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POLARITY ON C^*-ALGEBRAS

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Polarity, as a symmetric and antireflexive relation, was investigated on many algebraic and geometric structures. For instance, the disjointness on lattice ordered groups is a polarity with interesting properties described by G. Birkhoff, F. Šik, P. Conrad and P. Jaffard (see [4], [5], [8]). The basic properties of some polarities on C^*-algebras are investigated in this paper. Namely, the relation of polarities and ideals of C^*-algebras and the lattice characterization of sets of polars.

Let us introduce some notations. If A is *a C^*-algebra* (see [6]) with the unit element 1, then A_h is the set of all *hermitian elements* in A and A^+ is the set of all *positive elements* in A. $\mathscr{I}d(A)$, $\mathscr{L}(A)$ and $\mathscr{R}(A)$ denotes the set of all *closed ideals*, *closed left ideals* and *closed right ideals* in A. The set of all real (complex) numbers is denoted by $\mathbf{R}$ ($\mathbf{C}$).

An order ideal N of A is a subset in A^+ fulfilling $N + N \subseteq N$, $\alpha N \subseteq N$ for $\alpha \in \mathbf{R}^+$ and $0 \le y \le x$, $x \in N$, $y \in A \Rightarrow y \in N$. Effros [7] describes a bijection between closed left ideals and closed order ideals of a C^*-algebra. This bijection is possible to extend on a lattice isomorphism between closed left ideals in A and closed directed convex subgroups in A_h.

Recall that *a frame* is a complete lattice L fulfilling $a \wedge \bigvee b_\alpha = \bigvee (a \wedge b_\alpha)$ for all a, $\{b_\alpha\} \subseteq L$. *A quantale* is a complete lattice Q equipped with an associative binary operation $\cdot$ so that $a \cdot \bigvee b_\alpha = \bigvee (a \cdot b_\alpha)$, $(\bigvee b_\alpha) \cdot a = \bigvee (b_\alpha \cdot a)$ and $1 \cdot a = a$, for all a, $\{b_\alpha\} \subseteq Q$. All unexplained facts concerning frames (quantales) can be found in [9] ([10]). Namely, a quantale Q is called *regular* if $a = \bigvee\{b \in Q\colon c \in Q$ exists such that $c \cdot b = 0$, $c \vee a = 1\}$ holds, for any $a \in Q$.

§ 1. Meets and ideals

R. Archbold in [1] and [2] gives conditions for the existence of meets of positive elements $a, b \in A$ in the partially ordered set A_h. A consequence of these conditions is the Shermann theorem [1] saying that a C^*-algebra is commutative iff its set of hermitian elements is a lattice.

1.1 Proposition. *Let A be a C^*-algebra, $a, b \in A^+$. Then the following assertions are equivalent*:

1. $a \wedge b = 0$ *in* A_h.
2. $\overline{AaA} \cdot \overline{AbA} = \{0\}$.
3. $\overline{aA} \cdot \overline{bA} = \{0\}$.
4. $\overline{AaA} \cap \overline{AbA} = \{0\}$.
5. $\overline{Aa} \cdot \overline{Ab} = \{0\}$.

Proof. $1 \Leftrightarrow 2$: see [1], Th. 1.

$2 \Leftrightarrow 3$: There holds $\overline{aA} \cdot \overline{bA} = \{0\} \Leftrightarrow \overline{\overline{aA} \cdot \overline{bA}} = \{0\} \Leftrightarrow \overline{aA \cdot bA} = \{0\} \Leftrightarrow aA \cdot bA = \{0\}$ and $\overline{AaA} \cdot \overline{AbA} = \{0\} \Leftrightarrow \overline{\overline{AaA} \cdot \overline{AbA}} = \{0\} \Leftrightarrow \overline{AaA \cdot AbA} = \{0\} \Leftrightarrow AaA \cdot AbA = \{0\}$. It follows that $\overline{AaA} \cdot \overline{AbA} = \{0\} \Rightarrow AaA \cdot AbA = \{0\} \Rightarrow aA \cdot bA = \{0\} \Rightarrow \overline{aA} \cdot \overline{bA} = \{0\}$. Finally, we have $\overline{aA} \cdot \overline{bA} = \{0\} \Rightarrow aA \cdot bA = \{0\} \Rightarrow (AaA) \cdot (AbA) \subseteq A[(aA)(AbA)] \subseteq A \cdot (aA \cdot bA) = \{0\} \Rightarrow \overline{AaA} \cdot \overline{AbA} = \{0\}$.

$2 \Leftrightarrow 5$: We can prove similarly as $2 \Leftrightarrow 3$.

$3 \Leftrightarrow 4$: Two-sided closed ideals form a frame and thus $\overline{AaA} \cdot \overline{AbA} = \overline{AaA} \cap \overline{AbA}$.

Remark. It follows from the proof that the Proposition 1.1 holds even if (right, left) ideals in parts 2, 3, 5 are not closed. Further, the equivalence of assertions 2, 3, 4, 5 holds for $a, b \in A$, in general.

1.2. Lemma. *If* $a, b \in A^+$, $\alpha, \beta \in \mathbf{R}^+$ *and* $a \wedge b = 0$ *in* A^+, *then* $\alpha a \wedge \beta b = 0$ *in* A^+.

