Mathematica Slovaca

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Mathematica Slovaca, Vol. 45 (1995), No. 1, 79--89

Persistent URL: http://dml.cz/dmlcz/128593

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COMPARISON THEOREMS FOR DIFFERENTIAL EQUATIONS WITH DEVIATING ARGUMENT

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(Communicated by Milan Medveď)

ABSTRACT. Kusano, Naito and Tanaka have recently shown that we can deduce oscillatory and asymptotic behavior of the equation

$$L_n u(t) + p(t)u(t) = 0$$

from the oscillation of a set of the second order equations

$$\left(\frac{z'(t)}{r_i(t)}\right)' + \hat{q}_i(t)z(t) = 0\,.$$

In this paper, the above-mentioned result will be extended to a class of delay differential equations of the form

$$L_n u(t) + p(t)u\big[g(t)\big] = 0$$

for which asymptotic behavior is derived from the oscillation of the second order delay equations

$$\left(\frac{z'(t)}{r_i(t)}\right)' + q_i(t)z\big[\tau_i(t)\big] = 0\,.$$

Let us consider the differential equations

$$L_n u(t) + p(t)u(t) = 0\,, \qquad \text{and} \tag{1}$$

$$L_n u(t) + p(t)u\big[g(t)\big] = 0\,, \tag{2}$$

where $n \geq 3$, and L_n denotes the disconjugate differential operator

$$L_n = \frac{1}{r_n(t)}\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{r_{n-1}(t)}\frac{\mathrm{d}}{\mathrm{d}t}\cdots\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{r_1(t)}\frac{\mathrm{d}}{\mathrm{d}t}\frac{\cdot}{r_0(t)}\,. \tag{3}$$

It is assumed that $r_i(t)$, $0 \leq i \leq n$, $p(t)$, and $g(t)$ are continuous and positive on $[t_0,\infty)$, $g(t) \to \infty$ as $t \to \infty$, and

$$\int^{\infty} r_i(s)\,\mathrm{d}s = \infty \qquad \text{for} \quad 1 \leq i \leq n-1\,. \tag{4}$$

AMS Subject Classification (1991): Primary 34C10.
Key words: Differential equation, Deviating argument, Comparison theorem.

The operator L_n satisfying (4) is said to be in canonical form. It is well known that any differential operator of the form (3) can always be represented in a canonical form in an essentially unique way (see Trench [11]). In the sequel, we will assume that the operator L_n is in canonical form.

We introduce the notation:

$$L_0u(t) = \frac{u(t)}{r_0(t)},$$
$$L_iu(t) = \frac{1}{r_i(t)}\frac{\mathrm{d}}{\mathrm{d}t}L_{i-1}u(t), \qquad 1 \le i \le n.$$

The domain $\mathcal{D}(L_n)$ of L_n is defined to be the set of all functions $u\colon [T_u,\infty) \to \mathbb{R}$ such that $L_iu(t)$, $0 \le i \le n$, exist and are continuous on $[T_u,\infty)$. A nontrivial solution of (2) is called *oscillatory* if it has arbitrarily large zeros; otherwise it is called *nonoscillatory*. Equation (2) is said to be *oscillatory* if all of its solutions are oscillatory.

If $u(t)$ is a nonoscillatory solution of (2), then, according to a generalization of a lemma of Kiguradze [4; Lemma 3], there is an integer ℓ, $0 \le \ell \le n-1$, such that $\ell \not\equiv n \pmod 2$ and

$$\begin{aligned} u(t)L_iu(t) &> 0 \qquad \text{on} \quad [t_1,\infty), \quad 0 \le i \le \ell, \\ (-1)^{i-\ell}u(t)L_iu(t) &> 0 \qquad \text{on} \quad [t_1,\infty), \quad \ell+1 \le i \le n. \end{aligned} \tag{5}$$

A function $u(t)$ satisfying (5) is said to be a function of degree ℓ. The set of all nonoscillatory solutions of degree ℓ of (2) is denoted by $\mathcal{N}_\ell$. If we denote by $\mathcal{N}$ the set of all nonoscillatory solutions of (2), then

$$\mathcal{N} = \mathcal{N}_0 \cup \mathcal{N}_2 \cup \dots \cup \mathcal{N}_{n-1} \qquad \text{if } n \text{ is odd,}$$

and

$$\mathcal{N} = \mathcal{N}_1 \cup \mathcal{N}_3 \cup \dots \cup \mathcal{N}_{n-1} \qquad \text{if } n \text{ is even.}$$

It is known that equation (1) has always a nonoscillatory solution of degree 0 ($\mathcal{N}_0 \neq \emptyset$), see Hartman and Wintner [3]. Therefore, the extreme situation described in the following definition is of a particular interest.

