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Dedicated to Academician Štefan Schwarz on the occasion of his 80th birthday

REPRESENTATION OF OBSERVABLES IN QUANTUM MECHANICAL MODELS

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ABSTRACT. A quantum mechanical model suggested by J. Pykacz is developed. In particular, the notion of an observable is defined and a possibility how to build the probability theory in this model is shown.

There are several algebraic approaches to quantum mechanics. The most known is the quantum logics (orthomodular posets) theory (for references see e.g. [5]). In the last five years also a theory of fuzzy quantum posets has been developed (a review is contained in [2]). Another approach very closed to the latter was suggested following recently by J. Pykacz [6]. The aim of the present article is to propose the notion of an observable (in the framework of the theory) and to represent observables by random variables. Of course, the results are much more general and include also quantum logics (orthocomplemented posets) as a special case.

Recall that J. Pykacz starts with a family $F \subset \langle 0,1\rangle^X$ satisfying the following conditions:

(i) $1_X \in F$,

(ii) $f \in F \implies 1 - f \in F$,

(iii) $f_n \in F \ (n = 1, 2, \dots)\ f_n \le 1 - f_m \ (n \ne m) \implies \sum_{n=1}^{\infty} f_n \in F$.

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It is interesting that these axioms are very similar to the axioms of F-quantum posets. The only difference is in the third condition, when $\sup_n f_n$ is used instead of $\sum f_n$. Of course, in our concept only the assumption $1_X \in F$ will be used. Other properties of fuzzy quantum logics will be manifested by means of properties of considered observables. Namely, while fuzzy quantum posets are based on the Zadeh connective $f \vee g = \max(f, g)$, the fuzzy quantum logics are based on the Giles one: $f \vee g = \min(f + g, 1)$.

1. Definition. *Let P be a partially ordered set with the greatest element 1 and two binary operations $+$ and $-$. By an observable with values in P we mean any mapping $x\colon \mathcal{B}(R) \to P$ ($\mathcal{B}(R)$ is the family of all Borel subsets of R) satisfying the following conditions:*

(i) $x(R) = 1$;
(ii) *if* $A, B \in \mathcal{B}(R)$, $A \cap B = \emptyset$, *then* $x(A \cup B) = x(A) + x(B)$;
(iii) *if* $A \subset B$, $A, B \in \mathcal{B}(R)$, *then* $x(B \setminus A) = x(B) - x(A)$;
(iv) *if* $A_n \in \mathcal{B}(R)$ $(n = 1, 2 \dots)$ $A_n \nearrow A$, *then* $x(A_n) \nearrow x(A)$

$$(i.e.\ x(A_n) \leqq x(A_{n+1})\ (n = 1, 2, \dots)\ and\ x(A) = \bigvee_{n=1}^{\infty} x(A_n)).$$

2. Examples. Every Pykacz fuzzy quantum logic satisfies all the conditions stated above, if $f + g(x) = \min\bigl(f(x) + g(x), 1\bigr)$ $f - g(x) = f(x) - \min\bigl(f(x), g(x)\bigr)$. Especially, if Q is a q-σ-algebra of subsets of X (i.e. $A \in Q \implies X \setminus A \in Q$ and $A_n \in Q$ $(n = 1, 2, \dots)$, $A_n \cap A_m = \emptyset$ $(m \neq n \implies \bigcup_{n=1}^{\infty} A_n \in Q)$, then $F = \{\chi_A;\ A \in Q\}$ is a fuzzy quantum logic. If $\zeta\colon X \to R$ is a Q-measurable mapping, then the function $x\colon \mathcal{B}(R) \to F$, defined by $x(A) = \chi_{\zeta^{-1}(A)}$, is an observable.

More generally, if L is a Riesz space (or more generally an l-group) and $u \in L$ is a positive element, then $P = \{t \in L;\ 0 \le t \le u\}$ satisfies the stated axioms.

Another example is an orthomodular lattice P ([5]), where $u + v = u \vee v$, $u - v = u \wedge v'$.

3. Definition. *An observable $y\colon \mathcal{B}(R) \to P$ is called weakly Boolean if the following implication holds:*

(v) $y(A) \le y(B) \implies y(A \cup B) = y(B)$, $y(A \cap B) = y(A)$.

It is called Boolean, if

(vi) $y(A \cup B) = y(A) \vee y(B)$ *for every* $A, B \in \mathcal{B}(R)$.

If y is Boolean, then it is weakly Boolean. Namely, if $y(A) \leqq y(B)$, then

$$y(A \cup B) = y(A) \vee y(B) = y(B),$$
$$y(A \cap B) = 1 - y(A^c \cup B^c) = 1 - y(A^c) \vee y(B^c) = 1 - y(A^c) = y(A).$$

4. THEOREM. *Let $x, y\colon \mathcal{B}(R) \to P$ be observables, y be weakly Boolean, $x(\mathcal{B}(R)) \subset y(\mathcal{B}(R))$. Then there exists a Borel measurable mapping $T\colon R \to R$ such that*

$$x(A) = y(T^{-1}(A))$$

for every $A \in \mathcal{B}(R)$.

