities how to define g on A' such that $\mathrm{I}(A', f) = \mathrm{I}(A', g)$. Let us deal with the following cases:

a) $\operatorname{card} A' \leqslant 2$. Then g on A' can be defined in an arbitrary way, only A' must be a connected component of (A, g) (according to [4], 3.8). In this case also $\mathrm{DC}(A', f) = \mathrm{DC}(A', g)$.

b) $\operatorname{card} A' > 2$ and there are $a, b \in A'$ such that $a = f(b)$, $\{x \in A' \colon x \in \operatorname{dom} f, f(x) = b\} = \emptyset$ and either $f(a) = a$ or $a \notin \operatorname{dom} f$. In view of [4], 3.8, we obtain that if $x \in A' - \{a\}$, then $x \in \operatorname{dom} g$ and $g(x) = f(x)$ and either $g(a) = a$ or $a \notin \operatorname{dom} g$. It is obvious that in both cases (1) is valid.

c) (A', f) is isomorphic to some of the partial monounary algebras considered in [4], 2.1–2.7. In these sections of [4] we have described (up to isomorphism) all (A', g) with $\mathrm{I}(A', f) = \mathrm{I}(A', g)$. In each of these cases it is easy to see that (1) holds as well. □

In view of 1.6 and 1.7 we obtain

1.8. Theorem. *Let* $(A, f), (A, g) \in \mathcal{U}$. *Then* $\mathrm{DC}(A, f) = \mathrm{DC}(A, g)$ *if and only if* $\mathrm{I}(A, f) = \mathrm{I}(A, g)$.

Let $(A, f) \in \mathcal{U}$. According to 1.8, the conditions given in [4], Theorem 3.8, give a characterization of all $(A, g) \in \mathcal{U}$ such that (A, f) and (A, g) have common systems of directed convex subsets.

2. Auxiliary results

In what follows we shall study up-directed convex subsets of partial monounary algebras.

2.1. Lemma. *Let* $(A, f) \in \mathcal{U}$, $x, y \in A$. *Then* x *and* y *belong to the same connected component of* (A, f) *if and only if there is* $M \in \mathrm{DuC}(A, f)$ *with* $\{x, y\} \subseteq M$.

Proof. Suppose that x and y belong to the same connected component of (A, f). Then there are $m, n \in \mathbb{N} \cup \{0\}$ with $f^n(x) = f^m(y)$.
Put

$$M = \{x, y\} \cup \{f^i(x) \colon i \in \mathbb{N}, f^{i-1}(x) \in \operatorname{dom} f\} \cup \{f^i(y) \colon i \in \mathbb{N}, f^{i-1}(y) \in \operatorname{dom} f\}.$$

According to the definition, $M \in \mathrm{DuC}(A, f)$. Conversely, let there be $M \in \mathrm{DuC}(A, f)$ such that $\{x, y\} \subseteq M$. Since M is up-directed, there are $u \in M$ and paths X, Y such that X goes from x into u and Y goes from y into u. Thus x and u (y and u) are in the same connected component of (A, f) and therefore x and y belong to the same connected component of (A, f). □

2.2. Corollary. *Let* (A,f), $(A,g) \in \mathcal{U}$, $\mathrm{DuC}(A,f) = \mathrm{DuC}(A,g)$. *Then* (A,f) *and* (A,g) *have the same partition into connected components.*

2.3. Notation. Let $\mathcal{U}_c$ be the class of all connected partial monounary algebras. Further, let $\mathcal{W}$ be the class of all connected partial monounary algebras which contain a cycle with more than two elements, and let $\mathcal{V} = \mathcal{U}_c - \mathcal{W}$.

For $(A,f) \in \mathcal{U}_c$, $x, y \in A$, the symbol $L_f(x,y)$ denotes the least up-directed convex subset B of (A,f) such that $\{x,y\} \subseteq B$.

2.4. Lemma. *Let* $(A,f) \in \mathcal{U}_c$ *and let* $C \subseteq A$, $\operatorname{card} C > 2$. *Then* C *is a cycle of* (A,f) *if and only if* $L_f(x,y) = C$ *for each* $x, y \in C$, $x \neq y$.

Proof. Assume that C is a cycle of (A,f). It follows from the definition of up-directed convex subsets that $L_f(x,y) = C$ for each $x, y \in C$, $x \neq y$. Now suppose that C is not a cycle and that

$$L_f(x,y) = C \text{ for each } x, y \in C, x \neq y. \tag{1}$$

Since $\operatorname{card} C > 2$, we can take fixed $x, y \in C$, $x \neq y$. If x and y belong to a cycle D, then $L_f(x,y) = D$, $D = C$, a contradiction. Since (A,f) is connected, we can assume (without loss of generality) that x does not belong to any cycle. If $x = f^n(y)$ for some $n \in \mathbb{N}$, then $L_f(x,y) = \{y, f(y), \ldots, f^n(y) = x\} = C$. Then $f(y) \in C$ and $f(y) \neq y$. Thus by (1), $L_f(y, f(y)) = C$. Then $\{y, f(y)\} = C$, which is a contradiction, since $\operatorname{card} C > 2$ in view of the assumption. Therefore $x \notin \{y\} \cup \{f^i(y) \colon i \in \mathbb{N}, f^{i-1}(y) \in \operatorname{dom} f\}$. Put $z = f(x)$. We have $L_f(y,z) \subseteq \{y,z\} \cup \{f^i(y) \colon i \in \mathbb{N}, f^{i-1}(y) \in \operatorname{dom} f\} \cup \{f^i(z) \colon i \in \mathbb{N}, f^{i-1}(z) \in \operatorname{dom} f\}$. If $x \in L_f(y,z)$, then there is $i \in \mathbb{N} \cup \{0\}$ with $x = f^i(z)$, i.e. $x = f^{i+1}(x)$ and x belongs to a cycle, a contradiction. Hence $x \notin L_f(y,z)$ and we get $C = L_f(x,y) \neq L_f(f(x),y)$. Further, $f(x) \in L_f(x,y) = C$, therefore (1) implies $L_f(f(x),y) = C$, a contradiction. □

2.5. Corollary. *Let* (A,f), $(A,g) \in \mathcal{U}_c$ *and* $\mathrm{DuC}(A,f) = \mathrm{DuC}(A,g)$. *If* $C \subseteq A$ *and* $\operatorname{card} C > 2$, *then* C *is a cycle of* (A,f) *if and only if* C *is a cycle of* (A,g).

