Czechoslovak Mathematical Journal

Ethiraju Thandapani; S. Pandian

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Czechoslovak Mathematical Journal, Vol. 45 (1995), No. 4, 757–766

Persistent URL: http://dml.cz/dmlcz/128557

Czechoslovak Mathematical Journal, 45 (120) 1995, Praha

OSCILLATION THEOREMS FOR CERTAIN SECOND ORDER PERTURBED NONLINEAR DIFFERENCE EQUATIONS

E. THANDAPANI and S. PANDIAN, Salem

(Received May 25, 1994)

I. INTRODUCTION

In this paper we discuss the oscillatory behavior of the solutions of the perturbed second order nonlinear difference equation

$$\Delta(a_n h(y_{n+1})\Delta y_n) + Q(n, y_{n+1}) = P(n, y_{n+1}, \Delta y_n), \quad n \in \mathbb{N} \tag{1}$$

where $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$, Δ is the forward difference operator defined by $\Delta y_n = y_{n+1} - y_n$, $\{a_n\}$ is a real sequence with $a_n > 0$ for all $n \in \mathbb{N}$, $h\colon \mathbb{R} \to \mathbb{R} = (-\infty, \infty)$, $Q\colon \mathbb{N} \times \mathbb{R} \to \mathbb{R}$ and $P\colon \mathbb{N} \times \mathbb{R}^2 \to \mathbb{R}$ are continuous functions. The problem of determining oscillation criteria for less general equations has received a fair amount of attention on the last few years, see for example [4–13] and the references cited therein. To a large extent this is due to the realization that difference equations are important in applications. An excellent discussion of known oscillation criteria as well as some suggestions for future study and many references can be found in the recent monograph by Agarwal [1].

By a solution of equation (1) we mean a nontrivial sequence $\{y_n\}$ satisfying equation (1) for all $n \in \mathbb{N}$. A solution $\{y_n\}$ of (1) is said to be oscillatory if it is neither eventually positive nor eventually negative, and nonoscillatory otherwise.

For equations with perturbation terms such as equation (1), relatively few oscillation criteria are known, see for example [7] and [9]. In many instances our results will include, as special cases, known oscillation theorems established in [4, 5, 9, 11]. Examples illustrating some of our theorems are also presented. The results obtained here are motivated by those in [2, 3, 14].

2. Main results

In the sequel we assume that there exist real sequences $\{q_n\}$, $\{p_n\}$, $p_n \geqslant 0$, and continuous functions $q, f_1, f_2 \colon \mathbb{R} \to \mathbb{R}$ such that

$$Q(n,u)/f_1(u) \geqslant q_n \quad \text{and} \quad P(n,u,v)/f_2(u)g(v) \leqslant p_n \tag{2}$$

for $u, v \neq 0$, where

$$uf_i(u) > 0 \quad \text{for all } u \neq 0,\ i = 1, 2, \tag{3}$$

$$f_2(u)/f_1(u) \leqslant K \quad \text{for } u \neq 0,\ K > 0, \tag{4}$$

$$f_1(u) - f_2(v) = g_1(u,v)(u-v) \quad \text{for } u, v \neq 0, \tag{5}$$

g_1 is nonnegative function,

$$0 < g(v) \leqslant c \quad \text{for some constant } c. \tag{6}$$

$$\text{Also we assume that} \quad 0 < c_1 \leqslant h(u) \quad \text{for some constant } c_1 \tag{7}$$

and

$$\sum^{\infty} \frac{1}{a_n} = \infty. \tag{8}$$

We begin with the following theorem.

