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Dedicated to Professor Dr. Beloslav Riečan with admiration for his many profound contributions to measure theory and logic

SHARP AND FUZZY ELEMENTS OF AN RC-GROUP

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ABSTRACT. Effect algebras serve as algebraic models for logical calculi and thereby provide semantic interpretations for both sharp and fuzzy logical propositions. Most of the effect algebras that are so employed can be realized as intervals in partially ordered abelian groups, called CB-groups, that are enriched by a family of order-preserving endomorphisms called compressions. For a special class of CB-groups called RC-groups, we show that every element of the positive cone can be decomposed uniquely as a sum of a finite chain of sharp elements and a fuzzy element of the unit interval that dominates no nonzero sharp element. The category of RC-groups includes the additive groups of bounded measurable functions on σ-fields of sets, abelian ℓ-groups with Heyting MV-algebras as their unit intervals, and the self-adjoint parts of AW*-algebras.

1. Introduction

Let $\mathcal{B}$ be a σ-field of subsets of a nonempty set X and define $E(X,\mathcal{B},\mathbb{R})$ to be the set of all $\mathcal{B}$-measurable functions $e\colon X \to \mathbb{R}$ such that $0 \le e(x) \le 1$ for all $x \in X$ ([19]).

DEFINITION 1.1. If $M \in \mathcal{B}$, the characteristic set function (or indicator function) of M is denoted by $\chi_M \in E(X,\mathcal{B},\mathbb{R})$. The mappings ${}^\perp\colon E(X,\mathcal{B},\mathbb{R}) \to E(X,\mathcal{B},\mathbb{R})$ and $'\colon E(X,\mathcal{B},\mathbb{R}) \to E(X,\mathcal{B},\mathbb{R})$ are defined for $e \in E(X,\mathcal{B},\mathbb{R})$ by $e^\perp(x) := 1 - e(x)$ for all $x \in X$ and $e' := \chi_{e^{-1}(0)}$.

Elements $e \in E(X,\mathcal{B},\mathbb{R})$ can be regarded as (possibly) *fuzzy subsets* of X in the sense of L. Zadeh [26]. According to this interpretation, if $e \in E(X,\mathcal{B},\mathbb{R})$

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and $x \in X$, the real number $e(x)$ represents the "degree of membership", on a scale from 0 to 1, of x in the (possibly) fuzzy set e. If $e(x) = 1$, then x is a full-fledged member of e; if $e(x) = 0$, then x is a full-fledged member of the (possibly) fuzzy complement $e^{\perp}$ of e.

Elements $e \in E(X, \mathcal{B}, \mathbb{R})$ can manifest varying levels of "fuzziness". No fuzziness at all is associated with the characteristic set function χ_M of a set $M \in \mathcal{B}$; indeed each $x \in X$ has either full status as a member of χ_M or as a member of its complement $\left(\chi_M\right)^{\perp} = \chi_{(X \setminus M)}$ according to whether $\chi_M(x) = 1$ or $\chi_M(x) = 0$. On the other hand, if $b \in E(X, \mathcal{B}, \mathbb{R})$ and $b(x) < 1$ for all $x \in X$, then no element in X has full status as a member of b.

DEFINITION 1.2. An element $p \in E(X, \mathcal{B}, \mathbb{R})$ is *sharp* iff it has the form $p = \chi_M$ for some $M \in \mathcal{B}$. The set of all sharp elements in $E(X, \mathcal{B}, \mathbb{R})$ will be denoted by $P(X, \mathcal{B}, \mathbb{R})$.[1]

An element $b \in E(X, \mathcal{B}, \mathbb{R})$ is *blunt* iff $b(x) < 1$ for all $x \in X$.

The constant functions $\chi_\emptyset$ and χ_X in $P(X, \mathcal{B}, \mathbb{R})$, which we shall denote simply by 0 and 1, represent the sharp empty set and the sharp universal set X, respectively. The elements $f \in E(X, \mathcal{B}, \mathbb{R})$ such that both f and $f^{\perp}$ are blunt are "totally fuzzy" in the sense that no element $x \in X$ has full status as a member of either f or $f^{\perp}$. We note that 0 is the only element of $E(X, \mathcal{B}, \mathbb{R})$ that is both sharp and blunt.

The system $E(X, \mathcal{B}, \mathbb{R})$ is richly endowed with mathematical structure. It is a distributive lattice under the pointwise partial order with the pointwise minimum and maximum of $e, f \in E(X, \mathcal{B}, \mathbb{R})$ as the infimum $e \wedge f$ and supremum $e \vee f$ of e and f. The constant functions 0 and 1 are the smallest and largest elements of $E(X, \mathcal{B}, \mathbb{R})$. The unary operations $e \mapsto e^{\perp}$ and $e \mapsto e'$ serve as two notions of "complementation" or "negation" on $E(X, \mathcal{B}, \mathbb{R})$; both are order inverting, and $e \mapsto e^{\perp}$, but not $e \mapsto e'$, is of period two. The system $E(X, \mathcal{B}, \mathbb{R})$ carries partially defined binary operations $f \ominus e$, defined only when $e \leq f$, and $e \oplus f$, defined only when $e \leq f^{\perp}$, according to $(f \ominus e)(x) := f(x) - e(x)$ and $(e \oplus f)(x) := e(x) + f(x)$ for all $x \in X$. It also carries binary operations $(e, f) \mapsto e \hat{+} f$ and $(e, f) \mapsto e \supset f$ defined by $(e \hat{+} f)(x) := \min\{e(x) + f(x), 1\}$ and $e \supset f := \left(e \ominus (e \wedge f)\right)' \vee f$.

Under the pointwise partial order and with $p \mapsto p^{\perp}$ as the Boolean complementation, $P(X, \mathcal{B}, \mathbb{R})$ is a σ-complete Boolean algebra isomorphic to the σ-field $\mathcal{B}$. Under the partial binary operation $\ominus$, $E(X, \mathcal{B}, \mathbb{R})$ is a D-poset ([22]); under the partial binary operation $\oplus$, it is an effect algebra ([1], [15], [18]); it is a Heyting algebra with $\supset$ as the Heyting implication connective and $e \mapsto e'$ as the Heyting (or intuitionistic) negation connective[2] ([6]); and with $\hat{+}$ as

[1] This notation is chosen to correspond to the more general notion of a "projection" in a CB-group — see Lemma 4.7.

[2] For $d, e, f \in E(X, \mathcal{B}, \mathbb{R})$, $d \wedge e \leq f \iff d \leq (e \supset f)$ and $e' = (e \supset 0)$.

the MV-sum and $e \mapsto e^{\perp}$ as the MV-negation, $E(X, \mathcal{B}, \mathbb{R})$ is an MV-algebra ([23], [25]).