DEFINITION 1. Equation (2) is said to have *property* (A) if for n even $\mathcal{N} = \emptyset$ (i.e. (2) is oscillatory), and for n odd $\mathcal{N} = \mathcal{N}_0$.

Kusano and Naito [7], and Tanaka [10] have established sufficient conditions for equation (1) to have property (A). Their results generalize those of Lovelady [5] for equations of the form $y^{(n)} + p(t)y = 0$. Kusano, Naito and Tanaka have compared equation (1) with the set of the second order equations of the form

$$\left(\frac{y'(t)}{r_i(t)}\right)' + \hat{q}_i(t)y(t) = 0,$$

where $\hat{q}_i(t)$ have been constructed from $r_i(t)$, $0 \le i \le n$ and $p(t)$. For details, see [7] and [10].

The objective of this paper is to improve the above-mentioned results. We compare equation (1) with the set of the second order delay equations of the form

$$\left(\frac{z'(t)}{r_i(t)}\right)' + q_i(t)z\big[\tau_i(t)\big] = 0\,,$$

where $q_i(t)$ and $\tau_i(t)$ will be defined bellow, and then we extend our results to differential equations with deviating argument (2).

We begin with formulating some preparatory results which are needed for proofs of the main theorems.

Let $i_k \in \{1,\dots,n-1\}$, $1 \le k \le n-1$, and $t,s \in [t_0,\infty)$. We define

$$I_0 = 1\,,$$
$$I_k(t,s;r_{i_k},\dots,r_{i_1}) = \int_s^t r_{i_k}(x)I_{k-1}(x,s;r_{i_{k-1}},\dots,r_{i_1})\,\mathrm{d}x\,.$$

It is easy to verify that for $1 \le k \le n-1$

$$\begin{aligned} I_k(t,s;r_{i_k},\dots,r_{i_1}) &= (-1)^k I_k(s,t;r_{i_1},\dots,r_{i_k})\,,\\ I_k(t,s;r_{i_k},\dots,r_{i_1}) &= \int_s^t r_{i_1}(x)I_{k-1}(t,x;r_{i_k},\dots,r_{i_2})\,\mathrm{d}x\,. \end{aligned} \tag{6}$$

For simplicity of notation, we put

$$\begin{aligned} J_i(t,s) &= r_0(t)I_i(t,s;r_1,\dots,r_i)\,, \qquad & J_i(t) &= J_i(t,t_0)\,,\\ K_i(t,s) &= r_n(t)I_i(t,s;r_{n-1},\dots,r_{n-i})\,, \qquad & K_i(t) &= K_i(t,t_0)\,. \end{aligned}$$

Lemma 1. *Let ℓ be an integer such that $1 \le \ell \le n-1$ and $\ell \not\equiv n \pmod 2$. Equation (2) has a solution of degree ℓ if and only if the differential inequality*

$$\big\{L_n y(t) + p(t)y\big[g(t)\big]\big\}\operatorname{sgn} y\big[g(t)\big] \le 0 \tag{7}$$

has a solution of degree ℓ.

This lemma exhibits an important relationship between differential equation (2) and differential inequality (7). For a proof, see Kusano and Naito [8].

LEMMA 2. *If $u \in \mathcal{D}(L_n)$, then for $0 \le i \le k \le n-1$ and $t, s \in [T_u, \infty)$, one has:*

$$\begin{aligned} L_i u(t) = \sum_{j=i}^{k} (-1)^{j-i} L_j u(s) I_{j-i}(s,t;r_j,\dots,r_{i+1}) \\ + (-1)^{k-i+1} \int_t^s I_{k-i}(x,t;r_k,\dots,r_{i+1}) r_{k+1}(x) L_{k+1}u(x) \, \mathrm{d}x \,. \end{aligned} \tag{8}$$

This lemma is a generalization of Taylor's formula. The proof is immediate. The following theorem is an extension of a result of T r e n c h [12].

THEOREM 1. *Let*

$$\int^{\infty} K_{n-i-1}(t) J_{i-1}(t) p(t) \, \mathrm{d}t = \infty \tag{9}$$

for $i = 2, 4, \dots, n-1$ if n is odd, and for $i = 1, 3, \dots, n-1$ if n is even. Then equation (1) *has property* (A).