Proof. We have $\alpha a \geq 0$, $\beta b \geq 0$ and let $d \in A$, $\alpha a, \beta b \geq d \geq 0$ hold. Then $\alpha a - d \geq 0$, $\beta b - d \geq 0$ and if $\alpha \neq 0 \neq \beta$, then $a - \alpha^{-1} d = \alpha^{-1}(\alpha a - d) \geq 0$, $b - \beta^{-1} d = \beta^{-1}(\beta b - d) \geq 0$. It implies $a \geq \alpha^{-1} d \geq 0$, $b \geq \beta^{-1} d \geq 0$ and $0 = a \wedge b \geq \gamma d \geq 0$, where $\gamma = \min\{\alpha^{-1}, \beta^{-1}\}$. It means that $\gamma d = 0$, $\gamma \neq 0$ and together $d = 0$, $\alpha a \wedge \beta b = 0$ in A^+.

1.3. Proposition. *Let* A *be a* C^**-algebra and* $a, b \in A^+$. *Then* $a \wedge b = 0$ *in* A^+ *iff* $\overline{Aa} \cap \overline{Ab} = \{0\}$.

Proof. $\Leftarrow$: If $0 \leq x \leq a, b$ for $x \in A$, then $x \in \overline{Aa} \cap \overline{Ab} = \{0\}$. $\Rightarrow$: The smallest closed order ideal $\tilde{a}$ containing a has the form $\tilde{a} = \{c \in A : 0 \leq c \leq \lambda a$ for some $\lambda \in \mathbf{R}^+\}$. If $a \wedge b = 0$ in A^+, then with regard to 1.2 $\tilde{a} \cap \tilde{b} = \{0\}$ holds. Theorem 2.4 from [7] implies the existence of a bijection p between closed left ideals in A and closed order ideals in A such that $p(I) = I^+$ for $I \in \mathscr{L}(A)$ and $p^{-1}(\tilde{a}) = \overline{Aa}$, $p^{-1}(\tilde{b}) = \overline{Ab}$. The consequence is $\overline{Aa} \cap \overline{Ab} = \{0\}$.

1.4. Corollary. *Let* A *be a* C^**-algebra and* $a, b \in A^+$. *Then there holds*: 1. $a \wedge b = 0$ *in* $A_h \Rightarrow a \wedge b = 0$ *in* A^+.

2. *If* $a \wedge b$ *in* A_h *exists, then* $a \wedge b = 0$ *in* A^+ *implies* $a \wedge b = 0$ *in* A_h.

Proof. 1. It follows from 1.2, 1.3. and the fact $\overline{Aa} \cap \overline{Ab} \subseteq \overline{Aa} \cdot \overline{Ab}$ because $\mathscr{L}(A)$ is an idempotent quantale.

2. It is clear.

Remark. A simple example in the C^*-algebra of real square matrices of rang 2 shows that the conversion of the assertion 1. from 1.4. is not true and thus $\overline{Aa} \cap \overline{Ab} = \{0\}$ does not imply $\overline{Aa \cdot Ab} = \{0\}$ for $a, b \in A^+$, in general.

1.5. Lemma. *Let A be a C^*-algebra and $a \in A$. Then there holds*:
1. $\overline{Aa^*} = \overline{a^*A}, \overline{aA^*} = \overline{Aa^*}$.
2. $\overline{Aa} = \overline{A|a|}, \overline{a^*A} = \overline{|a|A}$.

Proof. 1. $x \in \overline{Aa^*} \Leftrightarrow x^* \in \overline{Aa} \Leftrightarrow \{y_a a\} \to x^* \in \overline{Aa} \Leftrightarrow \{y_a a\} \to x^*$, where $\{y_a\} \subseteq \subseteq A$ is a suitable sequence $\Leftrightarrow \{a^* \cdot y_a^*\} \to x \Leftrightarrow x \in \overline{a^*A}$. The second formula follows from the first.

2. We have $|a|^2 = a^* \cdot a \in \overline{Aa}$ and the Corollary 2.2 from [7] implies $|a| \in \overline{Aa}$, i.e., $\overline{A|a|} \subseteq \overline{Aa}$. If $a = u|a|$ is a polar decomposition of a, $u \in A^{**}$ and if $\{u_a\} \subseteq A$ is a sequence which weakly converges to u, then $\{u_a|a|\} \subseteq \overline{A|a|}$ weakly converges to a, $\overline{A|a|}$ is closed with regard to the weak convergence (A is dense in A^{**} in the weak topology), i.e., $a \in \overline{A|a|}$ and $\overline{Aa} \subseteq \overline{A|a|}$. Together $\overline{Aa} = \overline{A|a|}$ and $\overline{a^*A} = \overline{Aa^*} = \overline{A|a|^*} = \overline{|a|A}$ hold.

1.6. Corollary. *Let A be a C^*-algebra and a, $b \in A$. Then there holds*:
1. $|a| \wedge |b| = 0$ *in* $A^+ \Leftrightarrow \overline{Aa} \cap \overline{Ab} = \{0\}$.
2. $|a| \wedge |b| = 0$ *in* $A_h \Leftrightarrow \overline{Aa \cdot Ab} = \{0\}$.

Proof follows from 1.1., 1.3. and 1.5.