MV-algebras were originally introduced by C. Chang [3] to serve as algebraic models for the multivalued logics of Łukasiewicz. In [23], D. Mundici proved that every MV-algebra E can be realized as the unit interval in an enveloping abelian ℓ-group (lattice-ordered group) G with a distinguished order unit u. For the MV-algebra $E(X, \mathcal{B}, \mathbb{R})$, the enveloping group is the abelian ℓ-group $G(X, \mathcal{B}, \mathbb{R})$ of all bounded $\mathcal{B}$-measurable functions $f\colon X \to \mathbb{R}$ under pointwise addition and pointwise partial order. Thus, with the constant function 1 as order unit, $E(X, \mathcal{B}, \mathbb{R})$ is the "unit interval" $\{e \in G(X, \mathcal{B}, \mathbb{R}) :\ 0 \le e \le 1\}$ in $G(X, \mathcal{B}, \mathbb{R})$, and we have

$$P(X, \mathcal{B}, \mathbb{R}) \subseteq E(X, \mathcal{B}, \mathbb{R}) \subseteq G(X, \mathcal{B}, \mathbb{R}).$$

The passage from the system $E(X, \mathcal{B}, \mathbb{R})$ to the abelian ℓ-group $G(X, \mathcal{B}, \mathbb{R})$ with order unit 1 offers several advantages. For instance, the partially defined operations $e \oplus f$ and $f \ominus e$ are naturally extended to the addition $g+h$ and subtraction $h-g$ operations on $G(X, \mathcal{B}, \mathbb{R})$, and the MV-sum of $e, f \in E(X, \mathcal{B}, \mathbb{R})$ is given by the simple formula $e \hat{+} f = (e+f) \wedge 1$. Furthermore, there is a vast literature dealing with abelian ℓ-groups, and the well-developed theory for this class of groups can be brought to bear in the study of MV-algebras and the logical calculi for which they are models.

In this article, we take $G(X, \mathcal{B}, \mathbb{R})$ as a prototype and proceed to study sharp, and fuzzy elements in more general structures called unital groups ([7; Definition 2.3]), CB-groups ([16; Definition 3.4]), and RC-groups ([16; Definition 4.3 (iv)]. The unit intervals in these structures can serve as semantic models for various classes of (possibly nonstandard) logical calculi, including quantum-logical calculi ([4], [5], [20], [24]).

The prototype $G(X, \mathcal{B}, \mathbb{R})$ is not only an additive abelian ℓ-group with order unit 1, it is also a ring, and even an associative and commutative linear algebra with unity element 1 under the pointwise product of functions. In the interest of obtaining a desired level of generality, we shall largely ignore the multiplicative structure of the prototype $G(X, \mathcal{B}, \mathbb{R})$ except for special products pf of sharp elements $p \in P(X, \mathcal{B}, \mathbb{R})$ and functions $f \in G(X, \mathcal{B}, \mathbb{R})$. In this connection, we note that $pf = 0 \iff p \le f'$ and

$$P(X, \mathcal{B}, \mathbb{R}) = \{p \in G(X, \mathcal{B}, \mathbb{R}) :\ p = p^2\}.$$

DEFINITION 1.3. The additive subgroup of $G(X, \mathcal{B}, \mathbb{R})$ consisting of the functions $f \in G(X, \mathcal{B}, \mathbb{R})$ such that $f(x) \in \mathbb{Z}$ (the system of integers) for all $x \in X$ is denoted by $G(X, \mathcal{B}, \mathbb{Z})$.

Evidently, $G(X, \mathcal{B}, \mathbb{Z})$ is the subgroup of $G(X, \mathcal{B}, \mathbb{R})$ generated by the sharp elements $p \in P(X, \mathcal{B}, \mathbb{R})$. Because the functions in $G(X, \mathcal{B}, \mathbb{R})$ are bounded and

$\mathcal{B}$-measurable, $G(X,\mathcal{B},\mathbb{Z})$ consists of all $\mathbb{Z}$-valued functions f that assume only finitely many values and have the property that $f^{-1}(z)\in\mathcal{B}$ for every $z\in\mathbb{Z}$ (cf. Example 3.2 below). Under pointwise partial order, $G(X,\mathcal{B},\mathbb{Z})$ is again an abelian ℓ-group, $1\in G(X,\mathcal{B},\mathbb{Z})$, and the unit interval

$$E(X,\mathcal{B},\mathbb{Z}) := \{p\in G(X,\mathcal{B},\mathbb{Z}):\ 0\le p\le 1\} = P(X,\mathcal{B},\mathbb{R})$$

is precisely the set of sharp elements in $E(X,\mathcal{B},\mathbb{R})$.

The following lemmas and theorem are easily verified.

Lemma 1.4. $P(X,\mathcal{B},\mathbb{R}) = \{p\in E(X,\mathcal{B},\mathbb{R}):\ p\wedge p^{\perp}=0\}$.

Lemma 1.5. *Let $0\le t\in G(X,\mathcal{B},\mathbb{R})$ with $t\ne 0$. Then $t\in G(X,\mathcal{B},\mathbb{Z})$ iff there is a finite sequence $p_1,p_2,\dots,p_n$ of nonzero sharp elements in $P(X,\mathcal{B},\mathbb{R})$ such that $p_1\ge p_2\ge\cdots\ge p_n$ and $t=p_1+p_2+\cdots+p_n$. Moreover, if $t\in G(X,\mathcal{B},\mathbb{Z})$, the sequence $p_1\ge p_2\ge\cdots\ge p_n$ is uniquely determined by t.*

Lemma 1.6. *If $0\le b\in G(X,\mathcal{B},\mathbb{R})$, then b is a blunt element of $E(X,\mathcal{B},\mathbb{R})$ iff the only sharp element $p\in P(X,\mathcal{B},\mathbb{R})$ such that $p\le b$ is $p=0$.*

Theorem 1.7. *Let $f\in G(X,\mathcal{B},\mathbb{R})$ with $0\le f$. Then there exists a uniquely determined element $0\le t\in G(X,\mathcal{B},\mathbb{Z})$ and a uniquely determined blunt element $b\in E(X,\mathcal{B},\mathbb{R})$ such that $f=t+b$.*

If X is a singleton set and $\mathcal{B}=\{\emptyset,X\}$, then $G(X,\mathcal{B},\mathbb{R})$ may be identified with the totally ordered additive abelian group $\mathbb{R}$. Then the unit interval $E(X,\mathcal{B},\mathbb{R})$ is just the standard unit interval $[0,1]$ in $\mathbb{R}$, the subgroup $G(X,\mathcal{B},\mathbb{Z})$ is just the totally ordered additive group $\mathbb{Z}$ of integers, the sharp elements in $[0,1]$ are 0 and 1, and the blunt elements are real numbers in the half open interval $[0,1)$. Thus, Theorem 1.7 just says that every real number $r\ge 0$ can be written uniquely as $r=t+b$ where $0\le t\in\mathbb{Z}$ and $0\le b<1$.