The proof immediately follows from [7; Theorem A] and [10; Theorem 1].

The following theorem covers the case when condition (9) is violated. For convenience, we introduce the following notations:

$$q_i(t) = r_{i+1}(t) \int_t^{\infty} K_{n-i-2}(x,t) J_{i-1}\bigl(x, \tau_i(t)\bigr) p(x) \, \mathrm{d}x \,, \qquad i = 1, 2, \dots, n-3 \,; \tag{10}$$

$$q_{n-1}(t) = r_n(t) J_{n-2}\bigl(t, \tau_{n-1}(t)\bigr) p(t) \,; \tag{11}$$

$$\hat{q}_{n-1}(t) = r_{n-2}(t) \int_t^{\infty} J_{n-3}(s,t) K_0(s,t) p(s) \, \mathrm{d}s \,; \tag{12}$$

where $\tau_i(t) \colon [t_0, \infty) \to \mathbb{R}$, $1 \le i \le n-1$, are continuous and satisfy

$$\begin{aligned} \tau_i(t) \to \infty \text{ as } t \to \infty \,, \quad \tau_i(t) \le t \,, \qquad \text{for} \quad i = 1, 2, \dots, n-1 \,; \\ \tau_{n-1}(t) \not\equiv t \qquad \text{on any} \quad [t_1, \infty) \,, \ t_1 \ge t_0 \,. \end{aligned} \tag{13}$$

THEOREM 2. *Suppose that the integrals in* (9) *converge. Assume that the second order delay equations*

$$\left(\frac{z'(t)}{r_i(t)} \right)' + q_i(t) z\bigl[\tau_i(t)\bigr] = 0 \tag{E$_i$}$$

are oscillatory for $i = 2, 4, \dots, n-3$ *if* n *is odd, and for* $i = 1, 3, \dots, n-3$ *if* n *is even, and further suppose that either the second order delay equation*

$$\left(\frac{z'(t)}{r_{n-1}(t)}\right)' + q_{n-1}(t) z\big[\tau_{n-1}(t)\big] = 0 \qquad (\mathrm{E}_{n-1})$$

is oscillatory, or the second order equation

$$\left(\frac{z'(t)}{r_{n-1}(t)}\right)' + \hat{q}_{n-1}(t) z(t) = 0 \qquad (\hat{\mathrm{E}}_{n-1})$$

is oscillatory. Then equation (1) *has property* (A).

Proof. Without loss of generality, we may assume that $u(t)$ is a positive solution of (1). Then there exists an integer $\ell \in \{0, 1, \dots, n-1\}$, $\ell \not\equiv n \pmod 2$, and a number t_1 such that (5) holds for $t \ge t_1$. We claim that ℓ must be equal to 0 (if n is odd). Assume that $1 \le \ell \le n-3$. By Lemma 2, with $i = \ell + 1$, $k = n-1$, and $s \ge t \ge t_1$, taking (1) into account and then letting $s \to \infty$, we obtain for $t \ge t_1$

$$-L_{\ell+1} u(t) \ge \int_t^\infty r_n(x) I_{n-\ell-2}(x, t; r_{n-1}, \dots, r_{\ell+2}) p(x) u(x) \, \mathrm{d}x \,, \qquad (14)$$

and if $\ell \ge 2$, then putting $i = 0$, $k = \ell - 2$, and $t \ge s = t_1$

$$L_0 u(t) \ge \int_{t_1}^t I_{\ell-2}(t, x; r_1, \dots, r_{\ell-2}) r_{\ell-1}(x) L_{\ell-1} u(x) \, \mathrm{d}x \,. \qquad (15)$$

For details the reader is referred to [7] or to [10]. Combining (14) with (15) we have

$$\begin{aligned} -L_{\ell+1} u(t) &\ge \int_t^\infty r_n(x) I_{n-\ell-2}(x, t; r_{n-1}, \dots, r_{\ell+2}) p(x) r_0(x) \cdot \\ &\qquad \cdot \int_{t_1}^x I_{\ell-2}(x, s; r_1, \dots, r_{\ell-2}) r_{\ell-1}(s) L_{\ell-1} u(s) \, \mathrm{d}s \, \mathrm{d}x \\ &\ge \int_t^\infty r_n(x) I_{n-\ell-2}(x, t; r_{n-1}, \dots, r_{\ell+2}) p(x) r_0(x) \cdot \\ &\qquad \cdot \int_{\tau_\ell(t)}^x I_{\ell-2}(x, s; r_1, \dots, r_{\ell-2}) r_{\ell-1}(s) L_{\ell-1} u(s) \, \mathrm{d}s \, \mathrm{d}x \,, \end{aligned}$$