1.7. Proposition. *Let A be a C^*-algebra, $\mathscr{L}(A)(\mathscr{C}(A_h))$, (respectively) be the complete lattice of all closed left ideals in A (closed directed convex subgroups in A_h with the property (S): $a \in C \in \mathscr{C}(A_h)$, $\lambda \in \mathbf{R} \Rightarrow \lambda a \in C$, respectively).*

Then the mapping $f: \mathscr{L}(A) \to \mathscr{C}(A_h)$ such that $f(B) = B \cap A_h$ for $B \in \mathscr{L}(A)$ is a lattice isomorphism.

Proof. If $B \in \mathscr{L}(A)$, then $B \cap A_h$ is a closed subgroup in A_h and $B^+ = (A_h \cap B)^+ = A^+ \cap B$ because $(A_h \cap B)^+ \subseteq B^+ \subseteq A^+ \cap B \subseteq (A_h \cap B)^+$ holds. B^+ is an order ideal and thus each element $a \in A_h \cap B$ has the form $a = a^+ - a^-$, where $a^+, a^- \in A^+ \cap B$. B^+ is convex, $|a| \in B^+$, $|a| \geq a^+, a^- \geq 0$ and it implies $A_h \cap B \in \mathscr{C}(A_h)$. Further, we have $A_h \cap B = B^+ - A^+$ and thus $f = g \cdot h$, where $h(B) = b^+$ for $B \in \mathscr{L}(A)$ is a bijection (see [7]) and $g(B^+) = B^+ - B^+ = A_h \cap B$ is also a bijection. The mappings g, h and g^{-1}, h^{-1} preserve the inclusion and thus f is an isomorphism of complete lattices.

1.8. Lemma. *Let $B \in \mathscr{L}(A)$. Then B is a two-sided ideal in A iff $B = (A_h \cap \cap B) + i(A_h \cap B)$.*

Proof. $\Rightarrow$: Clearly $(A_h \cap B) + i(A_h \cap B) \subseteq B$ holds. If $B \in \mathscr{I}d(A)$, then B is selfadjoint (see [7], Remark after 2.8) and for each $b \in B$ we have $b = b^r + ib^i$, where $b^r = \frac{1}{2}(b + b^*)$, $b^i = \frac{1}{2i}(b - b^*)$, i.e., $B \subseteq (A_h \cap B) + i(A_h \cap B)$.

$\Leftarrow$: For each $b \in B$ we have $b = b_1 + ib_2$, where b_1, $b_2 \in A_h \cap B$ and $b^* = b_1 - ib_2 \in B$. It means that B is self-adjoint, i.e., $B \in \mathscr{I}d(A)$.

1.9. Corollary. *Let A be a C^*-algebra. Then the mapping $\tilde{f}: \mathscr{I}d(A) \to \tilde{\mathscr{C}}(A_h)$ such that $\tilde{f}(B) = B \cap A_h$ for $B \in \mathscr{I}d(A)$ is an isomorphism of frames $\mathscr{I}d(A)$ of closed ideals in A and $\tilde{\mathscr{C}}(A_h)$ of invariant closed directed convex subgroups in A_h with the property (S).*

Proof. If $B \in \mathscr{I}d(A)$, then $B \cap A_h \in \tilde{\mathscr{C}}(A_h)$. Let $C \in \tilde{\mathscr{C}}(A_h)$. Then C^+ is an invariant closed order ideal in A. With regard to [7], Th. 2.8. there exists an ideal $B \in \mathscr{I}d(A)$ such that $A_h \cap B = C$. The mapping $\tilde{f}$ is a restriction of the mapping f from 1.7. on $\mathscr{I}d(A)$ and $\tilde{f}$ is also a bijection.

1.10. Corollary. *If A is a C^*-algebra, then there holds:*

1. *Invariant closed convex directed subgroups in A_h form a frame.*
2. *$\tilde{f}(B \cdot C) = \tilde{f}(B) \cap \tilde{f}(C)$, for $B, C \in \mathscr{I}d(A)$.*
3. *$\tilde{f}(B) = A_h \cap B$ and $\tilde{f}^{-1}(C) = C + iC$, for $B \in \mathscr{I}d(A)$ and $C \in \tilde{\mathscr{C}}(A_h)$.*

Proof. 1. is clear.

2. $\tilde{f}(B \cdot C) = \tilde{f}(B \cap C) = (B \cap C) \cap A_h = \tilde{f}(B) \cap \tilde{f}(C)$.

3. $C + iC$ is a subgroup in A closed with respect to scalar multiplication. C^+ is an invariant closed order ideal and $B \in \mathscr{I}d(A)$ exists such that $B^+ = C^+$ (see [7], 2.8). Then $A_h \cap B = C^+ - C^- = C$ and $\tilde{f}^{-1}(C) = B = C + iC$ follows from 2.8.

§ 2. Polarities

A polarity is a symmetric and antireflexive binary relation. Some properties of polarities were investigated by F. Šik in [12] and [13]. Let us describe some polarities on C^*-algebras.

Definition. *$\varkappa$-polarity* (*δ-polarity*, respectively) is a binary relation on a C^*-algebra A with the following property:

$$a \varkappa b \Leftrightarrow |a| \wedge |b| = 0 \text{ in } A_h$$

($a \delta b \Leftrightarrow |a| \wedge |b| = 0$ in A^+, respectively), for $a, b \in A$.