The main theorems of this article, Theorem 5.8 and 5.9, constitute a generalization to RC-groups of Theorem 1.7. In Section 2 below, we review the basic definitions underlying the notion of an RC-group, using $G(X,\mathcal{B},\mathbb{R})$ as a running example to illustrate the concepts involved. In Section 3 we give some examples of RC-groups, including the self-adjoint part of an AW*-algebra and unital abelian ℓ-groups with Heyting MV-algebras as their unit intervals. In Section 4, we review some basic properties of CB-groups and RC-groups, and in Section 5, we state and prove our main theorems. Background material, more examples, and more details of the basic theory can be found in [1], [7], [8], [9], [10], [11], [12], [13], [14], [16].

2. Basic definitions

Let G be an additively-written partially ordered abelian group with positive cone $G^+ := \{g \in G : 0 \le g\}$. If, as a partially ordered set, G is a lattice, then G is said to be an abelian *ℓ-group.* An element $u \in G^+$ is called an *order unit* ([17; p. 4]) iff each element in G is dominated by an integer multiple of u.[3] An order unit u is *generative* iff every element in G^+ is a sum of a finite sequence of elements in G^+ each of which is dominated by u ([1; Definition 3.2]).

A *unital group* is a partially ordered abelian group G together with a specified generative order unit $u \in G^+$ called the *unit* ([7; Definition 2.3]). If G is a unital group with unit u, then $E := \{e \in G : 0 \le e \le u\}$ is called the *unit interval* in G, and elements $e \in E$ are called *effects.*[4] For instance, if $\mathcal{B}$ is a σ-field of subsets of a nonempty set X, then $G(X, \mathcal{B}, \mathbb{R})$ is a unital group with unit 1 and unit interval $E(X, \mathcal{B}, \mathbb{R})$.

Let G be a unital group with unit interval E. By a *sub-effect algebra* of E we mean a subset $P \subseteq E$ such that

(i) $0, u \in P$,
(ii) $p \in P \implies u - p \in P$,
(iii) if $p, q \in P$, then $p + q \in E \implies p + q \in P$.

A sub-effect algebra P of E is said to be *normal* iff, whenever $d, e, f \in E$ with $d + e + f \in E$, the conditions $d + e \in P$ and $d + f \in P$ imply that $d \in P$ ([11; Definition 1]). For instance, the set $P(X, \mathcal{B}, \mathbb{R})$ of sharp elements in $E(X, \mathcal{B}, \mathbb{R})$ is a normal sub-effect algebra of $E(X, \mathcal{B}, \mathbb{R})$.

By definition, a *retraction* on G with *focus* p, is an order-preserving group endomorphism $J: G \to G$ such that

(i) $p = J(u) \in E$,
(ii) if $e \in E$, then $e \le p \implies J(e) = e$.

If J is a retraction on G, then J is idempotent, i.e., $J \circ J = J$ ([9; Lemma 2.2]). Let J be a retraction on G with focus p. If, for all $e \in E$, $J(e) = 0 \implies e \le u - p$, then J is called a *compression* on G. A retraction $J: G \to G$ is said to be *direct* iff $g \in G^+ \implies J(g) \le g$. Every direct retraction on G is a compression.

A *compression base* for the unital group G is a family $(J_p)_{p \in P}$ of compressions on G, indexed by their own foci, such that the set P is a normal sub-effect algebra of the unit interval E in G and, for all $p, q, r \in P$, $p + q + r \in P \implies J_{p+r} \circ J_{q+r} = J_r$. A *CB-group* is a unital group G with a specified compression base $(J_p)_{p \in P}$ ([13; Definition 2.3]). A *proper* CB-group is

[3] Some authors refer to such an element as a *strong order unit* or simply a *strong unit.*
[4] This terminology is borrowed from the quantum theory of measurement ([2]).

a CB-group G such that every direct compression on G belongs to its compression base. A CB-group that is also an ℓ-group is called a *CBℓ-group.*

If G is a CB-group with compression base $(J_p)_{p\in P}$, then elements $p \in P$ are called *projections.* With the partial order inherited from G, P is a regular orthomodular poset with $p \mapsto u - p$ as orthocomplementation ([20], [24]).

EXAMPLE 2.1. If $\mathcal{B}$ is a σ-field of subsets of the nonempty set X, then each sharp element $p \in P(X,\mathcal{B},\mathbb{R}) \subseteq E(X,\mathcal{B},\mathbb{R})$ determines a direct compression J_p with focus p on $G(X,\mathcal{B},\mathbb{R})$ according to $J_p(f) := pf$ (pointwise product) for all $f \in G(X,\mathcal{B},\mathbb{R})$. Every retraction J on $G(X,\mathcal{B},\mathbb{R})$ has the form $J = J_p$ where $p = J(1) \in P(X,\mathcal{B},\mathbb{R})$, and $G(X,\mathcal{B},\mathbb{R})$ is a proper CB-group with compression base $(J_p)_{p\in P(X,\mathcal{B},\mathbb{R})}$.

As per the following example, any unital group can be organized into a proper CB-group in at least one way.

EXAMPLE 2.2. If G is a unital group and $(J_p)_{p\in P}$ is the family of all direct compressions on G, indexed by their own foci, then G is a proper CB-group with compression base $(J_p)_{p\in P}$. A CB-group G for which the compression base consists precisely of all direct compressions on G is called a *direct* CB-group. In a direct CB-group G with compression base $(J_p)_{p\in P}$, the projections P form a Boolean algebra. The CB-group $G(X,\mathcal{B},\mathbb{R})$ in Example 2.1 is a direct CB-group.