for all $t \geq t_2$, where $t_2 \geq t_1$ is chosen so that $\tau_\ell(t) \geq t_1$ for $t \geq t_2$. Since $L_{\ell-1}u(t)$ is increasing, we conclude from above that

$$-L_{\ell+1}u(t) \geq L_{\ell-1}u\big[\tau_\ell(t)\big] \int_t^\infty r_n(x) I_{n-\ell-2}(x,t;r_{n-1},\dots,r_{\ell+2}) p(x) r_0(x) \cdot$$

$$\cdot \int_{\tau_\ell(t)}^x I_{\ell-2}(x,s;r_1,\dots,r_{\ell-2}) r_{\ell-1}(s)\, \mathrm{d}s\, \mathrm{d}x\,.$$

Let $y(t)$ be given by

$$y(t) = L_{\ell-1}u(t)\,.$$

Note that $y(t) > 0$, and, in view of the above inequality,

$$-L_{\ell+1}u(t) \geq y\big[\tau_\ell(t)\big] \int_t^\infty K_{n-\ell-2}(x,t) J_{\ell-1}\big(x,\tau_\ell(t)\big) p(x)\, \mathrm{d}x\,. \tag{16}$$

That (16) also holds for $\ell = 1$ follows from (14) and the fact that $L_0u(t) \geq L_0u\big[\tau_1(t)\big]$. Noting that

$$\left(\frac{y'(t)}{r_\ell(t)}\right)' = r_{\ell+1}(t) L_{\ell+1}u(t)\,,$$

we see from (16) that

$$\left(\frac{y'(t)}{r_\ell(t)}\right)' + q_\ell(t) y\big[\tau_\ell(t)\big] \leq 0\,, \qquad \text{for} \quad t \geq t_2\,.$$

Lemma 1 implies that equation (E_ℓ) has an eventually positive solution. But this contradicts our assumption.

Let $\ell = n-1$. First suppose that equation (E_{n-1}) is oscillatory. We easily see that

$$-L_n u(t) = p(t) u(t) \tag{17}$$

and, by Lemma 2, we have

$$L_0 u(t) \geq \int_{t_1}^t I_{n-3}(t,x;r_1,\dots,r_{n-3}) r_{n-2}(x) L_{n-2}u(x)\, \mathrm{d}x\,. \tag{18}$$

Combining (17) with (18) and taking (13) into account we have

$$-L_n u(t) \geq p(t) r_0(t) \int_{\tau_{n-1}(t)}^t I_{n-3}(t,x;r_1,\dots,r_{n-3}) r_{n-2}(x) L_{n-2}u(x)\, \mathrm{d}x\,.$$

Since $L_{n-2}u(t)$ is increasing, we obtain from the above that

$$\begin{aligned}-L_n u(t) &\geq L_{n-2}u\big[\tau_{n-1}(t)\big]p(t)r_0(t)\int\limits_{\tau_{n-1}(t)}^{t} I_{n-3}(t,x;r_1,\dots,r_{n-3})r_{n-2}(x)\,\mathrm{d}x \\ &= L_{n-2}u\big[\tau_{n-1}(t)\big]p(t)J_{n-2}\big(t,\tau_{n-1}(t)\big)\,,\end{aligned} \tag{19}$$

for all large t, $t \geq t_2$. We see that $y(t) = L_{n-2}u(t) > 0$ satisfies

$$\left(\frac{y'(t)}{r_{n-1}(t)}\right)' = r_n(t)L_n u(t)\,.$$

Therefore we have from (19) that

$$\left(\frac{y'(t)}{r_{n-1}(t)}\right)' + q_{n-1}^{-}(t)y\big[\tau_{n-1}(t)\big] \leq 0\,, \qquad \text{for} \quad t \geq t_2\,.$$

Again, by Lemma 1, one gets that equation (E_{n-1}) has an eventually positive solution, contradicting the hypotheses.