2.1. Proposition. *$\varkappa$-polarity and δ-polarity are symmetric and antireflexive binary relations on C^*-algebra A with the following properties:*

1. *$a \varkappa b \Leftrightarrow \overline{AaA} \cdot \overline{AbA} = \{0\}$,*
2. *$a \varkappa b \Leftrightarrow \overline{Aa} \cap \overline{Ab} = \{0\}$,*
3. *$a \varkappa b \Leftrightarrow a^* \varkappa b^*$,*
4. *$a \varkappa b \Rightarrow a \delta b$,*
5. *$a \varkappa b \Leftrightarrow |a|^{2n} \varkappa b$, $a \delta b \Leftrightarrow |a|^{2n} \delta b$,*

for a, $b \in A$ and each positive integer n.

Proof. $\varkappa$ and δ are symmetric relations and their antireflexivity follows from the fact that $a = 0 \Leftrightarrow |a| = 0$ for $a \in A$.

1. and 2. follows from 1.6. and 1.1.

3. Propositions 1.1. and 1.5. implys $a\varkappa b \Leftrightarrow \overline{Aa} \cdot \overline{Ab} = \{0\} \Leftrightarrow (\overline{Aa} \cdot \overline{Ab})^* = \overline{Ab^*} \cdot \overline{Aa^*} = \overline{b^*A} \cdot \overline{a^*A} = \{0\} \Leftrightarrow b^* \varkappa a^* \Leftrightarrow a^* \varkappa b^*$.

4. follows from 1.4.

5. follows from 1., 3., 1.5. and the fact $\overline{Aa} = \overline{A|a|^2}$ for $a \in A$. Namely, $|a|^2 \in \overline{A|a|} \Rightarrow \overline{A|a|^2} \subseteq \overline{A|a|}$ and $|a|^2 \in \overline{A|a|^{2+}} \Rightarrow |a| \in \overline{A|a|^{2+}}$ (see [7], 2.2) $\Rightarrow \overline{A|a|} \subseteq \overline{A|a|^2}$.

Definition. Let φ be a polarity on a non-empty set M and $B \subseteq M$. Then $B'_\varphi = \{m \in M\colon m\varphi b$ for any $b \in B\}$. If $B = (B'_\varphi)'_\varphi = B''_\varphi$, then B is called *a φ-polar in M*. The set of all φ-polars in M is denoted by $\varphi(M)$.

The mapping $B \to B''_\varphi$ for $B \subseteq M$ is a closure operator and $B'''_\varphi = B'_\varphi$ holds. We shall write b'_φ instead of $\{b\}'_\varphi$ for $b \in M$. Let us investigate these notions for $\varkappa$ and δ polarity on C^*-algebras.

2.2. Proposition. *$\varkappa$-polars are closed ideals in a C^*-algebra A.*

Proof follows from the fact that $a\varkappa b \Leftrightarrow Aa \cdot Ab = \{0\}$, from the properties of left ideals in A and from the continuity of the multiplication in A.

Remark. A simple example in the C^*-algebra of real square matrices of rang *2* shows that δ-polars are not closed with respect to the addition.

2.3. Proposition. *If A is a C^*-algebra and $B \subseteq A$, then there holds:*
1. *$B''_\varkappa$ is the greatest ideal $C \in \mathscr{I}d(A)$ with respect to the property $C \cap B'_\varkappa = \{0\}$.*
2. *If $\langle B \rangle$ is the ideal generated by B in A, then $B'_\varkappa = \langle B \rangle'_\varkappa$.*

Proof. 1. $B''_\varkappa \cap B'_\varkappa = \{0\}$ holds and if $C \in \mathscr{I}d(A)$ such that $C \cap B'_\varkappa = \{0\}$, then for $c \in A$ and $b \in B'_\varkappa$ there holds $Ac \cdot Ab \subseteq c \cap B'_\varkappa = \{0\}$, i.e., $c\varkappa b$ and $C \subseteq B''_\varkappa$.

2. follows from 2.2.

2.4. Proposition. *If A is a C^*-algebra, then the mapping $B \to B''_\varkappa$ in the frame $\mathscr{I}d(A)$ is a nucleus and $\varkappa(A)$ is a frame. $\varkappa(A)$ is a complete Boolean algebra in which the complement of a $\varkappa$-polar B is $B'_\varkappa$, $\bigwedge\{B_\lambda \in \varkappa(A)\colon \lambda \in \Lambda\} = \bigcap\{B_\lambda \in \varkappa(A)\colon \lambda \in \Lambda\}$ and $\bigvee\{B^\lambda \in \varkappa(A)\colon \lambda \in \Lambda\} = (\bigcup\{B_\lambda \in \varkappa(A)\colon \lambda \in \Lambda\})''_\varkappa$.*

Proof follows from 2.2, 2.3. and from the properties of regular elements in frames.

2.5. Proposition. *The set $\delta(A)$ of all δ-polars of a C^*-algebra A is a complete Boolean algebra in which the complement of a δ-polar B is B'_δ, $\bigwedge\{B_\lambda \in \delta(A)\colon \lambda \in \Lambda\} = \bigcap\{B_\lambda \in \delta(A)\colon \lambda \in \Lambda\}$ and $\bigvee\{B_\lambda \in \delta(A)\colon \lambda \in \Lambda\} = (\bigcup\{B_\lambda \in \delta(A)\colon \lambda \in \Lambda\})''_\delta$.*

Proof. δ-polarity on A is a symmetric and antireflexive binary relation; and let us introduce a quasiorder $\leq$ on A in the following way: $a \leq b \Leftrightarrow \overline{Aa} \subseteq \overline{Ab}$ for $a, b \in A$. Then $\delta(A)$ is a complete Boolean algebra with respect to the introduced operations if the following conditions are fulfilled (see [12], Th. 1.4,4):

a) $x \geq y$, $x\delta y \Rightarrow 0 \geq y$, b) $0\delta 0$,
c) $x\delta y$, $z \leq x \Rightarrow z\delta y$, d) x non $\delta y \Rightarrow \exists z \in A$,
z non ≤ 0, $z \leq x$, $z \leq y$, for $x, y, z \in A$.