Theorem 1. *Let conditions* (2)–(8) *be fulfilled. If*

$$\sum^{\infty} (q_n - \mu p_n) = \infty. \tag{9}$$

where $\mu = cK$, then all solutions of equation (1) *are oscillatory.*

Proof. Suppose $\{y_n\}$ is a nonoscillatory solution of equation (1), say $y_n \neq 0$ for $n \geqslant n_0 \in \mathbb{N}$. Then

$$\begin{aligned}
\Delta\left(\frac{a_n h(y_{n+1})\Delta y_n}{f_1(y_n)}\right) \leqslant{} & \frac{-Q(n,y_{n+1})}{f_1(y_{n+1})} + \frac{P(n,y_{n+1},\Delta y_n)}{f_1(y_{n+1})} \\
& - \frac{a_n h(y_{n+1}) g_1(y_n, y_{n+1})(\Delta y_n)^2}{f_1(y_n) f_2(y_{n+1})} \\
\leqslant{} & -(q_n - \mu p_n).
\end{aligned}$$

Summing the above inequality from n_0 to $n-1$, we get

$$\frac{a_n h(y_{n+1})\Delta y_n}{f_1(y_n)} \leqslant \frac{a_{n_0} h(y_{n_0+1})\Delta y_{n_0}}{f_1(y_{n_0})} - \sum_{s=n_0}^{n-1} (q_s - \mu p_s)$$

or

$$\frac{a_n \Delta y_n}{f_1(y_n)} \leqslant \frac{a_{n_0} h(y_{n_0+1})\Delta y_{n_0}}{c_1 f_1(y_{n_0})} - \frac{1}{c_1} \sum_{s=n_0}^{n-1} (q_s - \mu p_s). \tag{10}$$

We assume that $y_n > 0$ for $n \geqslant n_1 \geqslant n_0 \in \mathbb{N}$; the proof for the case $y_n < 0$, $n \geqslant n_1$ is similar and will be omitted. In view of condition (9) it follows from (10) that there exists $n_2 \geqslant n_1$ such that $\Delta y_n \leqslant 0$ for $n \geqslant n_2$. It also follows from condition (9) that there exists an integer $n_3 \geqslant n_2$ such that

$$\sum_{s=n_3}^{n-1} (q_s - \mu p_s) \geqslant 0 \quad \text{for} \quad n \geqslant n_3.$$

Now summing equation (1) and using condition (2) we have

$$\begin{aligned} a_n h(y_{n+1})\Delta y_n &\leqslant a_{n_3} h(y_{n_3+1})\Delta y_{n_3} - \sum_{s=n_3}^{n-1} f_1(y_{s+1})(q_s - \mu p_s) \\ &= a_{n_3} h(y_{n_3+1})\Delta y_{n_3} - f_1(y_{n+1}) \sum_{s=n_3}^{n-1} (q_s - \mu p_s) \\ &\quad + \sum_{s=n_3}^{n-1} \Delta f_1(y_{s+1}) \left[\sum_{t=n_3}^{s} (q_t - \mu p_t) \right] \leqslant a_{n_3} h(y_{n_3+1})\Delta y_{n_3}. \end{aligned}$$

Hence

$$\Delta y_n \leqslant \frac{a_{n_3} h(y_{n_3+1})\Delta y_{n_3}}{c_1 a_n}.$$

From (8), it follows that $y_n \to \infty$ as $n \to \infty$ which is a contradiction. This completes the proof of the theorem. □

Remark. Theorem 1 generalizes Theorem 1 given in [9]. Theorem 1 also includes the result of [11] as a special case.

As an example of Theorem 1, consider the difference equation

$$\Delta\big(n(1 + y_{n+1}^2)\Delta y_n\big) + (10n + 4)y_{n+1}^3 = 4n\frac{y_{n+1}^3}{1 + y_{n+1}^2}, \quad n \geqslant 1. \tag{E$_1$}$$

Choose $f_1(u) = u^3$, $f_2(u) = u$, then all hypotheses of Theorem 1 are satisfied. That this equation (E_1) is oscillatory does not appear to be deducible from other known oscillation criteria.