P r o o f. Let Q be the set of all rationals. For every $r \in Q$ there is $A_r \in \mathcal{B}(R)$ such that

$$x((-\infty, r)) = y(A_r).$$

The condition (v) implies

$$y(A) \leq y(B) \implies \exists C \supset A, \quad y(B) = y(C).$$

Indeed, it suffices to put $C = A \cup B$. Similarly

$$y(A) \leq y(B) \implies \exists C \subset B, \quad y(A) = y(C).$$

Finally

$$y(A) \leq y(B) \leq y(C), \ \ A \subset C \implies \exists D, \ \ A \subset D \subset C, \quad y(B) = y(D).$$

In this case it suffices to put $D = C \cap (A \cup B) = (A \cap C) \cup (B \cap C) = A \cup (B \cap C)$, since then $A \subset D \subset C$ and

$$y(B \cap C) = y(B) \geq y(A),$$

hence

$$y(D) = y(A \cup (B \cap C)) = y(B \cap C) = y(B).$$

Therefore we can construct by induction a sequence $(B_r)_{r \in Q}$ such that

$$x((-\infty, r)) = y(B_r)$$

and

$$r < s, \ \ r, s \in Q \implies B_r \subset B_s \cdot$$

Now put

$$C_r = B_r \setminus \bigcap_{s\in Q} B_s \,.$$

Then by (iii) and (iv) and the equality $y\Big(\bigcap_{s\in Q} B_s\Big) = y\Big(\bigcap_{n=1}^{\infty} B_{-n}\Big) = \bigwedge_{n=1}^{\infty} y(B_{-n})$ $= \bigwedge_{s\in Q} y(B_s)$, we obtain

$$\begin{aligned} y(C_r) &= y(B_r) - y\Big(\bigcap_{s\in Q} B_s\Big) = y(B_r) - \bigwedge_{s\in Q} y(B_s) \\ &= x\big((-\infty, r)\big) - \bigwedge_{s\in Q} x\big((-\infty, s)\big) = x\big((-\infty, r)\big) - x\Big(\bigcap_{s\in Q} (-\infty, s)\Big) \\ &= x\big((-\infty, r)\big) - x(\emptyset) = x\big((-\infty, r)\big) \,. \end{aligned}$$

We have constructed a sequence $(C_r)_{r\in Q}$ such that

$$\begin{aligned} &x\big((-\infty, r)\big) = y(C_r)\,, \quad r \in Q\,, \\ &\bigcap_{r\in Q} C_r = \emptyset\,, \\ &r < s \implies C_r \subset C_s \,. \end{aligned}$$

Because of $\bigcap_{r\in Q} C_r = \emptyset$, the set

$$\{r \in Q\,;\ t \in C_r\}$$

is lower bounded and therefore there exists

$$T(t) = \inf\{r \in Q\,;\ t \in C_r\} \in R \,.$$

It is easy to see that

$$T^{-1}\big((-\infty, r)\big) = \bigcup \{C_s\,;\ s \in Q\,,\ s < r\} \in \mathcal{B}(R) \,.$$

Therefore

$$\begin{aligned} y\big(T^{-1}((-\infty, r))\big) &= y\Big(\bigcup \{C_s\,;\ s \in Q\,,\ s < r\}\Big) \\ &= \bigvee \{y(C_s)\,;\ s \in Q\,,\ s < r\} \\ &= \bigvee \{x\big((-\infty, s)\big)\,;\ s \in Q\,,\ s < r\} \end{aligned}$$

It follows that T is $\mathcal{B}(R)$-measurable. Moreover

$$\mathcal{K} = \{A \in \mathcal{B}(R);\ y(T^{-1}(A)) = x(A)\}$$

contains (by the condition (iii)) the set $\{\langle a, b) :\ a, b \in Q\}$ and then also (by the condition (ii)) the set $\mathcal{R} = \left\{\bigcup_{i=1}^{n} \langle a_i, b_i);\ n \in N,\ a_i, b_i \in Q\right\}$ which is a ring. Further $\mathcal{K}$ is a monotone family, therefore

$$\mathcal{K} \supset \mathcal{M}(\mathcal{R}) = \sigma(\mathcal{R}) = \mathcal{B}(R).$$

We obtained that every $A \in \mathcal{B}(R)$ belongs to $\mathcal{K}$, hence $y(T^{-1}(A)) = x(A)$.

Theorem 4 is an analog of similar theorems known in the theory of quantum logics (see e.g. [1]) as well as in the theory of fuzzy quantum posets ([3]). Now we show two possible applications.