Standing assumption. *In the sequel, G is a proper CB-group with unit u, unit interval E, and compression base $(J_p)_{p\in P}$.*

If $p \in P$, we define the set

$$C(p) := \{g \in G : \ g = J_p(g) + J_{u-p}(g)\},$$

and if $g \in C(p)$, we say that g is *compatible* with the projection p ([12; Definition 2.1(i)]). Evidently, $C(p)$ is a subgroup of G. If $g \in G$, we also define

$$CPC(g) := \bigcap\{C(p) : \ p \in P \ \text{and} \ g \in C(p)\} \qquad \text{([12; Definition 2.1(iii)]).}$$

Thus, $CPC(g)$ is a subgroup of G, and $h \in CPC(g)$ iff h is compatible with every projection p with which g is compatible. For the CB-group $G(X,\mathcal{B},\mathbb{R})$ in Example 2.1, and more generally, *in any direct CB-group, every element is compatible with every projection.*

If $g \in G$, we define

$$P^{\pm}(g) := \{p \in P \cap CPC(g) : \ g \in C(p) \ \text{and} \ J_{u-p}(g) \le 0 \le J_p(g)\}.$$

A projection $p \in P^{\pm}(g)$ splits $g = J_p(g) + J_{u-p}(g)$ into a "positive part" $J_p(g) \ge 0$ and a "negative part" $J_{u-p}(g) \le 0$. If $P^{\pm}(g)$ is nonempty for every

$g \in G$, then we say that the CB-group G has the *general comparability property* ([8; Definition 4.6]). If G has the general comparability property and $g \in G$, then

$$g^+ := J_p(g) \qquad \text{and} \qquad g^- := -J_{u-p}(g)$$

are independent of the choice of $p \in P^{\pm}(g)$ ([12; Theorem 3.2]). Moreover,

$$g = g^+ - g^- \qquad \text{and} \qquad 0 \le g^+, g^- .$$

If a direct CB-group has the general comparability property, then it is a CBℓ-group ([8; Theorem 4.9]) and for all $g \in G$, we have $g^+ = g \vee 0$ and $g^- = -(g \wedge 0)$.

EXAMPLE 2.3. In Example 2.1, if $f \in G(X, \mathcal{B}, \mathbb{R})$, let $M := f^{-1}\big([0, \infty)\big)$ and let $p := \chi_M \in P(X, \mathcal{B}, \mathbb{R})$. Then $p \in P(X, \mathcal{B}, \mathbb{R})^{\pm}(f)$, so $G(X, \mathcal{B}, \mathbb{R})$ has the general comparability property.

If $g \in G$, it is easy to see that $\bar{p}$ is the largest projection in the set $\{p \in P : \; g \in C(p) \text{ and } J_p(g) = 0\}$ iff $u - \bar{p}$ is the smallest projection in the set $\{q \in P : \; J_q(g) = g\}$. By definition, G is a *Rickart CB-group* iff, for every $g \in G$, there is a largest projection, denoted by g', in the set $\{p \in P : \; g \in C(p) \text{ and } J_p(g) = 0\}$ ([12; Definition 6.1]). Thus, if G is a Rickart CB-group, then the mapping $'\colon G \to P$, called the *Rickart mapping*, has the following property:

$$(\forall g \in G)(\forall p \in P)\big(p \le g' \iff \big(g \in C(p) \;\&\; J_p(g) = 0\big)\big).$$

EXAMPLE 2.4. In Example 2.1, $G(X, \mathcal{B}, \mathbb{R})$ is a Rickart CB-group with $g \mapsto g' := \chi_{g^{-1}(0)}$ as the Rickart mapping.[5]

As a consequence of [8; Theorem 6.3], the orthomodular poset P of projections in a Rickart CB-group is an orthomodular lattice ([21]) with $p \mapsto p' = u - p$ as the orthocomplementation.

DEFINITION 2.5. An *RC-group* is a Rickart CB-group with the general comparability property, and an *RCℓ-group* is an RC-group that is also an ℓ-group ([16; Definition 4.3 (iv)]).

As per Examples 2.3 and 2.4, our prototype $G(X, \mathcal{B}, \mathbb{R})$ is a direct RCℓ-group.

3. Examples of CB-groups and RC-groups

The following additional examples of CB-groups and RC-groups will help to fix ideas and indicate the scope of our theory.

[5] Note that $g \mapsto g'$ is the natural extension to $G(X, \mathcal{B}, \mathbb{R})$ of the Heyting negation mapping $e \mapsto e'$ on $E(X, \mathcal{B}, \mathbb{R})$.

EXAMPLE 3.1. Let $\mathcal{A}$ be a unital C*-algebra with unity element 1 and let $G(\mathcal{A}) := \{a \in \mathcal{A} : a = a^*\}$ be the additive subgroup of $\mathcal{A}$ consisting of all self-adjoint elements. Then $G(\mathcal{A})$ is a unital group with positive cone $G(\mathcal{A})^+ = \{aa^* : a \in \mathcal{A}\}$ and unit 1. Let $P(\mathcal{A}) := \{p \in G(\mathcal{A}) : p = p^2\}$ and for each $p \in P(\mathcal{A})$ define $J_p \colon G(\mathcal{A}) \to G(\mathcal{A})$ by $J_p(a) := pap$ for all $a \in G(\mathcal{A})$. Then J_p is a compression on $G(\mathcal{A})$ and, if J is any retraction on $G(\mathcal{A})$ and $p = J(1)$, then $p \in P(\mathcal{A})$ and $J = J_p$ ([9; Corollary 4.7]). The unital group $G(\mathcal{A})$ is a proper CB-group with compression base $(J_p)_{p \in P(\mathcal{A})}$. If $a \in G(\mathcal{A})$ and $p \in P(\mathcal{A})$, then $a \in C(p) \iff ap = pa$.

If $\mathcal{A}$ is a *Rickart C*-algebra*, i.e., if the right annihilating ideal of every element in $\mathcal{A}$ is a principal right ideal generated by a projection, then $G(\mathcal{A})$ is a Rickart CB-group. If $\mathcal{A}$ is an *AW*-algebra*, i.e. $\mathcal{A}$ is a Rickart C*-algebra and $P(\mathcal{A})$ is a complete (orthomodular) lattice, then $G(\mathcal{A})$ is an RC-group. In particular, if $\mathcal{A}$ is a von Neumann algebra, then $G(\mathcal{A})$ is an RC-group.

The following example generalizes the unital group $G(X, \mathcal{B}, \mathbb{Z})$ of Definition 1.3 in that $\mathcal{B}$ need only be a field of sets, not necessarily a σ-field.