In the next theorem we study the oscillation criteria for equation (1) subject to the conditions

$$\frac{f_1}{h} \quad \text{is a nondecreasing function on } \mathbb{R} \tag{11}$$

and

$$\int_\alpha^\infty \frac{h(x)}{f_1(x)}\,\mathrm{d}x < \infty \quad \text{and} \quad \int_{-\alpha}^{-\infty} \frac{h(x)}{f_1(x)}\,\mathrm{d}x < \infty \quad \text{for all } \alpha > 0. \tag{12}$$

Theorem 2. *Suppose that conditions* (2)–(8), (11) *and* (12) *hold, and in addition*

$$\sum^\infty (q_n - \mu p_n) < \infty, \tag{13}$$

$$\liminf_{n\to\infty} \sum_{s=n_0}^{n-1} (q_s - \mu p_s) \geqslant 0 \quad \textit{for large } n_0 \tag{14}$$

and

$$\lim_{n\to\infty} \sum_{s=n_0}^{n} \frac{1}{a_s}\left[\sum_{t=s}^{\infty}(q_t - \mu p_t)\right] = \infty \tag{15}$$

are satisfied. Then all solutions of equation (1) *are oscillatory.*

Proof. Let $\{y_n\}$ be a nonoscillatory solution of equation (1), say $y_n > 0$ for all $n \geqslant n_0 \in \mathbb{N}$. For any $n_1 \geqslant n_0$, summation of (1) yields

$$\frac{a_n h(y_{n+1})\Delta y_n}{f_1(y_{n+1})} \leqslant \frac{a_{n_1} h(y_{n_1+1})\Delta y_{n_1}}{f_1(y_{n_1+1})} - \sum_{s=n_1}^{n-1} (q_s - \mu p_s). \tag{16}$$

Now, if $\Delta y_n > 0$ for all $n \geqslant n_1 \geqslant n_0 \in \mathbb{N}$, we have from (16)

$$0 \leqslant \frac{a_{n_1} h(y_{n_1+1})\Delta y_{n_1}}{f_1(y_{n_1+1})} - \sum_{s=n_1}^{\infty} (q_s - \mu p_s).$$

Hence, for all $n \geqslant n_1$ we have

$$\sum_{s=n}^{\infty}(q_s - \mu p_s) \leqslant \frac{a_n h(y_{n+1})\Delta y_n}{h(y_{n+1})}$$

or

$$\frac{1}{a_n}\sum_{s=n}^{\infty}(q_s-\mu p_s)\leqslant \frac{h(y_{n+1})}{f_1(y_{n+1})}\Delta y_n. \tag{17}$$

Observe that for $y_n \leqslant x \leqslant y_{n+1}$ we have $\frac{h(x)}{f_1(x)} \geqslant \frac{h(y_n+1)}{f_1(y_n+1)}$ and it follows that

$$\int_{y_n}^{y_n+1}\frac{h(x)}{f_1(x)}\,\mathrm{d}x \geqslant \frac{h(y_{n+1})}{f_1(y_{n+1})}\Delta y_n.$$

Using the last inequality in (17) and summing from n_1 to n, we obtain

$$\sum_{s=n_1}^{n}\frac{1}{a_s}\left[\sum_{t=s}^{\infty}(q_t-\mu p_t)\right]\leqslant \int_{y_{n_1}}^{y_n+1}\frac{h(x)}{f_1(x)}\,\mathrm{d}x. \tag{18}$$

This contradicts (12) since the left sum diverges.