Let us prove these conditions for $\delta(A)$:
a) $x \geq y, x\delta y \Rightarrow \overline{Ax} \supseteq \overline{Ay}, \overline{Ax} \cap \overline{Ay} = \overline{\{0\}} \Rightarrow \overline{Ay} = \{0\} \Rightarrow y = 0$, b) clear, c) $x\delta y$, $z \leq x \Rightarrow \overline{Ax} \cap \overline{Ay} = \{0\}, \overline{Az} \subseteq \overline{Ax} \Rightarrow \overline{Az} \cap \overline{Ay} = \{0\} \Rightarrow z\delta y$, d) x non $\delta y \Rightarrow \exists z$, z non $\leq 0 \in \overline{Ax} \cap \overline{Ay} \Rightarrow \overline{Az} \subseteq \overline{Ax}, \overline{Az} \subseteq \overline{Ay}, 0 \neq z \Rightarrow z$ non $\leq 0, z \leq x, z \leq y$.

2.6. Corollary. *Let A be a C^*-algebra, $B, C \subseteq A$, $|B| = \{|b|: b \in B\}$ and let B' denote $B'_\varkappa$ or B'_δ and $\varphi(A)$ denote $\varkappa(A)$ or $\delta(A)$. Then there holds:*

1. $(\bigcup\{B_\lambda \in \varphi(A): \lambda \in \Lambda\})' = \bigcap\{B'_\lambda \in \varphi(A): \lambda \in \Lambda\}$ *and* $(\bigcap\{B_\lambda \in \varphi(A): \lambda \in \Lambda\})' = (\bigcup\{B'_\lambda \in \varphi(A): \lambda \in \Lambda\})''$.

2. $B' = |B|' = |B''|', |A| \wedge |B''| \subseteq B''$.

3. $(|B''| \wedge |C''|)'' = B'' \cap C''$.

Proof. 1. It follows from [12], B.3.

2. If $b \in B'$, $c \in B''$, then $|b| \wedge |c| = 0$ and $B' = |B|'$, $|B''|' = B''' = B'$. If $c \in |A| \wedge |B''|$, then $c = |a| \wedge |b|$ for $a \in A$, $b \in B''$ and $0 \leq |a| \wedge |b| \leq |b|$ holds. $\overline{Ab^+}$ is an order ideal and that fact implies $|a| \wedge |b| \in \overline{Ab}$ and $A(|a| \wedge |b|)$. $Ad \subseteq Ab \cdot Ad = \{0\}$ for each $d \in B'_\varkappa (\overline{A(|a| \wedge |b|)} \cap \overline{Ad} \subseteq \overline{Ab} \cap \overline{Ad} = \{0\}$ for each $d \in B'_\delta$). Finally, $|a| \wedge |b| \in B''$ holds.

3. From 2. it follows that $(|B''| \wedge |C''|'') \subseteq (|A| \wedge |B''|)'' \cap (|A| \wedge |C''|)'' \subseteq B'' \cap C''$. If $x \in B'' \cap C''$ and $y \in (|B''| \wedge |C''|)'$, then $|x| = |x| \wedge |x| \in |B''| \wedge |C''|$ and $|x| \wedge |y| = 0$ holds. It means that $x \in (|B''| \wedge |C''|)''$.

Let us proceed with some general considerations of polarities. Let $\emptyset \neq M$ be a set, $B(M) \subseteq \exp M$ be a complete Boolean algebra such that $\bigcup B(M) = M$. Then $\langle a \rangle_B = \bigwedge(P \in B(M): a \in P)$ is the smallest element from $B(M)$ containing a and let us define the polarity α_B on M: $a\alpha_B b \Leftrightarrow \langle a \rangle_B \cap \langle b \rangle_B = O_{B(M)}$ for a, $b \in M$.

There holds $O_{B(M)} = \{a \in M: a\alpha_B m$, for each $m \in M\}$.

2.7. Proposition. *1. If π is a polarity on M then there holds: $a\pi b \Rightarrow a''_\pi \cap b''_\pi = O_{\pi(M)} = \{x \in M: x\pi m$ for each $m \in M\}$.*

2. If π is a polarity on M such that $\pi(M)$ is a complete Boolean algebra, then $\pi = \alpha_\pi$.

Proof. 1. If $x \in a''_\pi \cap b''_\pi$, then $x \in b'_\pi$, i.e., $x\pi x$ and the rest follows from the antireflexivity of π.

2. If $a\pi b$, then $a\alpha_\pi b$. If $a\alpha_\pi b$, a non πb, then $z \notin O_{\pi(M)}$ exists (see [12], Th. 1.4,4), $z \in a''_\pi \cap b''_\pi$, a contradiction.