2.6. Notation. Let $(A,f) \in \mathcal{W}$. For $x \in A$ put $n(x) = \min\{i \in \mathbb{N} \cup \{0\} \colon f^i(x)$ belongs to a cycle $\}$. If $c \in A$, $n(c) = 0$ (i.e., c belongs to a cycle), then we denote

$$A_f(c) = \{x \in A \colon f^{n(x)}(x) = c\}.$$

2.7. Lemma. *Let* $(A,f) \in W$ *and assume that* C *is a cycle of* (A,f). *Let* $c \in C$. *Then* $x \in A_f(c)$ *if and only if the following condition is satisfied:*

(i) *c is the unique element of C such that* $L_f(x,c) \cap C = \{c\}$.

Proof. Let $x \in A_f(c), n = n(x)$. Then $c = f^n(x)$ and $L_f(x,c) = \{x, f(x), \ldots, f^n(x)\}$. If $d \in C - \{c\}$, then $L_f(x,d) = \{x, f(x), \ldots, f^n(x)\} \cup C$, therefore $L_f(x,c) \cap C = \{c\}$ and $L_f(x,d) \cap C = C$.

Now assume that $x \in A$ and that (i) is valid. Then $x \in A_f(d)$ for some $d \in C$. As we have shown in the first part of the proof, this yields

(1) d is the unique element of C such that $L_f(x,d) \cap C = \{d\}$. Therefore $d = c$ in view of (i) and (1), hence $x \in A_f(c)$. □

From 2.5 and 2.7 we obtain

2.8. Corollary. *Let (A,f), $(A,g) \in \mathcal{U}_c$ and $\mathrm{DuC}(A,f) = \mathrm{DuC}(A,g)$. Assume further that C with $\operatorname{card} C > 2$ is a cycle of (A,f). Then C is a cycle of (A,g) and, for each $c \in C$, the relation $A_f(c) = A_g(c)$ is valid.*

3. The class $\mathcal{V}$: basic lemmas

This section deals with up-directed convex subsets of partial monounary algebras which belong to the class $\mathcal{V}$, i.e. of such connected (A,f) for which one of the following conditions is satisfied:

(1) $\operatorname{dom} f \neq A$,

(2) $\operatorname{dom} f = A$ and (A,f) contains a cycle C with $\operatorname{card} C \leqslant 2$,

(3) $\operatorname{dom} f = A$ and (A,f) contains no cycle.

3.1. Notation. Let $(A,f) \in \mathcal{V}$, $x, y \in A$. Put

$$\begin{aligned} k_f(x,y) &= \min\{i \in \mathbb{N} \cup \{0\} : f^i(x) \in \{y\} \cup \{f^j(y) : j \in \mathbb{N}, f^{j-1}(y) \in \operatorname{dom} f\}\}, \\ m_f(y,x) &= \min\{j \in \mathbb{N} \cup \{0\}; f^{k_f(x,y)}(x) = f^j(y)\}, \\ d_f(x,y) &= k_f(x,y) + m_f(y,x). \end{aligned}$$

3.2. Lemma. *Let $(A,f) \in \mathcal{V}$, $x, y \in A$. If $x = f^j(y)$ for some $j \in \mathbb{N}$ and $x \notin \{y, f(y), \ldots, f^{j-1}(y)\}$, then $d_f(x,y) = j$.*

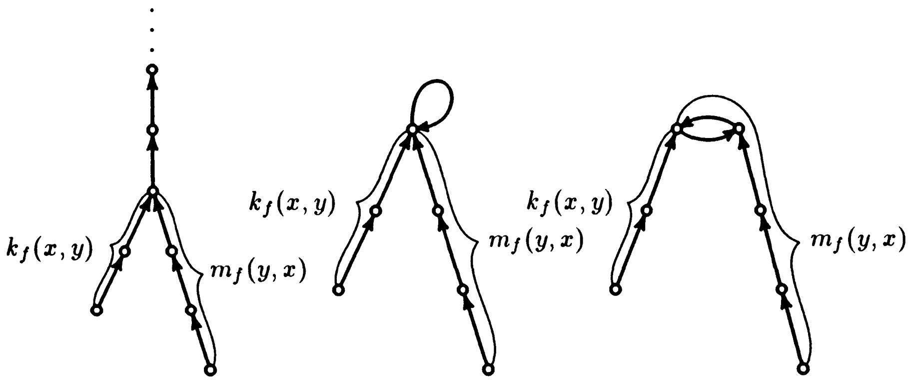
Proof. By 3.1, $k_f(x,y) = 0$, $m_f(y,x) = j$ and $d_f(x,y) = 0 + j = j$. □

3.3. Lemma. *Let $(A,f) \in \mathcal{V}$, $x, y \in A$. If $k = k_f(x,y)$, $m = m_f(y,x)$, then*

$$L_f(x,y) = \{x, f(x), \ldots, f^k(x)\} \cup \{y, f(y), \ldots, f^m(y)\},$$

where all elements in the above sets are mutually distinct except $f^k(x) = f^m(y)$.