5. Definition. *We say that two observables x_1, x_2 have a joint observable h, if there exists a mapping $h\colon \mathcal{B}(R^2) \to P$ satisfying the following conditions:*

(i) $h(R^2) = 1$;
(ii) *if* $A, B \in \mathcal{B}(R^2)$, $A \cap B = \emptyset$, *then* $h(A \cup B) = h(A) + h(B)$;
(iii) *if* $A, B \in \mathcal{B}(R^2)$, $A \subset B$, *then* $h(B \setminus A) = h(B) - h(A)$;
(iv) *if* $A_n \in \mathcal{B}(R^2)$ $(n = 1, 2, \dots)$, $A_n \nearrow A$, *then* $h(A_n) \nearrow h(A)$:
(v) $h(A \times B) = x_1(A) \wedge x_2(B)$ *for every* $A, B \in \mathcal{B}(R)$.

6. Proposition. *If $x_1, x_2, y\colon \mathcal{B}(R) \to P$ are observables, y is Boolean and $x_1(\mathcal{B}(R)) \cup x_2(\mathcal{B}(R)) \subset y(\mathcal{B}(R))$, then there exists a joint observable of observables x_1, x_2.*

Proof. By Theorem 4, there exist Borel functions $T_1, T_2\colon R \to R$ such that

$$x_i = y \circ T_i^{-1}, \qquad i = 1, 2.$$

Then

$$h = y \circ (T_1, T_2)^{-1}$$

is the joint observable.

If two observables have a joint observable, then a calculus of observables can be constructed. E.g., $x_1 + x_2 = h \circ g^{-1}$, where $g\colon R^2 \to R$, $g(u, v) = u + v$. Actually

$$\begin{aligned} x_1 + x_2 = h \circ g^{-1} &= y \circ (T_1, T_2)^{-1} \circ g^{-1} \\ &= y \circ (g(T_1, T_2))^{-1} = y \circ (T_1 + T_2)^{-1}. \end{aligned}$$

Of course, the sum $x_1 + x_2$ could be defined directly by the formula $x_1 + x_2 = y \circ (T_1 + T_2)^{-1}$, but we are not able to prove that $y \circ (T_1 + T_2)^{-1}$ does not depend on the choice of y, T_1, T_2. The independence is clear, if y is Boolean, but $y \circ (T_1 + T_2)^{-1}$ could by defined also for y weakly Boolean. So the problem of the definition of $x_1 + x_2$ for weakly Boolean y is open.

7. DEFINITION. *A state on P is a mapping $m: P \to \langle 0, 1\rangle$ satisfying the following conditions:*

(i) $m(1) = 1$;
(ii) *if* $f, g, h \in P$, $f = g + h$, *then* $m(f) = m(g) + m(h)$;
(iii) *if* $f_n \in P$, $(n = 1, 2, \dots)$ $f \in P$, $f_n \nearrow f$, *then* $m(f_n) \nearrow m(f)$.

8. PROPOSITION. *If $m: P \to \langle 0, 1\rangle$ is a state, $x: \mathcal{B}(R) \to P$ is an observable, then the mapping $m_x: \mathcal{B}(R) \to \langle 0, 1\rangle$, defined by $m_x(A) = m\big(x(A)\big)$, is a probability measure.*

P r o o f. Evidently $m_x(R) = m\big(x(R)\big) = m(1) = 1$. Further, if $A_n \in \mathcal{B}(R)$ are pairwise disjoint, then (since $\bigcup_{i=1}^{n} A_i \subset \bigcup_{i=1}^{n+1} A_i$ $(n = 1, 2, \dots)$)

$$m_x\left(\bigcup_{n=1}^{\infty} A_n\right) = m\left(x\left(\bigcup_{n=1}^{\infty}\bigcup_{i=1}^{n} A_i\right)\right) = m\left(\bigvee_{n=1}^{\infty} x\left(\bigcup_{i=1}^{n} A_i\right)\right)$$
$$= m\left(\bigvee_{n=1}^{\infty}\sum_{i=1}^{n} x(A_i)\right) = \sup_n m\left(\sum_{i=1}^{n} x(A_i)\right)$$
$$= \sup_n \sum_{i=1}^{n} m\big(x(A_i)\big) = \sum_{n=1}^{\infty} m_x(A_n).$$

9. DEFINITION. *We say that two observables x_1, x_2 have a joint distribution $\mu: \mathcal{B}(R^2) \to \langle 0, 1\rangle$ if μ is a probability measure such that $\mu(A \times B) = m\big(x_1(A) \wedge x_2(B)\big)$ for every $A, B \in \mathcal{B}(R)$.*

10. PROPOSITION. *If observables x_1, x_2 have a joint observable, then they have also a joint distribution.*

P r o o f. If $h: \mathcal{B}(R^2) \to P$ is a joint observable, then $\mu = m \circ h: \mathcal{B}(R^2) \to \langle 0, 1\rangle$ is a joint distribution.

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