Let us put $\alpha \geq \beta$ foi the polarities α, β on M when there holds: $x\beta y \Rightarrow x\alpha v$ for $x, y \in M$.

2.8. Proposition. *Let M be a set, (K) be a given property of a system of subsets in M closed with respect to meets and covering M. Let $0 = \bigcap\{\langle m \rangle: m \in M\}$, where $\langle m \rangle$ is the smallest subset of M containing m and having the property (K). Then there holds:*

ϱ is the greatest polarity on M such that polars form a complete Boolean algebra, have the property (K) and $O_{\varrho(M)} = 0$ iff ϱ has the following property:

$$a\varrho b \Leftrightarrow \langle a\rangle \cap \langle b\rangle = 0. \qquad (*)$$

Proof. $\Leftarrow$: If β is a polarity on M, $\beta(M)$ is a complete Boolean algebra, $O_{\beta(M)} = O$ and β-polars have the property (K), then there holds: $a\beta b \Rightarrow a''_\beta \cap \cap b''_\beta = O_{\beta(M)}$ (see 2.7,1) $\Rightarrow a\varrho b$ for $a, b \in M$. It means that $\varrho \geq \beta$.

The relation ϱ defined by (*) is a polarity fulfilling the necessary conditions (see [12], Th. 1.4,4) which guarantee that $\varrho(M)$ is a complete Boolean algebra. There holds $O_{\varrho(M)} = 0$ (see 2.7,1) and we shall prove that polars from $\varrho(M)$ have the property (K). If $X \subseteq M$, then $X'_\varrho = \bigcap\{x'_\varrho : x \in X\}$ and $\langle x\rangle'_\varrho \subseteq x'_\varrho \subseteq \langle x'_\varrho\rangle$ holds. We have $\langle t\rangle \cap \langle z\rangle \subseteq \langle t\rangle \cap \langle x\rangle = 0$ for each $t \in x'_\varrho$ and $z \in \langle x\rangle$, i.e., $t \in \langle x\rangle'_\varrho$ and $x'_\varrho = \langle x\rangle'_\varrho$. Finally, $x''_\varrho \supseteq \langle x'_\varrho\rangle'_\varrho = x''_\varrho$ and thus $x'_\varrho = \langle x'_\varrho\rangle''_\varrho \supseteq \supseteq \langle x'_\varrho\rangle \supseteq x'_\varrho$, i.e., $x'_\varrho = \langle x'_\varrho\rangle$.

$\Rightarrow$: It follows from 2.7,2.

2.9. Corollary. *1. $\varkappa$-polarity is the greatest polarity on a C^*-algebra such that polars form a complete Boolean algebra and polars are ideals.*

2. δ-polarity is the greatest polarity on a C^*-algebra such that polars form a complete Boolean algebra and polars are left ideals.

3. The greatest polarity β on a C^*-algebra A such that polars form a complete Boolean algebra and polars are right ideals has the following properties: $a\beta b \Leftrightarrow \overline{aA} \cap \overline{bA} = \{0\}$ and $a\beta b \Rightarrow a^*\delta b^*$, for $a, b \in A$.

Proof follows from 2.8. We have $a\beta b \Leftrightarrow \overline{aA} \cap \overline{bA} = \{0\} \Leftrightarrow \overline{aA^*} \cap \overline{bA^*} = \{0\} \Leftrightarrow a^*\delta b^*$ (see 1.5).

Now, we investigate a polarity corresponding to the multiplication in C^*-algebras.

2.10. Proposition. *If ε is a binary relation on a C^*-algebra A such that $a\varepsilon b \Leftrightarrow a\cdot b = 0$ for $a, b \in A$, then there holds*:

1. $a\varepsilon b \Leftrightarrow \overline{Aa}\cdot\overline{bA} = \{0\}$.

2. $a\varepsilon b \Leftrightarrow b^*\varepsilon a^*$, $a\varepsilon b \Leftrightarrow |a|\,\varepsilon\,|b^*|$.

3. $a\varkappa b \Rightarrow a\varepsilon b \Rightarrow a\delta b^*$.

Proof. 1. $a\varepsilon b \Leftrightarrow Aa\cdot bA = \{0\} \Leftrightarrow \overline{\overline{Aa\cdot bA}} = \overline{Aa\cdot bA} = \{0\} \Leftrightarrow \overline{Aa}\cdot\overline{bA} = \{0\}$.

2. $b^*\varepsilon a^* \Leftrightarrow b^*\cdot a^* = 0 \Leftrightarrow a\cdot b = 0 \Leftrightarrow a\varepsilon b \Leftrightarrow \overline{Aa}\cdot\overline{bA} = \{0\} \Leftrightarrow \overline{A|a|}\cdot\overline{|b^*|A} = \{0\} \Leftrightarrow |a|\cdot|b^*| = 0 \Leftrightarrow |a|\varepsilon|b^*|$ (see 1.5).