Proof. Since $L_f(x,y)$ is the smallest up-directed convex subset of (A,f) containing x and y, the required relation follows immediately from 3.1. □



3.4. Corollary. *Let* $(A,f) \in \mathcal{V}$, $x, y \in A$. *Then* $d_f(x,y) = \operatorname{card} L_f(x,y) - 1$.

P r o o f. In view of 3.3 we obtain

$$\begin{aligned}\operatorname{card} L_f(x,y) &= \big(k_f(x,y)+1\big) + \big(m_f(y,x)+1\big) - 1\\ &= k_f(x,y) + m_f(y,x) + 1\\ &= d_f(x,y) + 1.\end{aligned}$$

□

3.5. Corollary. *Let* $(A,f) \in \mathcal{V}$, $x, y \in A$. *Then* $d_f(x,y) = d_f(y,x)$.

P r o o f. The assertion follows from 3.4, since $L_f(x,y) = L_f(y,x)$. □

3.6. Corollary. *Let* (A,f), $(A,g) \in \mathcal{V}$ *and suppose that* $\operatorname{DuC}(A,f) = \operatorname{DuC}(A,g)$. *Then* $d_f(x,y) = d_g(x,y)$.

P r o o f. If the assumption is valid, then $L_f(x,y) = L_g(x,y)$, thus 3.4 yields the assertion. □

3.7. Lemma. *Let* $(A,f) \in V$, x, y, $z \in A$. *If* $d_f(x,z) + d_f(z,y) = d_f(x,y)$, *then* $z \in L_f(x,y)$.

P r o o f. Let the assumption hold, $d_f(x,z) + d_f(z,y) - d_f(x,y)$. If $x = y$, then $d_f(x,y) = 0 = d_f(x,z)$ and $z = x$, $z \in L_f(x,y)$. The cases $z = x$ or $z = y$ are obvious. Suppose that x, y and z are distinct. First, let $y = f^n(x)$ for some $n \in \mathbb{N}$, $y \notin \{x, f(x), \dots, f^{n-1}(x)\}$. In view of 3.2, $d_f(x,y) = n$. Since $d_f(y,z) < n$, $f^i(z) \neq x$ for each $i \in \mathbb{N} \cup \{0\}$ (in the opposite case $y = f^{i+n}(z)$, $d_f(y,z) = i+n \geqslant n$).

Put $k = k_f(x,z)$, $u = f^k(x)$. Then $k \leqslant d_f(x,z) < n$ and $y = f^{n-k}(u)$. Further, if $m = m_f(z,x)$, then $u = f^m(z)$, $m < n$ and $y = f^{n-k}(u) = f^{n-k+m}(z)$. Then 3.2 implies

$$d_f(y,z) = n - k + m,$$

thus

$$\begin{aligned} n &= d_f(x,z) + d_f(z,y) = k_f(x,z) + m_f(z,x) + n - k + m \\ &= k + m + n - k + m = n + 2m, \\ m &= 0. \end{aligned}$$

Therefore $u = z = f^k(x)$ and we get that $u \in \{x, f(x), \ldots, f^n(x)\} = L_f(x,y)$.

Now suppose that $y \neq f^i(x)$, $x \neq f^i(y)$ for any $i \in \mathbb{N} \cup \{0\}$. Then $k_f(x,y) \neq 0 \neq m_f(y,x)$. Put $k = k_f(x,y)$, $m = m_f(y,x)$, $k_1 = k_f(x,z)$, $m_1 = m_f(z,x)$. If $k_1 \leqslant k$, then

$$\begin{aligned} d_f(x,z) &= k_f(x,z) + m_f(z,x) = k_1 + m_1, \\ d_f(z,y) &= m_1 + (k - k_1) + m, \\ n = d_f(x,y) &= d_f(x,z) + d_f(z,y) \\ &= k_1 + m_1 + m_1 + k - k_1 + m = (k+m) + 2m_1 = n + 2m_1, \end{aligned}$$

thus $m_1 = 0$ and $z = f^{k_1}(x)$. Since $k_1 \leqslant k$, we have

$$z \in L_f(x,y) = \{x, f(x), \ldots, f^k(x)\} \cup \{y, f(y), \ldots, f^m(y)\}.$$

Suppose that $k_1 > k$. Then

$$\begin{aligned} d_f(x,z) &= k_f(x,z) + m_f(z,x) = k_1 + m_1, \\ d_f(y,z) &= m_1 + (k_1 - k) + m, \\ n = d_f(x,y) &= d_f(x,z) + d_f(z,y) = k_1 + m_1 + m_1 + k_1 - k + m \\ &= (k_1 + m) + (k_1 - k) + 2m_1 > k + m + 2m_1 = n + 2m_1, \end{aligned}$$

which is a contradiction. □

3.8. Lemma. *Let (A,f), $(A,g) \in \mathcal{V}$ and suppose that $d_f(x,y) = d_g(x,y)$ for all x, $y \in A$. Then* $\mathrm{DuC}(A,f) = \mathrm{DuC}(A,g)$.