Now, suppose that equation $(\hat{\mathrm{E}}_{n-1})$ is oscillatory. Then, by [10; Theorem 2] and by [7; Theorem B], it follows that equation (1) has no nonoscillatory solution of degree $\ell = n-1$. This completes the proof. □

Kusano and Naito in [7; Theorem B] and Tanaka in [10; Theorem 2] have established comparison theorems to the effect that we can derive property (A) of the nth order equation from the oscillation of the second order equations. Those results are included in Theorem 2 (by putting $\tau_i(t) \equiv t$).

Moreover, in the examples stated below, we show that we often obtain better results if we deduce property (A) of equation (1) from the oscillation of second order delay equations than from that of the second order ordinary equations (without delay).

Now we are prepared to extend our results to equation (2). The main tool in our efforts is the following result, which is due to Kusano and Naito [8].

Lemma 3. *Let*

$$g(t) \in C^1\big([t_0,\infty)\big)\,, \quad g'(t) > 0\,, \quad g(t) \leq t\,. \tag{20}$$

Equation (2) *has property* (A) *if the equation*

$$L_n u(t) + \widetilde{p}(t)u(t) = 0 \tag{21}$$

has property (A), *where*

$$\widetilde{p}(t) = \frac{p\big[g^{-1}(t)\big]r_n\big[g^{-1}(t)\big]}{g'\big[g^{-1}(t)\big]r_n(t)}\,.$$

Applying Theorems 1 and 2 to equation (21) we obtain the following two corollaries

COROLLARY 1. *Let* (20) *hold. Further suppose that all the conditions of Theorem* 1 *are satisfied with* $p(t)$ *replaced by* $\widetilde{p}(t)$. *Then equation* (2) *has property* (A).

COROLLARY 2. *Let* (20) *hold. Further suppose that all the conditions of Theorem* 2 *are satisfied with* $p(t)$ *replaced by* $\widetilde{p}(t)$. *Then equation* (2) *has property* (A).

We show that the conclusions of Corollaries 1 and 2 can be strengthened as follows:

THEOREM 3. *Assume that equation* (2) *has property* (A). *Then every nonoscillatory solution* $u(t)$ *of* (2) *satisfies*

$$\lim_{t\to\infty} \frac{u(t)}{r_0(t)} = 0$$

if and only if

$$\int^{\infty} J_0\big(g(t)\big)K_{n-1}(t)p(t)\,\mathrm{d}t = \infty .$$

The proof of this theorem immediately follows from [6; Theorem 1] of Kitamura and Kusano.

For the special case of equation (2), namely, for the equation

$$\left(\frac{1}{r_2(t)}\left(\frac{1}{r_1(t)}u'(t)\right)'\right)' + p(t)u\big[g(t)\big] = 0\,, \tag{22}$$

we have the following result:

COROLLARY 3. *Let* (20) *hold. Further suppose that at least one of the following conditions holds:*

(i)

$$\int^{\infty}\left(\int_{t_0}^{t} r_1(s)\,\mathrm{d}s\right)\frac{p\big[g^{-1}(t)\big]}{g'\big[g^{-1}(t)\big]}\,\mathrm{d}t = \infty\,;$$

(ii) *the equation*

$$\left(\frac{z'(t)}{r_2(t)}\right)' + \left(\frac{p\big[g^{-1}(t)\big]}{g'\big[g^{-1}(t)\big]}\int_{\tau_2(t)}^{t} r_1(s)\,\mathrm{d}s\right)z\big[\tau_2(t)\big] = 0$$

with $\tau_2(t)$ *defined as in* (15), *is oscillatory;*

(iii) *the equation*

$$\left(\frac{z'(t)}{r_2(t)}\right)' + \left(r_1(t)\int_{t}^{\infty}\frac{p\big[g^{-1}(s)\big]}{g'\big[g^{-1}(s)\big]}\,\mathrm{d}s\right)z(t) = 0$$

is oscillatory.

Then equation (22) *has property* (A).

E x a m p l e 1. Let us consider the third order Euler equation

$$\left(t^{1/2}y''\right)' + \frac{a}{t^{5/2}}y = 0\,, \qquad t > 1\,, \quad a \in \mathbb{R}. \tag{23}$$

We put for this equation $\tau_2(t) = t/3$. Then, by Corollary 3, equation (23) has property (A) if the second order delay equation

$$\left(t^{1/2}y'(t)\right)' + \frac{2a}{3t^{3/2}}y(t/3) = 0$$

is oscillatory. By a generalization of the well-known criterion of H i l l e [2], it comes if

$$a > \frac{1}{8\sqrt{3}}\,,$$

and moreover, by Theorem 3, if $a > \dfrac{1}{8\sqrt{3}}$, then every nonoscillatory solution $y(t)$ of equation (23) satisfies $\lim\limits_{t\to\infty} y(t) = 0$. Note that we obtain a better result than T a n a k a 's criterion [10] provides.