If $\{\Delta y_n\}$ changes sign, then there exists a sequence $\{n_k\}$ such that $\Delta y_{n_k} < 0$. Choose k large enough for (14) to hold. We then have

$$\frac{a_n h(y_{n+1})\Delta y_n}{f_1(y_{n+1})}\leqslant \frac{a_{n_k}h(y_{n_k+1})\Delta y_{n_k}}{f_1(y_{n_k+1})}-\sum_{s=n_k}^{n-1}(q_s-\mu p_s)$$

so

$$\limsup_{n\to\infty}\frac{a_n h(y_{n+1})\Delta y_n}{f_1(y_{n+1})}\leqslant \frac{a_{n_k}h(y_{n_k+1})\Delta y_{n_k}}{f_1(y_{n_k+1})}+\limsup_{n\to\infty}\left[-\sum_{s=n_k}^{n-1}(q_s-\mu p_s)\right]<0,$$

which contradicts the fact that $\{\Delta y_n\}$ oscillates. Hence there exists an integer $n_2 \geqslant n_1$ such that $\Delta y_n < 0$ for all $n \geqslant n_2$. Condition (14) implies that for any integer $n_3 \geqslant n_0$ there exists $n_4 \geqslant n_3$ such that

$$\sum_{s=n_4}^{n-1}(q_s-ckp_s)>0$$

for all $n \geqslant n_4$. Choosing $n_4 \geqslant n_2$ as indicated and then summing equation (1), we have

$$\begin{aligned}a_n h(y_{n+1})\Delta y_n &\leqslant a_{n_4}h(y_{n_4+1})\Delta y_{n_4}-\sum_{s=n_4}^{n-1}f_1(y_{s+1})(q_s-\mu p_s)\\ &= a_{n_4}h(y_{n_4+1})\Delta y_{n_4}-f_1(y_{n+1})\sum_{s=n_4}^{n-1}(q_s-\mu p_s)\\ &\quad+\sum_{s=n_4}^{n-1}\Delta f_1(y_{s+1})\left[\sum_{t=n_4}^{s}(q_t-\mu p_t\right]\leqslant a_{n_4}h(y_{n_4+1})\Delta y_{n_4}\end{aligned}$$

$$h(y_{n+1}\Delta y_n \leqslant \frac{a_{n_4}h(y_{n_4+1})\Delta y_{n_4}}{a_n}. \tag{19}$$

Since $\{y_n\}$ is positive decreasing and h is positive and continuous, there exists a positive constant C and an integer $n_5 \geqslant n_4$ such that $0 < h(y_{n+1}) \leqslant C$, $n \geqslant n_5$. Hence

$$\Delta y_n \leqslant \frac{a_{n_4} h(y_{n_4+1}) \Delta y_{n_4}}{C a_n}, \qquad n \geqslant n_5.$$

Summing the above inequality and using (8) we get $y_n \to -\infty$ as $n \to \infty$, a contradiction. A similar proof holds when $\{y_n\}$ is eventually negative. □

Remark. If $h(u) \equiv 1$ and $f_1(u) = f_2(u)$, then Theorem 2 reduces to Theorem 2 given [9].

Corollary 3. *If conditions* (2)–(8) *and* (13)–(15) *hold, then all bounded solutions of equation* (1) *are oscillatory.*

P r o o f. Condition (10) was only used in the first part of the proof of Theorem 2. We had $y_n > 0$ and $\Delta y_n \geqslant 0$ for $n \geqslant n_1 \geqslant n_0 \in \mathbb{N}$, so by (5), $f_1(y_n) \geqslant f_1(y_{n_1})$ for $n \geqslant n_1$. From (15) and (18) we then obtain a contradiction to the boundedness of $\{y_n\}$. □

Theorem 4. *Let conditions* (2)–(7), (11) *and* (14) *hold. Assume that*

$$\int_0^{\pm\alpha} \frac{h(x)}{f_1(x)}\,\mathrm{d}x < \infty \quad \text{for every} \quad \alpha > 0 \tag{20}$$

and

$$\sum^{\infty} \left[\frac{M}{a_s} - \frac{1}{a_s} \sum_{t=n_0}^{s-1} (q_t - \mu p_t)\right] = -\infty \tag{21}$$