Proof. Let the assumption be satisfied and suppose that there is $M \in \mathrm{DuC}(A,f) - \mathrm{DuC}(A,g)$. First, let M be not convex in (A,g). Then there are

x, $y \in M$, $z \in A - M$ and there is a path in $G(A, g)$ going form x into y, containing y only once and containing z. Put $n = k_g(x, y)$, $j = k_g(x, z)$. Then $j < n$ and

$$y = g^n(x), \quad z = g^j(x),$$

$m_g(y, x) = 0$, $m_g(z, x) = 0$. By the assumption we get

$$\begin{aligned} j &= d_g(x, z) = d_f(x, z), \\ n &= d_g(x, y) = d_f(x, y), \\ n - j &= d_g(z, y) = d_f(z, y), \end{aligned}$$

thus $d_f(x, z) + d_f(z, y) = d_f(x, y)$. Lemma 3.7 implies that $z \in L_f(x, y)$. Since $M \in \mathrm{DuC}(A, f)$, x, $y \in M$, we have $L_f(x, y) \subseteq M$ and $z \in M$, a contradiction. Therefore M is convex in (A, g), hence M is not up-directed in (A, g) and there are x, $y \in M$, $x \neq y$, such that

$$M \cap \{g^i(x) \colon g^{i-1}(x) \in \operatorname{dom} g,\ i \in \mathbb{N},\ i \geqslant k_g(x, y)\} = \emptyset.$$

Put $k = k_g(x, y)$, $z = g^k(x)$. Then $z \notin M$. We have

$$d_g(x, z) + d_g(z, y) = d_g(x, y),$$

thus the assumption yields that

$$d_f(x, z) + d_f(z, y) = d_f(x, y).$$

It follows from 3.7 that $z \in L_f(x, y)$, hence $z \in M$, which is a contradiction. □

3.8. Corollary. *Let (A, f), $(A, g) \in \mathcal{V}$. Then $\mathrm{DuC}(A, f) = \mathrm{DuC}(A, g)$ if and only if $d_f(x, y) = d_g(x, y)$ for each x, $y \in A$.*

4. The class $\mathcal{V}$

We shall describe here two constructions which assign to each given partial monounary algebra $(A, f) \in \mathcal{V}$ some new partial monounary algebras. These constructions will be called "breaking (A, f) at one point" and "turning up (A, f) along a thread".

4.1.1. Construction. *Let $(A, f) \in \mathcal{V}$, $a \in A$. We define a partial mapping g of A into A as follows:*

(1) *if* $x \in \operatorname{dom} f$ *and* $x \neq f^i(a)$ *for each* $i \in \mathbb{N} \cup \{0\}$, *then* $x \in \operatorname{dom} g$ *and* $g(x) = f(x)$;

(2) *if* $a \in \operatorname{dom} f$, $x = f(a) \neq a$, *then* $x \in \operatorname{dom} g$ *and* $g(x) = a$;

(3) *if* $f^{i-1}(a) \in \operatorname{dom} f$, $x = f^i(a)$ *for some* $i \in \mathbb{N}$, $i > 1$ *and* $x \notin \{f^{i-1}(a), f^{i-2}(a)\}$, *then* $x \in \operatorname{dom} g$ *and* $g(x) = f^{i-1}(a)$;

(4) $a \in \operatorname{dom} g$, $g(a) = a$.

Notice that $(A, g) \in \mathcal{V}$ and it is a complete monounary algebra.

4.1.2. Definition. Let $(A, f) \in \mathcal{V}$, $a \in A$. If a partial mapping g of A into A is constructed as in 4.1.1, then we say that (A, g) is obtained by α-breaking (A, f) at a point $a \in A$.

4.2. Definition. Let $(A, f) \in \mathcal{V}$, $a \in A$. Assume that g is a partial mapping of A into A such that (1)–(3) from 4.1.1 are valid Consider the following conditions for g:

(β) If $a \in \operatorname{dom} f$ and $f(a) \neq a$, then $a \in \operatorname{dom} g$ and $g(a) = f(a)$.

(γ) $a \notin \operatorname{dom} g$.

If (β) holds, then the partial monounary algebra (A, g) is said to be obtained by β-breaking (A, f) at the point a. Similarly, if (γ) is valid, the we say that (A, g) is obtained by γ-breaking (A, f) at a. Let us remark that if either $a \notin \operatorname{dom} f$ or $f(a) = a$, then β-breaking (A, f) at a is not defined.

4.3. Definition. Let $(A, f) \in \mathcal{V}$, $a \in A$. If (A, g) is obtained by α-, β-, or γ-breaking (A, f) at a point $a \in A$, then we shall say that (A, g) is obtained by breaking (A, f) at $a \in A$. If there is $b \in A$ such that (A, g) is obtained by breaking (A, f) at this b, then (A, g) is said to be obtained by breaking (A, f).

4.4. Lemma. *Let* $(A, f) \in \mathcal{V}$ *and suppose that* (A, g) *is obtained by breaking* (A, f). *If* $x, y \in A$ *and* $d_f(x, y) = 1$, *then* $d_g(x, y) = 1$.

Proof. Assume that (A, g) is obtained by α-breaking (A, f) at $a \in A$ (the cases of β- or γ-breaking are quite analogous). Let $x, y \in A$, $d_f(x, y) = 1$. Then $x \neq y$ and either $y = f(x)$ or $x = f(y)$; we can suppose that $y = f(x)$. One of the following conditions is satisfied:

(1) $x \neq f^i(a)$ for each $i \in \mathbb{N} \cup \{0\}$,

(2) $x = a$,

(3) $x = f(a) \neq a$,

(4) $x = f^i(a)$ for some $i \in \mathbb{N}$, $i > 1$ and $x \notin \{f^{i-1}(a), f^{i-2}(a)\}$.