EXAMPLE 3.2. Let $\mathcal{B}$ be a field of subsets of a nonempty set X and let $G(X, \mathcal{B}, \mathbb{Z})$ be the partially ordered group under pointwise partial order and pointwise addition consisting of all bounded functions $f \colon X \to \mathbb{Z}$ such that $f^{-1}(z) \in \mathcal{B}$ for every $z \in \mathbb{Z}$. Then $G(X, \mathcal{B}, \mathbb{Z})$ is a unital ℓ-group with the constant function 1 as the unit. Let $P(X, \mathcal{B}, \mathbb{Z})$ be the subset of $G(X, \mathcal{B}, \mathbb{Z})$ consisting of the characteristic set functions χ_M of sets $M \in \mathcal{B}$. If $p \in P(X, \mathbb{R})$, define $J_p \colon G(X, \mathcal{B}, \mathbb{Z}) \to G(X, \mathcal{B}, \mathbb{Z})$ by $J_p(f) := pf$ (pointwise product) for all $f \in G(X, \mathcal{B}, \mathbb{Z})$. Then J_p is a compression with focus p on $G(X, \mathcal{B}, \mathbb{Z})$, every retraction J on $G(X, \mathcal{B}, \mathbb{Z})$ has the form $J = J_p$ where $p = J(1)$, and $G(X, \mathcal{B}, \mathbb{Z})$ is a proper RCℓ-group with compression base $(J_p)_{p \in P(X, \mathcal{B}, \mathbb{Z})}$.

In Example 3.2, $P(X, \mathcal{B}, \mathbb{Z})$ is isomorphic to $\mathcal{B}$ as a Boolean algebra of sets. By the Stone representation theorem, every Boolean algebra is isomorphic to a Boolean algebra of sets $\mathcal{B}$; hence, *every Boolean algebra can be realized as the Boolean algebra of projections in a proper RCℓ-group.*

EXAMPLE 3.3. A partially ordered abelian group H has the *Riesz interpolation property*, or is an *interpolation group* iff, whenever $a, b, c, d \in H$ with $a, b \le c, d$, there is an element $t \in H$ with $a, b \le t \le c, d$ ([17; Chapter 2]). Every abelian ℓ-group is an interpolation group. If H is an interpolation group and u is an order unit in H, then u is automatically generative, hence H is a unital group with unit u. If H is a unital interpolation group, then every retraction on H is direct, hence there is one and only one way to organize H into a proper CB-group, namely by choosing as the compression base all direct contractions on H, indexed by their own foci as in Example 2.2. By [8; Theorem 4.9], a proper CB-group with the interpolation and general comparability properties is a CBℓ-group.

The unit interval E in a unital ℓ-group G forms an MV-algebra, and, conversely, by Mundici's theorem ([23]), every MV-algebra can be realized as the unit interval E in a unital ℓ-group G that is uniquely determined by E up to an isomorphism of unital groups; moreover, as in Example 3.3, G can be organized into a proper CBℓ-group in only one way, namely as a direct CB-group. If E is the unit interval in a proper CBℓ-group, then E is a Heyting algebra iff G is an RCℓ-group ([14; Theorem 8.7]).

4. Properties of CB-groups and RC-groups

As an effort to keep this article somewhat self contained, we assemble in this section some known properties of CB-groups and RC-groups that will be needed for the proof of our main theorems in Section 5. We maintain our standing assumption that G is a proper CB-group with unit u, unit interval E, and compression base $(J_p)_{p\in P}$.

The development in [8], [10], [12] pertains to a restricted class of CB-groups, namely the so-called *compressible groups*, i.e., CB-groups in which every retraction is a compression and every compression belongs to the compression base. The CB-groups $G(X,\mathcal{B},\mathbb{R})$ in Example 2.1, $G(\mathcal{A})$ in Example 3.1, $G(X,\mathcal{B},\mathbb{Z})$ in Example 3.2, and any proper interpolation CB-group as in Example 3.3 are all compressible groups. The more general notion of a CB-group was introduced later in [11]. In [13] it was shown that the proofs of all of the basic properties developed earlier in [8], [10], [12] for compressible groups carry over almost verbatim for CB-groups. Therefore, in the present article, we can and do make free use of results from [8], [10], [12].

Lemma 4.1. *Let $g\in G$ and $p\in P$. Then:*

(i) $u, p, u-p \in C(p) = C(u-p)$.

(ii) $g\in C(p) \iff u-g\in C(p)$.

(iii) $u, g \in CPC(g) = CPC(u-g)$.

Proof.

(i) $J_p(u)=p$ and $J_{u-p}(u)=u-p$, whence $u=p+(u-p)=J_p(u)+J_{u-p}(u)$, i.e., $u\in C(p)$. Obviously, $p, u-p\in C(p)=C(u-p)$.

(ii) By (i), $u\in C(p)$. Hence (ii) follows from the fact that $C(p)$ is a subgroup of G.

(iii) By (i), $u\in C(p)$ for all $p\in P$, hence $u\in CPC(g)$. That $CPC(g)=CPC(u-g)$ follows from (ii), and $g\in CPC(g)$ is obvious. □

Lemma 4.2. *If $p, q \in P$, then $p \in C(q) \iff J_p \circ J_q = J_q \circ J_p \iff q \in C(p)$.*

Proof. See [8; Theorem 5.4]. □

In view of Lemma 4.2, for projections $p, q \in P$, we usually write the condition $p \in C(q)$ in the more symmetric form pCq. By [8; Theorem 5.4], pCq iff p and q are Mackey compatible elements in the orthomodular poset P ([24]).

Lemma 4.3. *Let $p, q \in P$. Then:*

(i) *If pCq, then the infimum $p \wedge q$ of p and q exists in P and $J_p(q) = J_q(p) = p \wedge q$.*
(ii) *If $p \leq q$, then $q - p \in P$, pCq, and $J_p \circ J_q = J_q \circ J_p = J_p$.*
(iii) *If pCq, then $C(p) \cap C(q) \subseteq C(p \wedge q)$.*
(iv) *If pCq, then $p \wedge (u - q) = 0 \implies p \leq q$.*

Proof. For (i), see [8; Corollary 5.6]. For (ii), see [8; Corollaries 5.2(iv), 5.5]. For (iii), see [12; Corollary 2.4]. Assume the hypotheses of (iv). Then $pC(u-q)$, whence by (i), $0 = p \wedge (u-q) = J_p(u-q) = p - J_p(q) = p - p \wedge q$, and it follows that $p = p \wedge q \leq q$. □

Lemma 4.4. *Suppose that G is a Rickart CB-group with Rickart mapping $g \mapsto g'$. Then:*

(i) $p \in P \implies p' = u - p$ *and* $p'' := (p')' = p$.
(ii) *If $g, h \in G$ with $0 \leq g \leq h$, then $h' \leq g'$.*
(iii) *If $e \in E$, then e'' is the smallest projection in $\{p \in P : \ e \leq p\}$.*

Proof. For (i), (ii), and (iii) see [12; Lemma 6.2: (iii), (vi), (viii)], respectively. □

By Lemma 4.4(i), if $p \in P$, then $p' = u - p$, whence, if $g \in G$, then $g' \in P$, so $g'' = u - g'$ and $g''' = u - (u - g') = g'$. Also, $p = p''$ and $J_p(p') = J_p(u - p) = J_p(u) - J_p(p) = p - p = 0$. We make routine use of these facts in our proofs in Section 5.