for every constant M. Then all solutions of equation (1) *are oscillatory.*

P r o o f. Let $\{y_n\}$ be a nonoscillatory solution of equation (1), say $y_n > 0$ for all $n \geqslant n_0 \in \mathbb{N}$; the proof for the case $y_n < 0$ for all $n \geqslant n_0 \in \mathbb{N}$ is similar and will be omitted. Since (14) holds we see from the proof of Theorem 2 that $\{\Delta y_n\}$ cannot change sign for arbitrarily large n. If $\Delta y_n \geqslant 0$ for all $n \geqslant n_1 \geqslant n_0 \in \mathbb{N}$, then (16) implies

$$\frac{h(y_{n+1})}{f_1(y_{n+1})} \Delta y_n \leqslant \frac{a_{n_1} h(y_{n_1+1}) \Delta y_{n_1}}{a_n f_1(y_{n_1+1})} - \frac{1}{a_n} \sum_{s=n_1}^{n-1} (q_s - \mu p_s)$$

and another summation yields

$$\sum_{s=n_1}^{n} \frac{h(y_{s+1})}{f_1(y_{s+1})} \Delta y_s \leqslant \sum_{s=n_1}^{n} \left[\frac{M}{a_s} - \frac{1}{a_s} \sum_{t=n_1}^{s-1} (q_t - \mu p_t)\right], \tag{22}$$

which contradicts (21) since the left hand side of (22) is nonnegative.

If $\Delta y_n < 0$ for $n \geqslant n_1$, we have from (22)

$$\int_{y_{n_1}}^{y_{n+1}} \frac{h(x)}{f_1(x)}\,\mathrm{d}x \leqslant \sum_{s=n_1}^{n} \frac{h(y_{s+1})}{f_1(y_{s+1})}\Delta y_s \leqslant \sum_{s=n_1}^{n}\left[\frac{M}{a_s} - \frac{1}{a_s}\sum_{t=n_1}^{s-1}(q_t - \mu p_t)\right]$$

or

$$\int_{y_{n+1}}^{y_{n_1}} \frac{h(x)}{f_1(x)}\,\mathrm{d}x \geqslant -\sum_{s=n_1}^{n}\left[\frac{M}{a_s} - \frac{1}{a_s}\sum_{t=n_1}^{s-1}(q_t - \mu p_t)\right],$$

which contradicts condition (20). This completes the proof of the theorem. □

Remark. Theorem 4 generalizes Theorem 4 given in [9]. For the equation

$$(\mathrm{E}_2)\qquad \Delta\big(n(n+1)y_{n+1}^2\Delta y_n\big) + \big[4(n+1)^2 + (n+2)y_{n+1}^2\big]y_{n+1}^{1/3} = \frac{2(n+2)y_n^{11/5}}{1+y_n^2}, \qquad n \geqslant 1$$

Let us put $f_1(u) = u^{1/3}$, $f_2(u) = u^{1/5}$. Then

$$\frac{Q(n, y_{n+1})}{f_1(y_{n+1})} \geqslant 4(n+1)^2, \qquad \frac{P(n, y_{n+1}, \Delta y_n)}{f_2(y_{n+1})} \geqslant 2(n+2)$$

and we see that all the hypotheses of Theorem 4 are satisfied. Hence equation (E_2) is oscillatory. Again the oscillation of the equation is not deducible from any of the previously known oscillation criteria.

Theorem 5. *If conditions* (2)–(8) *and* (14) *hold, and*

$$(23)\qquad \limsup_{n\to\infty} \sum_{s=n_0}^{n} (q_s - \mu p_s) = \infty \quad \textit{for all large } n_0$$

is satisfied, then all solutions of equation (1) *are oscillatory.*

P r o o f. Suppose that $y_n > 0$ for $n \geqslant n_1 \geqslant n_0 \in \mathbb{N}$, the proof for the case $y_n < 0$, $n \geqslant n_1$ is similar and will be omitted. Since (14) holds, we see from the proof of Theorem 2 that $\{\Delta y_n\}$ cannot change sign for arbitrarily large n. If $\Delta y_n \geqslant 0$ for $n \geqslant n_2$ for some $n_2 \geqslant n_1$, then from (16) and (23) we have

$$\liminf_{n\to\infty} \frac{a_n h(y_{n+1})\Delta y_n}{f_1(y_{n+1})} = -\infty,$$

which is a contradiction. Thus $\Delta y_n < 0$ for large n and we proceed as in the proof of Theorem 2.