3. We have $a\varkappa b \Leftrightarrow a\cdot b \in \overline{AaA} \cap \overline{AbA} = \{0\} \Rightarrow a\varepsilon b$ (see 1.1). If $a\varepsilon b$, then $|a|\cdot|b^*| = 0$ and if $z \in A|a| \cap A|b^*|$, then $z = m|a| = n|b^*|$ for suitable elements $m, n \in A$. It implies that $|z^*|^2 = z\cdot z^* = m|a|\,|b^*|n^* = 0$ and thus $z = 0$, $\overline{A|a|} \cap \overline{A|b^*|} = \{0\}$. If $p \in \overline{A|a|} \cap \overline{A|b^*|}$, then sequences $\{m_i\}, \{n_i\} \subseteq A$ exist such that $m_i|a| \to p$, $n_i|b^*| \to p$ and we have $\{0\} = \{m_i|a|\cdot|b^*|n_i^*\} \to p\cdot p^* \Rightarrow |p^*|^2 =$

$= 0 \Rightarrow |p^*| = 0 \Rightarrow p = 0$. According to 1.5 $Aa \cap Ab^* = A|a| \cap A|b^*| = \{0\}$ holds; thus $a\delta b^*$.

The relation ε is neither symmetric nor antireflexive, which is a reason to introduce the following relation:

Definition. *γ-polarity* on a C^*-algebra A is defined in the following way:

$$a\gamma b \Leftrightarrow a^* \cdot b = 0 \quad \text{for} \quad a, b \in A.$$

2.11. Proposition. *γ-Polarity is a polarity on A and has the following properties:* 1. $a\gamma b \Leftrightarrow |a^*|\, \varepsilon\, |b^*| \Leftrightarrow a^* \varepsilon b$.

2. $a\varkappa b \Rightarrow a^* \gamma b$ *and* $a\gamma b \Rightarrow a^* \delta b^*$, *for* $a, b \in A$.

Proof. We have $a\gamma b \Leftrightarrow a^* \cdot b = 0 \Leftrightarrow b^* \cdot a = 0 \Leftrightarrow b\gamma a$, $a\gamma a \Leftrightarrow |a|^2 = 0 \Rightarrow a = 0$. The rest follows from 2.10,2. and 3.

The polarity γ is derived from the operation $\circ$ on A such that $a \circ b = a^* \cdot b$ for $a, b \in A$, which J. Rosický [11] introduced. γ-Polars are closed right ideals in A that form a complete complemented lattice $\gamma(A)$ and they have similar properties as $\varkappa$-polars and δ-polars. Namely, the analogy of 2.3 is true.

If I is a left ideal in A, then I'_γ is a two-sided ideal in A. If we define a polarity on A that is similar as γ and that is defined by the formula $a \cdot b^* = 0$, then polars are left ideals in A.

2.12. Lemma (A generalization of Th. 1, [3]). *If* $(G,.)$ *is a groupoid,* $\overline{}$ *is a closure operator on G and* $X \circ Y = \overline{X \cdot Y}$, $\bigvee Y_i = \overline{\bigcup Y_i}$ *for* X, Y, $Y_i \subseteq G (i \in I)$, *then the following assertions are equivalent:*

1. $X \cdot \overline{Y} \subseteq X \circ Y$.

2. $X \circ Y = X \circ \overline{Y}$.

3. $X \cdot (\bigvee Y_i) \subseteq \bigvee (X \cdot Y_i)$

4. $X \circ (\bigvee Y_i) = \bigvee (X \circ Y_i)$.

Proof. 3 $\Rightarrow$ 4: $X \circ (\bigvee Y_i) = \overline{X \cdot \bigvee Y_i} = \overline{\bigvee (X \cdot Y_i)} \subseteq \overline{\bigcup \overline{(X \cdot Y_i)}} = \bigvee (X \circ Y_i)$, $\bigvee (X \circ Y_i) = \overline{\bigcup \overline{X \cdot Y_i}} \subseteq \overline{X . \bigcup Y_i} = X \circ \bigvee Y_i$.

4 $\Rightarrow$ 2: For $Y_i = Y (i \in I)$ it holds $X \circ \overline{Y} = X \circ \bigvee Y_i = \bigvee (X \circ Y_i) = X \circ Y$. 2 $\Rightarrow$ 1: $X \cdot \overline{Y} \subseteq X \circ \overline{Y} = X \circ Y$.

1 $\Rightarrow$ 3: $X \cdot \bigvee Y_i = X \cdot \overline{\bigcup Y_i} \subseteq \overline{X \cup Y_i} = \overline{X \circ (\bigcup Y_i)} = \overline{\bigcup (X \cdot Y_i)} = \bigvee X \circ Y_i$.

Remark. A similar lemma is true when we change the multipliers in operations $\circ$ and $\cdot$. For example 1. $\overline{Y} \cdot X \subseteq Y \circ X$.

2.13. Theorem. *On a C^*-algebra A the following assertions are equivalent:*

1. The set $\gamma(A)$ of all γ-polars on A is a complete Boolean algebra such that the complement of a γ-polar B is B'_γ,

$\bigvee \{B_\lambda \in \gamma(A) \colon \lambda \in \Lambda\} = (\bigcup \{B_\lambda \in \gamma(A) \colon \lambda \in \Lambda\})''_\gamma$ *and* $B \wedge C = (B \cdot C)''_\gamma$ *for γ-polars B, C.*

2. $a^* \cdot b = 0 \Leftrightarrow a \cdot b = 0$ *for each* a, $b \in A$.