Lemma 4.5. *Let G be an RC-group with Rickart mapping $g \mapsto g'$ and let $a \in G$. Then:*

(i) $(a^+)'' \in P^{\pm}(a)$.
(ii) $a^+ \in CPC(a)$.
(iii) $(a^+)' = 0 \implies 0 \leq a$.
(iv) *If $p \in P$, $a \in C(p)$, and $J_{u-p}(a) \leq 0 \leq J_p(a)$, then $a^+ = J_p(a)$.*

Proof. For (i), see [10; Theorem 3.1]. For (ii), see [12; Lemma 4.3(vii)]. For (iii), see [12; Theorem 6.5(v)]. For (iv), see [12; Lemma 4.2]. □

By Lemma 1.4, the sharp elements p in the prototype $E(X, \mathcal{B}, \mathbb{R})$ are characterized by the condition $p \wedge p^{\perp} = 0$, i.e., the only effect $e \in E(X, \mathcal{B}, \mathbb{R})$ such that $e \leq p, 1-p$ is $e = 0$. This notion is generalized as follows ([18]).

DEFINITION 4.6. An element $s \in E$ is *sharp* iff

$$\big(\forall e \in E\big)\big(e \le s, u-s \implies e = 0\big).$$

LEMMA 4.7. *Every projection $p \in P$ is sharp. Moreover, if G is an RC-group, then P is precisely the set of sharp elements in G.*

Proof. That every $p \in P$ is sharp follows from [9; Lemma 2.3(ii)], and the converse for an RC-group follows from [12; Theorem 5.7]. □

5. Decomposition into sharp and blunt elements

Standing assumption. *In this section, G is an RC-group with unit u, unit interval E, compression base $\big(J_p\big)_{p\in P}$, and Rickart mapping $g \mapsto g'$.*

The following definition generalizes Definition 1.2.

DEFINITION 5.1. An element $b \in G^+$ is *blunt* iff the only projection $p \in P$ such that $p \le b$ is $p = 0$.

LEMMA 5.2. *Let $p, q \in P$, let $b, c \in E$ and suppose that b is blunt. Then:*

(i) *If $c \le b$, then c is blunt.*

(ii) $b \le q \implies (q-b)' = q'$.

(iii) $b \in C(p) \implies \big((p'-b)^+\big)' = p$.

Proof. Part (i) is obvious.

(ii) Assume that $b \le q$. Then $0 \le q - b \le q \le u$, so $q - b \in E$, whence $q - b \le (q-b)''$ by Lemma 4.4(iii). In particular, $q - (q-b)'' \le b$. Also, by Lemma 4.4(ii), $0 \le q - b \le q$ implies that $q' \le (q-b)'$, which in turn implies that $(q-b)'' \le q'' = q$. Therefore, by Lemma 4.3(ii), $q - (q-b)'' \in P$. Since b is blunt, it follows that $q - (q-b)'' = 0$, i.e. $(q-b)'' = q$, and therefore, $(q-b)' = q'$.

(iii) We have $J_p(p'-b) = J_p(p') - J_p(b) = -J_p(b) \le 0$. Also, as $b \le u$, we have $J_{p'}(b) \le J_{p'}(u) = p'$, whence $J_{p'}(p'-b) = p' - J_{p'}(b) \ge 0$. Since $b \in C(p)$ and $p' \in C(p)$, it follows that $p' - b \in C(p)$, so Lemma 4.5(iv) implies that

$$(p'-b)^+ = J_{p'}(p'-b) = p' - J_{p'}(b).$$

Now $b \in C(p)$ implies that $0 \le J_{p'}(b) \le J_p(b) + J_{p'}(b) = b \le u$, hence $J_{p'}(b)$ is blunt by (i). Therefore, by (ii) with $q := p'$ and b replaced by $J_{p'}(b)$ we have $\big((p'-b)^+\big)' = \big(p' - J_{p'}(b)\big)' = p'' = p$. □

DEFINITION 5.3. Define $\kappa\colon G \to P$ by $\kappa g := \big((u-g)^+\big)'$ for all $g \in G$.

THEOREM 5.4. *Let $g \in G^+$. Then $\kappa g \in CPC(g)$, $g \in C(\kappa g)$, and κg is the largest projection in the set $\{p \in P:\ p \le g \in C(p)\}$.*

Proof. Let $k := \kappa g = \big((u-g)^+\big)'$, so that $k' = \big((u-g)^+\big)'' \in P^{\pm}(u-g)$ by Lemma 4.5(i). Therefore,

$$u - k = k' \in CPC(u-g)\,, \qquad u - g \in C(k') = C(u-k)\,, \tag{1}$$

and

$$k - J_k(g) = J_k(u-g) \le 0 \le J_{k'}(u-g) = k' - J_{k'}(g) = (u-g)^+\,. \tag{2}$$

By (1) and Lemma 4.1, $\kappa g = k \in CPC(g)$ and $g \in C(k) = C(\kappa g)$. As a consequence of (2),

$$k \le J_k(g) \qquad \text{and} \qquad J_{k'}(g) \le k'\,. \tag{3}$$

As $g \in G^+$, we have $J_{k'}(g) \in G^+$, and it follows from $g \in C(k)$ and (3) that $\kappa g = k \le J_k(g) \le J_k(g) + J_{k'}(g) = g$. Thus,

$$\kappa g \in \{p \in P:\ p \le g \in C(p)\}\,.$$

Now suppose $p \in P$ with $p \le g \in C(p)$. We have to show that $p \le k$. As $g \in C(p)$ and $k \in CPC(g)$, it follows that kCp, hence that $k'Cp$. Put $r := J_{k'}(p)$. By Lemma 4.3(i), $r = p \wedge k'$. Then, as $p \le g$,

$$r = J_{k'}(p) \le J_{k'}(g)\,,$$

whence, as $r \le k'$, Lemma 4.3(ii) implies that

$$r = J_r(r) \le J_r\big(J_{k'}(g)\big) = J_r(g). \tag{4}$$

Thus, by (2) and (4)

$$0 \le J_r\big((u-g)^+\big) = J_r\big(k' - J_{k'}(g)\big) = r - J_r(g) \le 0\,,$$

and it follows that

$$J_r\big((u-g)^+\big) = 0\,. \tag{5}$$

We have $g \in C(p)$, $g \in C(k) = C(k')$, and pCk', whence $g \in C(p \wedge k') = C(r)$ by Lemma 4.3(iii). Also, by Lemma 4.5(ii) and Lemma 4.1(iii),

$$(u-g)^+ \in CPC(u-g) = CPC(g)\,,$$

and therefore $(u-g)^+ \in C(r)$. Consequently, by (5), $r \le \big((u-g)^+\big)' = k$. But, $r \le k'$, therefore $p \wedge k' = r = 0$, and in view of the fact that pCk, we have $p \le k$ by Lemma 4.3(iv). □