If (1) is valid, then 4.1.1 (1) implies that $x \in \operatorname{dom} g$ and $g(x) = f(x) = y$, thus $d_g(x, y) = 1$. Let (2) hold. Then $a \in \operatorname{dom} f$, $y = f(a) \neq a$ and 4.1.1 (2) yields that $y \in \operatorname{dom} g$ and $g(y) = a = x$, therefore $d_g(x, y) = 1$. Assume that the condition (3) is satisfied. By 4.1.1 (2) $g(x) = a$. If $y = a$ then $d_g(x, y) = 1$. Let

$y \neq a$; then $y \notin \{f(a), a\}$ and $y = f^2(a)$. Thus 4.1.1 (3) implies that $y \in \operatorname{dom} g$ and $g(y) = f(a) = x$. Hence $d_g(x, y) = 1$. Now suppose that (4) holds. According to 4.1.1 (3), $x \in \operatorname{dom} g$ and $g(x) = f^{i-1}(a)$. If $y = f^{i-1}(a)$, then $y = g(x)$ and $d_g(x, y) = 1$. Let $y \notin \{f^i(a), f^{i-1}(a)\}$. We have $y = f^{i+1}(a)$. By 4.1.1 (3), $g(y) = f^i(a)$ and hence $g(y) = x$, $d_g(x, y) = 1$. □

4.5. Lemma. *Let $(A, f) \in \mathscr{V}$ and suppose that (A, g) is obtained by breaking (A, f). Then $d_f(x, y) = d_g(x, y)$ for each x, $y \in A$.*

Proof. Similarly as in 4.4 we restrict ourselves to the case when (A, g) is obtained by α-breaking (A, f); the proofs for the other two cases can be performed analogously.

Let x, $y \in A$, $d_f(x, y) = n$. If $n = 0$, then $x = y$ and $d_g(x, y) = 0$. If $n = 1$, then the assertion is obtained by 4.4. Suppose that $n > 1$ and assume that the assertion is valid for 0, 1, ..., $n - 1$. First, let $y = f^n(x)$. Then $y \notin \{x, f(x), \ldots, f^{n-1}(x)\}$. Put $z = f^{n-1}(x)$. Then $d_f(x, z) = n - 1$, $d_f(z, y) = 1$. By the induction hypothesis, $d_g(x, z) = n - 1$, $d_g(z, y) = 1$. These relations, according to the definition of d_g, imply that either $d_g(x, y) = n - 2$ or $d_g(x, y) = n$. If $d_g(x, y) = n - 2$ then $d_f(x, y) = n - 2$ by the induction hypothesis, which is a contradiction. Therefore $d_g(x, y) = n = d_f(x, y)$. Now suppose that neither $y = f^n(x)$ nor $x = f^n(y)$ holds. Since (A, f) is connected, this implies that $x \in \operatorname{dom} f$. Put $v = f(x)$. We obtain that $d_f(x, v) = 1$, $d_f(v, y) = n - 1$. As above, $d_g(x, v) = 1$, $d_g(v, y) = n - 1$, thus either $d_g(x, y) = n - 2$ or $d_g(x, y) = n$. By the induction hypothesis, if $d_g(x, y) = n - 2$, then $d_f(x, y) = n - 2$, which is a contradiction, and hence $d_g(x, y) = n$. □

4.6.1. Definition. Let $(A, f) \in \mathscr{V}$. A set $B \subseteq A$ is called a thread of (A, f), if it satisfies one of the following conditions:

(a) $B = \{b_i : i \in \mathbb{Z}\}$, $b_i \neq b_j$ for each i, $j \in \mathbb{Z}$, $i \neq j$, and $f(b_{i-1}) = b_i$ for each $i \in \mathbb{Z}$;

(b) $B = \{b_i : i \in \mathbb{N}\}$, $b_i \neq b_j$ for each i, $j \in \mathbb{N}$, $i \neq j$, $f(b_{i+1}) = b_i$ for each $i \in \mathbb{N}$ and either $b_1 \notin \operatorname{dom} f$ or b_1 belongs to a cycle of (A, f).

4.6.2. Definition. Let (A, f), $(A, g) \in \mathscr{V}$ and let B be a thread of (A, f). Then (A, g) is said to be obtained by turning up (A, f) along a thread B, if $\operatorname{dom} g = A$, $g(x) = f(x)$ for each $x \in A - B$ and whenever $b \in B$, then $b \neq g(b) \in B \cap \operatorname{dom} f$ and $f(g(b)) = b$.

4.7. Lemma. *Assume that $(A, f) \in \mathscr{V}$, $B \subseteq A$ and (A, g) is obtained by turning up (A, f) along a thread B. One of the following conditions is satisfied:*

(a) *$B = \{b_i : i \in \mathbb{Z}\}$, $b_i \neq b_j$ for each i, $j \in \mathbb{Z}$, $i \neq j$, and $f(b_{i-1}) = b_i = g(b_{i+1})$ for each $i \in \mathbb{Z}$;*

(b) *$B = \{b_i : i \in \mathbb{N}\}$, $b_i \neq b_j$ for each i, $j \in \mathbb{N}$, $i \neq j$, $f(b_{i+1}) = b_i$, $g(b_i) = b_{i+1}$ for each $i \in \mathbb{N}$ and either $b_1 \notin \operatorname{dom} f$ or b_1 belongs to a cycle of (A, f).*

Proof. Let the assumption hold. Then either (a) or (b) of 4.6.1 is valid. First, suppose that (a) of 4.6.1 is valid and let $b_i \in B$. By 4.6.2 we have $f(g(b_i)) = b_i = f(b_{i-1})$ and $f(b_{j-1}) = b_j$ for each $j \in \mathbb{Z}$, thus $g(b_i) = b_{i-1}$. Now let (b) of 4.6.1 hold, $b_i \in B$. According to 4.6.2, there is $j \in \mathbb{N}$, $j \neq i$, with $g(b_i) = b_j$. Then $b_i = f(g(b_i)) = f(b_j)$. If $j > 1$, then $f(b_j) = b_{j-1}$, thus $i = j - 1$ and $g(b_i) = b_j = b_{i+1}$, $i > 2$. Let $j = 1$. We have $b_i = f(b_1)$. By 4.6.1, b_1 belongs to a cycle. Since $f(b_2) = b_1$, this implies that either $f(b_1) = b_1$, or $f(b_1) = b_2$, or $f(b_1) = b$, $f(b) = b_1$, $b \notin B$. The relation $f(b_1) = b_1$ contradicts the fact that $j \neq i$, the relation $f(b_1) = b$ contradicts $b_i = f(b_1)$, therefore $f(b_1) = b_2$. We obtain $b_i = b_2$ and $g(b_2) = b_1$. □

4.8. Definition. Let (A, f), $(A, g) \in \mathcal{V}$. If (A, g) is obtained by turning up (A, f) along a thread B, then (A, g) is said to be obtained by turning up (A, f).