Next we discuss the oscillatory behavior of equation (1) subject to the condition

$$\frac{g_1(u,v)}{h(u)} \geqslant \lambda > 0 \quad \text{for} \quad u, v \neq 0. \tag{24}$$

□

Theorem 6. *Suppose that conditions* (2)–(8), (14) *and* (24) *hold. Assume there exists a positive non-decreasing sequence* $\{\beta_n\}$ *such that*

$$\limsup_{n\to\infty} \frac{1}{(n)^{(\alpha)}} \sum_{s=n_0}^{n-1} (n-s)^{(\alpha)}\beta_s \Big[(q_s-\mu p_s) - \frac{a_s}{4\lambda}\Big(\frac{\alpha}{(n-s+\alpha-1)} - \frac{\Delta\beta_s}{\beta_s}\Big)^2\Big] = \infty \tag{25}$$

for a positive integer $\alpha \geqslant 1$, *where* $(n)^{(\alpha)} = n(n-1)\dots(n-\alpha+1)$ *is the usual factorial notation. Then all solutions of equation* (1) *are oscillatory.*

Proof. Suppose that $y_n > 0$ for $n \geqslant n_1 \geqslant n_0 \in \mathbb{N}$; the proof for the case $y_n < 0$, $n \geqslant n_1$ is similar and will be omitted. Since (14) holds as before we see from the proof of Theorem 2 that $\{\Delta y_n\}$ cannot change sign for arbitrarily large n. Let $\Delta y_n \geqslant 0$ for $n \geqslant n_2$ for some integer $n_2 \geqslant n_1$.

Define

$$z_n = \frac{v_n\beta_n}{f_1(y_n)} \quad \text{where} \quad v_n = a_n h(y_{n+1})\Delta y_n.$$

Then for $n \geqslant n_2$ we have

$$\Delta z_n \leqslant -\beta_n(q_n-\mu p_n) + \frac{\Delta\beta_n v_{n+1}}{f_1(y_{n+1})} - \frac{\beta_n v_n g_1(y_n,y_{n+1})\Delta y_n}{f_1(y_n)f_1(y_{n+1})}, \quad n \geqslant n_2. \tag{26}$$

Using the inequalities $v_{n+1} \leqslant v_n$ and $f_1(y_n) \leqslant f_1(y_{n+1})$ we obtain from (26)

$$\Delta z_n \leqslant -\beta_n(q_n-\mu p_n) + \frac{\Delta\beta_n}{\beta_{n+1}} z_{n+1} - \frac{\lambda\beta_n}{\beta_{n+1}^2 a_n} z_{n+1}^2.$$

Since

$$\sum_{s=n_2}^{n-1} (n-s)^{(\alpha)}\Delta z_s = -(n-n_2)^{(\alpha)} z_{n_2} + \alpha \sum_{s=n_2}^{n-1} (n-s)^{(\alpha-1)} z_{s+1}$$

we get

$$\frac{1}{(n)^{(\alpha)}}\sum_{s=n_2}^{n-1}(n-s)^{(\alpha)}\beta_s(q_s-\mu p_s)$$

$$\leqslant \frac{(n-n_2)^{(\alpha)}z_{n_2}}{(n)^{(\alpha)}}-\frac{\alpha}{(n)^{(\alpha)}}\sum_{s=n_2}^{n-1}(n-s)^{(\alpha-1)}z_{s+1}$$