COROLLARY 5.5. *If $g \in G^+$, then $g - \kappa g \in G^+$, $\kappa(g - \kappa g) \in CPC(g)$, $g \in C\big(\kappa(g - \kappa g)\big)$, and $\kappa(g - \kappa g) \leq \kappa g$.*

Proof. Let $k := \kappa g$. By Theorem 5.4, k is the largest projection in the set $\{p \in P :\ p \leq g \in C(p)\}$. In particular, $k \leq g$, so $g - \kappa g = g - k \in G^+$. Define $\bar{k} := \kappa(g - \kappa g) = \kappa(g - k)$. By Theorem 5.4 with g replaced by $g - k$,

$$\bar{k} \in CPC(g - k)\,, \quad g - k \in C(\bar{k}) \quad \text{and} \quad \bar{k} \leq g - k \leq g\,.$$

As $g, k \in C(k)$, we have $g - k \in C(k)$, whence $\bar{k} \in C(k)$. Thus, $k \in C(\bar{k})$, and it follows that $g = (g - k) + k \in C(\bar{k})$. Consequently, $\bar{k} \leq g \in C(\bar{k})$, and it follows that $\bar{k} \leq k$. To prove that $\bar{k} \in CPC(g)$, suppose that $r \in P$ and $g \in C(r)$. Since $k \in CPC(g)$, we have $k \in C(r)$, whence $g - k \in C(r)$. But, $\bar{k} \in CPC(g - k)$, and therefore $\bar{k} \in C(r)$. □

THEOREM 5.6. *Let $b \in G^+$. Then the following conditions are mutually equivalent:*

(i) *b is blunt.*
(ii) *$\kappa b = 0$.*
(iii) *$b \in E$ and b is blunt.*

Proof. That (i) $\Longrightarrow$ (ii) follows from Theorem 5.4 with g replaced by b.

(ii) $\Longrightarrow$ (iii). If $\kappa b = \big((u - b)^+\big)' = 0$, then $0 \leq u - b$ by Lemma 4.5(iii), whence $0 \leq b \leq u$, i.e., $b \in E$. To prove that b is blunt, suppose $p \in P$ and $p \leq b$. Then $p \leq b \leq u$, so $p = J_p(p) \leq J_p(b) \leq J_p(u) = p$, and we have $J_p(b) = p$. Also, $0 \leq u - b \leq u - p \in P$, whence $u - b = J_{u-p}(u - b) = J_{u-p}(u) - J_{u-p}(b) = u - p - J_{u-p}(b)$, and it follows that $J_{u-p}(b) = b - p$. Consequently, $J_p(b) + J_{u-p}(b) = p + (b - p) = p$, i.e., $b \in C(p)$, and we have $p \leq b \in C(p)$. Therefore by Theorem 5.4, $p \leq \kappa b = 0$, so $p = 0$ and b is blunt.

Obviously, (iii) $\Longrightarrow$ (i). □

LEMMA 5.7. *Let $g \in G$ and let $\mathbb{N} := \{1, 2, 3, \ldots\}$.*

(i) *If $p_1 \geq p_2 \geq \cdots$ is a descending sequence of projections such that $\sum_{i=1}^{n} p_i \leq g$ for all $n \in \mathbb{N}$, then there exists $M \in \mathbb{N}$ such that $p_n = 0$ for all $n \geq M$.*
(ii) *If $p \in P$ and $np \leq g$ for every $n \in \mathbb{N}$, then $p = 0$.*

Proof.

(i) Since u is an order unit in G, there is a positive integer N such that $g \leq Nu$. Let $M := N+1$. Then $Mp_M \leq \sum_{i=1}^{M} p_i \leq g \leq Nu$, whence $Np_M + p_M = Mp_M = MJ_{p_M}(p_M) \leq NJ_{p_M}(u) = Np_M$, and it follows that $p_M = 0$.

(ii) In (i), put $p_i := p$ for all $i \in \mathbb{N}$. □

THEOREM 5.8. *Let $g \in G^+$. Then, either g is a blunt element of E, or there is a blunt element b of E and a finite descending sequence $p_1 \geq p_2 \geq \cdots \geq p_n$ of nonzero projections in $P \cap CPC(g)$ such that $g = p_1 + p_2 + \cdots + p_n + b$ and $g \in \bigcap_{i=1}^{n} C(p_i)$.*

Proof. Assume that g is not a blunt element of E. By recursion, define sequences $g_1, g_2, \ldots \in G$ and $p_1, p_2, \ldots \in P$ by

$$g_1 := g \in G^+, \qquad p_1 := \kappa g = \kappa g_1,$$

and for all $n = 1, 2, \ldots,$

$$g_{n+1} := g_n - p_n \qquad \text{and} \qquad p_{n+1} := \kappa g_{n+1}.$$

As $g = g_1$ is not blunt, $p_1 = \kappa(g) \neq 0$ by Theorem 5.6. We shall prove by induction that, for every $n = 1, 2, \ldots,$

$$g_1, g_2, \ldots, g_n \in G^+ \cap CPC(g), \tag{1}$$

$$p_1 \geq p_2 \geq \cdots \geq p_n, \tag{2}$$

$$p_1, p_2, \ldots, p_n \in P \cap CPC(g), \tag{3}$$

$$g = \sum_{i=1}^{n} p_i + g_{n+1}, \tag{4}$$

and

$$g \in \bigcap_{i=1}^{n} C(p_i). \tag{5}$$

By hypothesis, $g_1 = g \in G^+$, and it is clear that $g_1 = g \in CPC(g)$ and $g = p_1 + g_2$. By Theorem 5.4, we have

$$p_1 = \kappa g \in CPC(g) \qquad \text{and} \qquad g \in C(p_1);$$

hence (1)–(5) hold for $n = 1$. As our inductive hypothesis, we assume that (1)-(5) hold for a positive integer n. As $g_n \in G^+$, we can and do replace g in Corollary 5.5 by g_n and conclude that

$$g_{n+1} = g_n - p_n = g_n - \kappa g_n \in G^+, \qquad p_{n+1} = \kappa g_{n+1} \in CPC(g_n),$$
$$g_n \in C(p_{n+1}) \qquad \text{and} \qquad p_{n+1} \leq p_n.$$