4.9. Lemma. *Let (A, f), $(A, g) \in \mathcal{V}$ and suppose that (A, g) is obtained by turning up (A, f). If x, $y \in A$ and $d_f(x, y) = 1$, then $d_g(x, y) = 1$.*

Proof. Let (A, g) be obtained by turning up (A, f) along a thread B. If x, $y \in A$, $d_f(x, y) = 1$, then we can assume that $y = f(x)$. If $x \notin B$, then $g(x) = f(x) = y$ and hence $d_g(x, y) = 1$. Let $x \in B$, i.e., $x = b_i$ for some $i \in \mathbb{Z}$ ($i \in \mathbb{N}$). According to 4.7, either (a) or (b) of 4.7 is valid. Suppose that (a) holds. Then 4.7 implies that $f(b_i) = b_{i+1}$, $g(b_{i+1}) = b_i$, hence $g(y) = x$ and $d_g(x, y) = 1$. Now let (b) be valid. If $i > 1$, then $d_g(x, y) = 1$ similarly as if (a) holds. Let $i = 1$. Since $f(x) = y$, $d_f(x, y) = 1$, we obtain that $x = b_1$ belongs to a two-element cycle, because $(A, f) \in \mathcal{V}$. Then $f(y) = x$. If $y \notin B$, then $g(y) = f(y) = x$ (in view of 4.6.2), hence $d_g(x, y) = 1$. If $y \in B$, then $y = b_2$. According to 4.7, $g(b_1) = b_2 = y$, thus $d_g(x, y) = 1$. □

4.10. Lemma. *Let (A, f), $(A, g) \in \mathcal{V}$ and suppose that (A, g) is obtained by turning up (A, f). Then $d_f(x, y) = d_g(x, y)$ for each x, $y \in A$.*

Proof. Analogously as 4.5. □

4.11. Lemma. *Let (A, f), $(A, g) \in \mathcal{V}$ and suppose that $d_f(x, y) = d_g(x, y)$ for each x, $y \in A$. If $z \in \operatorname{dom} g$ and $g(z) \neq z$, then either $z \in \operatorname{dom} f$ and $g(z) = f(z)$, or $g(z) \in \operatorname{dom} f$ and $f(g(z)) = z$.*

Proof. Let the assumption hold and let $z \in \operatorname{dom} g$, $g(z) \neq z$. Then $1 = d_g(z, g(z)) = d_f(z, g(z))$. Put $y = g(z)$. Assume that either $z \notin \operatorname{dom} f$ or $z \in \operatorname{dom} f$, $g(z) \neq f(z)$ (i.e., $y \neq f(z)$). The relation $d_f(z, y) = 1$ implies that then $y \in \operatorname{dom} f$ and $f(y) = z$. Thus $g(z) \in \operatorname{dom} f$ and $f(g(z)) = z$. □

4.12. Lemma. *Let* (A, f), $(A, g) \in \mathcal{V}$, $g \neq f$. *If* $d_f(x, y) = d_g(x, y)$ *for each* x, $y \in A$, *then* (A, g) *is obtained either by turning up or by breaking* (A, f).

Proof. Assume that $d_f(x, y) = d_g(x, y)$ for each x, $y \in A$. If $\operatorname{card} A = 1$, then obviously (A, g) is obtained by breaking (A, f). Let $\operatorname{card} A > 1$. Then $\operatorname{dom} g \neq \emptyset$. Since $g \neq f$, there exists $b_1 \in A$ such that either

$$(1.1) \qquad b_1 \in \operatorname{dom} f - \operatorname{dom} g$$

or

$$(1.2) \qquad b_1 \in \operatorname{dom} g \text{ and either } b_1 \notin \operatorname{dom} f \text{ or } b_1 \in \operatorname{dom} f, \quad g(b_1) \neq f(b_1).$$

Let us introduce the following elements by induction: Let $i \in \mathbb{N}$, $i > 1$. If $b_1, \ldots, b_{i-1}$ are defined and $b_{i-1} \in \operatorname{dom} g$, $g(b_{i-1}) \notin \{b_1, \ldots, b_{i-1}\}$, then put $b_i = g(b_{i-1})$. (In the opposite case we stop introducing b_i's.) Further, let B be the set of all b_i defined above. One of the possibilities (a)–(c) occurs:

(a) (A, g) contains a one-element cycle or $\operatorname{dom} g \neq A$: then $B = \{b_1, \ldots, b_n\}$ and either $g(b_n) = b_n$ or $b_n \notin \operatorname{dom} g$; $g(b_i) = b_{i+1}$ for each $i \in \{1, \ldots, n-1\}$, if $n > 1$;

(b) (A, g) contains a two-element cycle: then $B = \{b_1, \ldots, b_n\}$, $n > 1$ and $g(b_n) = b_{n-1}$; $g(b_i) = b_{i+1}$ for each $i \in \{1, \ldots, n-1\}$;

(c) (A, g) contains no cycle and $\operatorname{dom} g = A$: then $B = \{b_i : i \in \mathbb{N}\}$, $g(b_i) = b_{i+1}$ for each $i \in \mathbb{N}$.

Let us show by induction that

$$(2) \qquad \text{if } b_i \in B, \quad \text{where } i \in \mathbb{N}, \quad i > 1, \quad \text{then } b_i \in \operatorname{dom} f \quad \text{and} \quad f(b_i) = b_{i-1}$$

is valid.