To describe better the situation in which not all second order equations (E_i) are oscillatory, we use the following definition and in the sequel we suppose that $k_1, k_2, \dots, k_m \in \{1, 2, \dots, n-1\}$, where $m \ge 1$ are all mutually different such that $n \not\equiv k_i \pmod 2$, $1 \le i \le m$.

DEFINITION 2. We say that equation (2) has *property* $\mathrm{A}_{k_1,\cdots,k_m}$ if

$$\mathcal{N} = \mathcal{N}_0 \cup \mathcal{N}_{k_1} \cup \dots \cup \mathcal{N}_{k_m} \qquad \text{if } n \text{ is odd,}$$

and

$$\mathcal{N} = \mathcal{N}_{k_1} \cup \dots \cup \mathcal{N}_{k_m} \qquad \text{if } n \text{ is even.}$$

THEOREM 4. *Assume that* (20) *holds. Let* (9) *be satisfied for* $i \in \{1, 3, \dots, n-1\} - \{k_1, \dots, k_m\}$ *if* n *is even, and for* $i \in \{2, 4, \dots, n-1\} - \{k_1, \dots, k_m\}$ *if* n *is odd. Then equation* (2) *has property* $\mathrm{A}_{k_1,\dots,k_m}$.

THEOREM 5. *Assume that* (20) *holds and the integrals in* (9) *converge. Let* $q_i(t)$ *and* $\tau_i(t)$, $1 \le i \le n-1$, *be defined as in* (10), (11) *and* (13). *Then equation* (2) *has property* $\mathrm{A}_{k_1,\dots,k_m}$ *if equations* (E_i) *are oscillatory for* $i \in \{1, 3, \dots, n-1\} - \{k_1, \dots, k_m\}$ *if* n *is even, and for* $i \in \{2, 4, \dots, n-1\} - \{k_1, \dots, k_m\}$ *if* n *is odd.*

The proofs of Theorem 4 and 5 follow from Corollary 1 and 2, taking [1; Theorem 9] into account.

R e m a r k 1. If equation (E_{n-1}) is replaced by equation $(\hat{\mathrm{E}}_{n-1})$, then Theorem 5 remains valid.

E x a m p l e 2. Let us consider the fifth order delay equation

$$\left(t^{-1}y^{(4)}(t)\right)' + \frac{a}{t^{\frac{7}{2}}}y\left(\sqrt{t}\right) = 0\,, \qquad t > 1\,, \quad a > 0\,. \tag{24}$$

We put $\tau_2(t) = \lambda t$ for some $\lambda \in (0,1)$. Then, by Theorem 2, equation (24) has not any solution of degree $\ell = 2$ if the second order delay equation

$$y''(t) + \frac{a}{t^2}\left(\frac{1}{4} - \frac{2}{15}\lambda\right)y(\lambda t) = 0 \tag{25}$$

is oscillatory. By a generalization of the criterion of H i l l e [2], it sets in if

$$a\left(\frac{\lambda}{4} - \frac{2}{15}\lambda^2\right) > \frac{1}{4}\,. \tag{26}$$

If we put $\lambda = \dfrac{15}{16}$, then (26) reduces to $a > \dfrac{32}{15}$. Note that we have obtained better result than T a n a k a 's criterion [10] provides. On the other hand, by Theorem 2, equation (24) has no solution of degree $\ell = 4$ if the second order equation

$$\left(t^{-1}y'(t)\right)' + \frac{17a}{60t^3}y(t) = 0$$

is oscillatory, which, by H i l l e 's criterion, comes if $a > \dfrac{60}{17}$. Finally, by Theorem 2 and 5,

if $a > \dfrac{60}{17}$, then equation (24) has property (A),

if $\dfrac{32}{15} < a < \dfrac{60}{17}$, then equation (24) has property A_4,

if $a > 0$, then equation (24) has property $\mathrm{A}_{2,4}$,

and moreover, by Theorem 3, if $a > \dfrac{60}{17}$, then every nonoscillatory solution $y(t)$ of equation (24) satisfies $\lim\limits_{t\to\infty} y(t) = 0$.

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Received June 18, 1992
Revised March 22, 93

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