As $g_n \in CPC(g)$, $p_n \in CPC(g)$, and $CPC(g)$ is a subgroup of G, it follows that

$$g_{n+1} = g_n - p_n \in CPC(g).$$

As $p_{n+1} \in CPC(g_n)$ and $g_n \in CPC(g)$, it follows that

$$p_{n+1} \in CPC(g).$$

By (4),

$$g = \left(\sum_{i=1}^{n} p_i\right) + p_{n+1} + g_{n+1} - p_{n+1} = \sum_{i=1}^{n+1} p_i + g_{n+2}\,.$$

We have $g_n \in C(p_{n+1})$ and, since $p_{n+1} \le p_n \le \cdots \le p_1$, we also have $p_i \in C(p_{n+1})$ for $i = 1, 2, \dots, n$. By (4), $g = \sum\limits_{i=1}^{n} p_i + g_n - p_n$, and it follows that $g \in C(p_{n+1})$. Therefore,

$$g \in \bigcap_{i=1}^{n+1} C(p_i)\,,$$

and our inductive argument is complete.

As $g_{n+1} \in G^+$, it follows from (4) that $\sum\limits_{i=1}^{n} p_i \le g$ for all $n = 1, 2, \dots$, whence by Lemma 5.7(i), there is a smallest positive integer M such that $p_M = 0$. Moreover, $p_1 \ne 0$ implies that $M > 1$. Thus, with $n := M - 1$ and $b := g_{n+1}$, we have a descending sequence $p_1 \ge p_2 \ge \cdots \ge p_n$ of nonzero projections in $P \cap CPC(g)$ such that $g = p_1 + p_2 + \cdots + p_n + b$ and $g \in \bigcap\limits_{i=1}^{n} C(p_i)$. Moreover, $\kappa b = \kappa g_{n+1} = p_{n+1} = p_M = 0$, so b is blunt by Lemma 5.6. □

If $g = p_1 + p_2 + \cdots + p_n + b$ as in Theorem 5.8, then, as $g \in \bigcap\limits_{i=1}^{n} C(p_i)$ and $p_1 + p_2 + \cdots + p_n \in \bigcap\limits_{i=1}^{n} C(p_i)$, it follows that $b \in \bigcap\limits_{i=1}^{n} C(p_i)$. Therefore, by the following theorem, the decomposition $g = p_1 + p_2 + \cdots + p_n + b$ is uniquely determined by g.

THEOREM 5.9. *Suppose that $p_1 \ge p_2 \ge \cdots \ge p_n$ is a descending sequence of nonzero projections in P, b is a blunt element of E, $b \in \bigcap\limits_{i=1}^{n} C(p_i)$, and $g = p_1 + p_2 + \cdots + p_n + b$. Then, $p_1 = \kappa g$, and for $m = 2, \dots, n$, $p_m = \kappa\Big(g - \sum\limits_{i=1}^{m-1} p_i\Big)$.*

P r o o f. Define $g_1 := g$ and by recursion for $m = 1, 2, 3 \dots, n-1$,

$$g_{m+1} := g_m - p_m = \left(\sum_{i=m+1}^{n} p_i\right) + b = g - \sum_{i=1}^{m} p_i\,.$$

Since $p_i \le p_1$ for $1 < i \le n$, we have $0 \le J_{p_1'}(p_i) \le J_{p_1'}(p_1) = 0$. Also, $b \in E$ implies that $J_{p_1'}(b) \le p_1'$, and it follows that

$$J_{p_1'}(u - g_1) = J_{p_1'}(p_1' - p_2 - \cdots - p_n - b) = p_1' - J_{p_1'}(b) \ge 0\,. \tag{1}$$

Also, $J_{p_1}(p_1') = 0$, whence

$$J_{p_1}(u - g_1) = -J_{p_1}(p_2 + \cdots + p_n + b) \le 0\,. \tag{2}$$

Moreover, $p_1', p_2, \dots, p_n, b \in C(p_1)$, so $u - g_1 \in C(p_1)$, and it follows from (1), (2) and Lemma 4.5(iv) that

$$(u - g_1)^+ = J_{p_1'}(u - g_1) = p_1' - J_{p_1'}(b). \tag{3}$$

As $0 \le b \in C(p_1)$, we have $J_{p_1'}(b) \le J_{p_1'}(b) + J_{p_1}(b) = b$, hence $J_{p_1'}(b)$ is a blunt element of E. Therefore, by (3) and Lemma 5.2(iii),

$$\kappa g_1 = \left((u - g_1)^+\right)' = (p_1' - J_{p_1'}(b))' = p_1 .$$

The same argument applied to $g_2 = p_2 + \cdots + p_n + b$ shows that $\kappa g_2 = p_2$, and continuing in this way by induction, we conclude that $\kappa g_i = p_i$ for $i = 1, 2, \dots, n$. □

Recall that a partially ordered abelian group H is *archimedean* iff, whenever $g, h \in H$ and $nh \le g$ for all $n \in \mathbb{N} = \{1, 2, \dots\}$, it follows that $h \le 0$ ([17; p. 20]). By [12; Lemma 3.5], the RC-group G is archimedean iff, for every $g, h \in G^+$, the condition $nh \le g$ for all $n \in \mathbb{N}$ implies that $h = 0$.[6] Using some of the results developed above, we now show that it is only necessary to check the latter condition for the case in which $h \in E$ and $g = u$.

THEOREM 5.10. *The RC-group G is archimedean iff, for every $h \in E$, the condition $nh \le u$ for all $n \in \mathbb{N}$ implies that $h = 0$.*

P r o o f. If G is archimedean, $h \in E$, and $nh \le u$ for all $n \in \mathbb{N}$, then clearly $h = 0$. Conversely, assume that $h \in E$ with $nh \le u$ for all $n \in \mathbb{N}$ implies $h = 0$. Let $g, h \in G^+$ with $nh \le g$ for all $n \in \mathbb{N}$. If $p \in P$ and $p \le h$, then $np \le g$ for all $n \in \mathbb{N}$, whence $p = 0$ by Lemma 5.7(ii), and it follows that h is blunt. Thus, by Theorem 5.6, $h \in E$. As u is an order unit in G, there exists $N \in \mathbb{N}$ such that $g \le Nu$, and we have $mh \le g \le Nu$ for all $m \in \mathbb{N}$. Letting $m = nN$, we find that $Nnh \le Nu$ for all $n \in \mathbb{N}$. Therefore, by [8; Lemma 4.8(ii)], $nh \le u$ for all $n \in \mathbb{N}$, and it follows from our hypothesis that $h = 0$. □

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[6] See [10; Example 4.1] for an example of an RC-group that is not archimedean.

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