Assume that $b_2 \in B$. Then (1.2) holds. Next, $g(b_1) = b_2$, $b_2 \neq b_1$ and 4.11 yields that either $b_1 \in \operatorname{dom} f$ and $g(b_1) = f(b_1)$, or $g(b_1) \in \operatorname{dom} f$ and $f(g(b_1)) = b_1$. In the first case we have got a contradiction to (1.2), in the second the required assertion is valid. Now let $i \in \mathbb{N}$, $i > 2$, $b_i \in B$ and suppose that if $j \in \mathbb{N}$, $1 < j < i$, $b_i \in B$, then $b_j \in \operatorname{dom} f$ and $f(b_j) = b_{j-1}$. Then $b_{i-1} \in \operatorname{dom} f$, $f(b_{i-1}) = b_{i-2} \neq b_{i-1}$. By 4.11, either $b_{i-1} \in \operatorname{dom} f$ and $g(b_{i-1}) = f(b_{i-1})$ (i.e., $b_i = b_{i-1}$, a contradiction), or $g(b_{i-1}) \in \operatorname{dom} f$, $f(g(b_{i-1})) = b_{i-1}$. Thus $b_i \in \operatorname{dom} f$, $f(b_i) = b_{i-1}$.

Analogously as B, we can define a set C as follows: let $c_1 = b_1$ and let $i \in \mathbb{N}$, $i > 1$. If $c_1, \ldots, c_{i-1}$ are defined and $c_{i-1} \in \operatorname{dom} f$, $f(c_{i-1}) \notin \{c_1, \ldots, c_{i-1}\}$, then put $c_i = f(c_{i-1})$. (By (1.1) or (1.2), $c_i \notin B$.) The set C is the set of all such c_i's. (As above, we stop the process, if we cannot define the next c_i.) It can be proved as prove that the following condition is satisfied:

(3) if $c_i \in C$, where $i \in \mathbb{N}$, $i > 1$, then $c_i \in \operatorname{dom} g$ and $g(c_i) = c_{i-1}$. Further, one of the conditions analogous to (a)–(c) is valid (with g, B, b_i replaced by f, C, c_i).

Put $D = B \cup C$. If (c) holds, then D is a thread of (A, f). If (a) or (b) is valid, i.e., B is finite, according to (a)–(c) we obtain that one of the following conditions is satisfied:

$$b_n \notin \operatorname{dom} g, \tag{4.1}$$

$$g(b_n) = b_n, \tag{4.2}$$

$$g(b_{n-1}) = b_n, \quad g(b_n) = b_{n-1}. \tag{4.3}$$

To complete the proof let us now

$$g(x) = f(x) \text{ for each } x \in A - D. \tag{5}$$

Let $x \in A - D$. There is a unique $k \in \mathbf{N}$ with $f^k(x) \in D$, $f^{k-1}(x) \notin D$. First, let $k = 1$, $f(x) = e$. According to the definition of the set D we obtain that $x \in \operatorname{dom} f$, $f(x) \neq x$. Application of 4.11 (with f and g interchanged) yields that either

$$x \in \operatorname{dom} g \quad \text{and} \quad g(x) = f(x) \tag{6.1}$$

or

$$f(x) \in \operatorname{dom} g \quad \text{and} \quad g(f(x))x. \tag{6.2}$$

Suppose that (6.2) is valid. Then $e \in \operatorname{dom} g$ and $g(e) = x$, which is a contradiction, since if $e \in \operatorname{dom} g$, then $g(e) \in D$, $x \notin D$.

Now let $k > 1$. By the induction hypothesis, $g(f(x)) = f(f(x))$. Put $f(x) = y$. We have $f(x) \neq x$, thus 4.11 (again with f and g interchanged) implies that either (6.1) or (6.2) holds. If we suppose the validity of (6.2), then $y \in \operatorname{dom} g$, $g(y) = x$ and

$$x = g(y) = g(f(x)) = f^2(x).$$

Thus x belongs to a cycle of (A, f). The set $C \subseteq D$ was constructed in such a way that each element of a cycle of (A, f) belongs to C; therefore $x \in C \subseteq D$, which is a contradiction.

If D is the thread of (A, f) defined above and (5) holds, then (A, g) is obtained by turning up (A, f) along D. If (4.1) and (5) hold, then (A, g) is obtained by γ-breaking (A, f) at the point b_n. If (4.2) and (5) hold, then (A, g) is obtained by α-breaking (A, f) at b_n. If (4.3) and (5) hold, then (A, g) is obtained by β-breaking (A, f) at the point b_n. □

4.13 Lemma. *Let (A, f), $(A, g) \in \mathcal{V}$, $g \neq f$. Then* $\mathrm{DuC}(A, f) = \mathrm{DuC}(A, g)$ *if and only if (A, g) is obtained from (A, f) either by turning up or by breaking.*

Proof. Let us consider the following conditions:

(i) $\mathrm{DuC}(A, f) = \mathrm{DuC}(A, g)$,

(ii) $d_f(x, y) = d_g(x, y)$ for each x, $y \in A$,

(iii) (A, g) is obtained either by turning up or by breaking (A, f). The relation (i) $\Longleftrightarrow$ (ii) was proved in 3.8. Further, the implication (ii) $\Longrightarrow$ (iii) was shown in 4.12 and the converse implication, (iii) $\Longrightarrow$ (ii), follows from 4.5 and 4.10. □

5. The class $\mathscr{W}$ and the general case

In this section we shall first study (partial) monounary algebras (A, g) such that, if (A, f) is a connected monounary algebra possessing a cycle with more than two elements, then $\mathrm{DuC}(A, f) = \mathrm{DuC}(A, g)$. Let us remark that C is a cycle of (A, f) if and only if C is a cycle of (A, g) (by 2.5). Further, the general case is investigated.