3. A is commutative.

Proof. $1 \Rightarrow 2$: If $a, b \in A$, $a^* \cdot b = 0$, then $a \in b'_\gamma$ implies $a \cdot b \in b'_\gamma \cdot b''_\gamma \subseteq b'_\gamma \wedge \wedge b''_\gamma = \{0\}$, i.e., $a \cdot b = 0$. If $a \cdot b = 0$, then similarly $(a^*)^* \cdot b = 0 \Rightarrow a^* \in b'_\gamma \Rightarrow \Rightarrow a^* \cdot b \in b'_\gamma \cdot b''_\gamma \subseteq b'_\gamma \wedge b''_\gamma = \{0\} \Rightarrow a^* \cdot b = 0$.

$2 \Rightarrow 1$: a) We shall prove that $(h \cdot k)'_\gamma = (k \cdot h)'_\gamma$ for each $h, k \in A$. We have $a \cdot b = 0 \Leftrightarrow a^* \cdot b = 0 \Leftrightarrow b^* \cdot a = 0 \Leftrightarrow b \cdot a = 0$ for $a, b \in A$ and further $x \gamma hk \Leftrightarrow \Leftrightarrow x^* \cdot h \cdot k = 0 \Leftrightarrow (x \cdot h) \cdot k = 0 \Leftrightarrow k(x \cdot h) = 0 \Leftrightarrow k^* \cdot x \cdot h = 0 \Leftrightarrow (x^* \cdot k)^* \cdot h = 0 \Leftrightarrow \Leftrightarrow x^* k \cdot h = 0 \Leftrightarrow x \gamma k \cdot h$, for $x \in A$. It implies that $z \in (X \cdot Y)'_\gamma \Leftrightarrow z \gamma xy$ (for each $x \in X$, $y \in Y) \Leftrightarrow z \gamma yx \Leftrightarrow z \in (Y \cdot X)'_\gamma$, for any $X, Y \subseteq A$.

If we introduce $X \circ Y = (X \cdot Y)''_\gamma$ for each $X, Y \subseteq A$, then $X \circ Y = Y \circ X$ holds.

b) $\gamma(A)$ is a closure system and let us prove $X \circ Y = X \circ Y''_\gamma$ for each $X, Y \subseteq A$. According to 2.12 it is sufficient to prove that $X \cdot Y''_\gamma \subseteq X \circ Y$. If $a \in X \cdot Y''_\gamma$, then $a = x \cdot c$ for suitable $x \in X$ and $c \in Y''_\gamma$. We have $b^*(x \cdot y) = 0 \Rightarrow bxy = 0 \Rightarrow \Rightarrow (bx)^* y = 0 \Rightarrow bx \in Y'_\gamma$ for $x \in X$, $y \in Y$, $b \in (X \cdot Y)'_\gamma$. Further, $(bx)^* c = 0 \Rightarrow 0 = = bxc = ba \Rightarrow b^* a = 0 \Rightarrow a \gamma b \Rightarrow a \in (X \cdot Y)''_\gamma$.

It means that $(\gamma(A), \circ, \vee)$ is a multiplicative lattice (see 2.12, $2 \Rightarrow 4$), $X \circ A = X$ holds for any $X \in \gamma(A)$ because γ-polars are right ideals in A. These facts and 2.12,2 imply that $\circ$ is associative. Namely, $X \circ (Y \circ Z) = X \circ (Y \cdot Z)''_\gamma = = X \circ (Y \cdot Z) = [X \cdot (Y \cdot Z)]''_\gamma = [(X \cdot Y) \cdot Z]''_\gamma = (X \cdot Y) \circ Z = (X \cdot Y)''_\gamma \circ Z = (X \circ Y) \circ \circ Z$ for each $X, Y, Z \subseteq A \cdot \gamma(A)$ is a regular quantale and [10], Th. 2.5 implies that $\gamma(A)$ is a frame. It means that $X \circ Y = X \cap Y$ for each $X, Y \in \gamma(A)$. Finally, $\gamma(A)$ is a complemented distributive complete lattice, i.e., $\gamma(A)$ is a complete Boolean algebra.

$2 \Rightarrow 3$: [8], Proposition 3.3 implies the existence of a set $\{X_i \colon i \in I\} \subseteq \mathscr{R}(A)$ for each $Y \in \mathscr{R}(A)$ such that $Z_i \circ X_i = 0$, $Z_i \vee Y = A$ and $Y = \bigvee (X_i \colon i \in I)$ for suitable $Z_i \in \mathscr{R}(A)$ and $i \in I$. We have $Z_i \cdot X_i = 0$ (see 2.) and therefore $\mathscr{R}(A)$ is a regular quantale. $\mathscr{R}(A)$ is a frame (see [10], Th. 2.5) and A is commutative.

$3 \Rightarrow 3$: For $a, b \in A$ there holds $a^* \cdot b = 0 \Rightarrow (ab)^* ab = 0 \Rightarrow |ab|^2 = \Rightarrow a \cdot b = 0$ and further $ab = 0 \Rightarrow (a^* \cdot b)^* \cdot a^* \cdot b = 0 \Rightarrow |a^* b|^2 = 0 \Rightarrow a^* b = 0$.

Remarks. 1. $\mathscr{R}(A)$ is a frame iff A is commutative (see [6], 2.5.7).

2. If $\gamma(A)$ is a complete Boolean algebra, then $\gamma = \varepsilon$.

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ПОЛЯРНОСТИ В C^*-АЛГЕБРАХ

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Резюме

В этой статье исследуются основные свойства полярностей в C^*-алгебрах, а именно, отношение полярностей и идеалов в C^*-алгебрах и решеточная характеристика множества поляр.