$$+\frac{1}{(n)^{(\alpha)}}\sum_{s=n_2}^{n-1}\frac{\Delta\beta_s}{\beta_{s+1}}z_{s+1}(n-s)^{(\alpha)}-\frac{1}{(n)^{(\alpha)}}\sum_{s=n_2}^{n-1}\frac{(n-s)^{(\alpha)}\lambda\beta_s}{a_s\beta_{s+1}^2}z_{s+1}^2$$

$$\leqslant \frac{(n-n_2)^{(\alpha)}z_{n_2}}{(n)^{(\alpha)}}-\frac{1}{(n)^{(\alpha)}}\sum_{s=n_2}^{n-1}\frac{(n-s)^{(\alpha)}\lambda\beta_s}{a_s\beta_{s+1}^2}$$

$$\times\left[z_{s+1}^2+\frac{a_s\beta_{s+1}}{\lambda}\left(\frac{\alpha}{n-s+\alpha-1}-\frac{\Delta\beta_s}{\beta_s}\right)z_{s+1}\right]$$

$$\leqslant \frac{(n-n_2)^{(\alpha)}z_{n_2}}{(n)^{(\alpha)}}+\frac{1}{(n)^{(\alpha)}}\sum_{s=n_2}^{n-1}\frac{(n-s)^{(\alpha)}a_s\beta_s}{4\lambda}\left(\frac{\alpha}{n-s+\alpha-1}-\frac{\Delta\beta_s}{\beta_s}\right)^2$$

or

$$\frac{1}{(n)^{(\alpha)}}\sum_{s=n_2}^{n-1}(n-s)^{(\alpha)}\beta_s\left[(q_s-\mu p_s)-\frac{a_s}{4\lambda}\left(\frac{\alpha}{n-s+\alpha-1}-\frac{\Delta\beta_s}{\beta_s}\right)^2\right]$$

$$\leqslant \frac{(n-n_2)^{(\alpha)}z_{n_2}}{(n)^{(\alpha)}}\to z_{n_2}\quad\text{as}\quad n\to\infty,$$

which contradicts (25). If $\Delta y_n<0$ for large n, we proceed as in the proof of Theorem 2. This completes the proof of the theorem. □

Corollary 7. *If condition* (25) *is replaced by*

$$\limsup_{n\to\infty}\frac{1}{(n)^{(\alpha)}}\sum_{s=n_0}^{n-1}(n-s)^{(\alpha)}\beta_s(q_s-\mu p_s)=\infty,$$

$$\limsup_{n\to\infty}\frac{1}{(n)^{(\alpha)}}\sum_{s=n_0}^{n-1}\frac{(n-s)^{(\alpha)}\beta_s a_s}{(n-s+\alpha-1)^2}\left[\alpha-(n-s+\alpha-1)\left(\frac{\Delta\beta_s}{\beta_s}\right)\right]^2<\infty$$

for a positive integer $\alpha\geqslant 1$, then all solutions of equation (1) *are oscillatory.*

Remark. Theorem 6 and Corollary 7 extend Theorem 5 and Corollary 6 of [11], respectively. As an example, the difference equation

$$(\mathrm{E}_3)\qquad \Delta\big(n(1+y_{n+1}^2)\Delta y_n\big)+(5n+2)(y_{n+1}+y_{n+1}^3)=\frac{4ny_{n+1}^3}{1+y_{n+1}^2}$$

satisfies the conditions of Corollary 7. Hence all solutions of (E_3) are oscillatory. Here we put $f_1(u) = u + u^3$, $f_2(u) = u^3$ and observe that

$$\frac{Q(n, y_{n+1})}{f_1(y_{n+1})} = (5n + 2) \quad \text{and} \quad \frac{P(n, y_{n+1}, \Delta y_n)}{f_2(y_{n+1})} \leqslant 4n.$$

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Authors' address: Department of Mathematics, Madras University P.G. Centre Salem-636 011, Tamil Nadu, India.