5.1. Notation. *Let $(A, f) \in \mathscr{W}$ and assume that C is a cycle of (A, f). Let Θ be an equivalence relation A such that*

$$x\Theta = \begin{cases} x, & \text{if } x \notin C, \\ C, & \text{if } x \in C. \end{cases}$$

Then Θ is a congruence relation of (A, f) and it determines a monounary algebra $(A', f') = (A, f)/\Theta$. (If $x\Theta = \{x\}$ for $x \in A$, we shall also write $x\Theta = x$.)

5.2. Lemma. *Let (A, f), $(A, g) \in \mathscr{W}$ and suppose that C is a cycle of (A, f) and of (A, g). Then* $\mathrm{DuC}(A, f) = \mathrm{DuC}(A, g)$ *implies that* $\mathrm{DuC}(A', f') = \mathrm{DuC}(A', g')$.

Proof. Let $\mathrm{DuC}(A, f) = \mathrm{DuC}(A, g)$, $B \in \mathrm{DuC}(A', f')$. (Notice that C is an element of A', but C is a convex set of (A, f).) If $C \notin B$, then $B \in \mathrm{DuC}(A, f) = \mathrm{DuC}(A, g)$, and the relations $C \notin B$, $B \in \mathrm{DuC}(A, g)$ imply that $B \in \mathrm{DuC}(A', g')$. Let $C \in B$. Denote $B_1 = B - C$. Since $B \in \mathrm{DuC}(A', f')$, the set $B_1 \cup C$ belongs to $\mathrm{DuC}(A, f)$. Then $(B_1 \cup C)/\Theta = B \in \mathrm{DuC}(A', g')$, because $B_1 \cup C \in \mathrm{DuC}(A, g)$. Therefore $\mathrm{DuC}(A', f') \subseteq \mathrm{DuC}(A', g')$. The converse inclusion can be proved analogously. □

5.3. Lemma. *Let (A, f), $(A, g) \in \mathscr{W}$,* $\mathrm{DuC}(A, f) = \mathrm{DuC}(A, g)$. *If x does not belong to a cycle of (A, f), then $g(x) = f(x)$.*

Proof. Suppose that C is a cycle of (A, f) (and hence of (A, g), too, in view of 2.5). According to 5.2 we have

$$\mathrm{DuC}(A', f') = \mathrm{DuC}(A', g'). \tag{1}$$

First, suppose that $g' \neq f'$.

Since $(A', f'), (A', g') \in \mathscr{V}$, $g' \neq f'$, the relation (1) and 4.10 imply that (A', g') is obtained either by turning up or by breaking (A', f'). However, (A', f') and (A', g') contain the same one-element cycle C, $f'(C) = C = g'(C)$, thus (A', g') is obtained by α-breaking (A', f') at the point C, and then $g' = f'$. Therefore

$$(3) \qquad g'(x) = f'(x) \quad \text{for each} \quad x \in A - C.$$

Let $x \in A - C$, $g(x) \neq f(x)$. Then $\{g(x), f(x)\} \subseteq C$. Put $g(x) = c_1$, $f(x) = c_2$. By 2.6, $x \in A_g(c_1)$ and $x \in A_f(c_2)$, which contradicts 2.8. □

5.4. Notation. *Let $(A, f) \in \mathscr{W}$, let $C = \{c_1, c_2, \ldots, c_n\}$ be a cycle of (A, f),* card $C = n$. *The set of all permutations of $1, 2, \ldots, n$ will be denoted by S_n. If $\varepsilon \in S_n$, put $f_\varepsilon(x) = f(x)$ for each $x \in A - C$, $f_\varepsilon(c_{\varepsilon(i)}) = c_{\varepsilon(i+1)}$ for each $i \in \{1, 2, \ldots, n-1\}$, $g_\varepsilon(c_{\varepsilon(n)}) = c_{\varepsilon(1)}$. For $\varepsilon \in S_n$, (A, f_ε) is said to be obtained from (A, f) by permuting a cycle.*

5.5. Lemma. *Let $(A, f) \in \mathscr{W}$. If $(A, g) \in \mathscr{U}$, then* $\mathrm{DuC}(A, f) = \mathrm{DuC}(A, g)$ *if and only if (A, g) is obtained from (A, f) by permuting a cycle.*

Proof. Suppose that $\mathrm{DuC}(A, f) = \mathrm{DuC}(A, g)$. If C is a cycle of (A, f), then C is a cycle of (A, g) in view of 2.5 and according to 5.3, $g(x) = f(x)$ for each $x \in A - C$. Therefore $g = f_\varepsilon$ for some $\varepsilon \in S_n$. Conversely, if $g = g_\varepsilon$ for some $\varepsilon \in S_n$, then it is obvious that $\mathrm{DuC}(A, f) = \mathrm{DuC}(A, g)$. □

In the following theorem, the operations on connected components of (A, f) and (A, g) are denoted by the symbols f or g, respectively.

5.6. Theorem. *Let (A, f) and (A, g) be partial monounary algebras. Then (A, f) and (A, g) have the same systems of up-directed convex subsets if and only if the following conditions are satisfied:*

(i) *(A, f) and (A, g) have the same partition into connected components,*

(ii) *if B is a connected component of (A, f) and the partial operations g and f on B are distinct, then (B, g) is obtained from (B, f) by turning up, breaking or permuting a cycle.*

Proof. Let $\mathrm{DuC}(A, f) = \mathrm{DuC}(A, g)$. Then (i) holds by 2.2 and (ii) by 4.13 and 5.5. If (i) and (ii) hold, then the relation $\mathrm{DuC}(A, f) = \mathrm{DuC}(A, g)$ follows from 4.13 and 5.5 (notice that by permuting a two-element cycle on a component B we have $g = f$ on B). □

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