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Mária Jurečková; Ferdinand Chovanec

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ON SOME PROPERTIES OF SUBMEASURES ON MV-ALGEBRAS

Mária Jurečková — Ferdinand Chovanec

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ABSTRACT. In this paper we study some properties of a submeasure on MV-algebras. We show that the nonatomicity,the Saks property and the Darboux property are equivalent properties of a submeasure on MV-algebras.

1. Introduction

Let $\mathcal{S}$ be a σ-algebra and $\mu\colon \mathcal{S} \to [0,\infty)$ be a measure on $\mathcal{S}$, i.e.,

(i) $\mu(\emptyset) = 0$;

(ii) $\mu\left(\bigcup\limits_{n=1}^{\infty} A_n\right) = \sum\limits_{n=1}^{\infty} \mu(A_n)$, whenever $(A_n)_{n=1}^{\infty} \subset \mathcal{S}$,

such that $A_i \cap A_j = \emptyset$, $i \neq j$, and $\bigcup\limits_{n=1}^{\infty} A_n \in \mathcal{S}$.

We say that μ is *nonatomic* if, for arbitrary $A \in \mathcal{S}$ such that $\mu(A) > 0$, there exists $B \in \mathcal{S}$, $B \subset A$ such that $0 < \mu(B) < \mu(A)$.

A measure μ has the *Darboux property* if, for any $A \in \mathcal{S}$ and any $t \in \mathbb{R}$ such that $0 < t < \mu(A)$ there exists $B \in \mathcal{S}$, $B \subset A$, such that $\mu(B) = t$.

It is known that the fact that μ is a nonatomic measure on a σ-algebra $\mathcal{S}$ is a sufficient condition for μ having the Darboux property ([6]). Generalizations of this proposition can be found in many directions. For example, Olejček in [8] showed that the preceding assertion for a finitely additive measure is false in

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general and gave some sufficient conditions for a finitely additive measure having the Darboux property. An interesting result can be found in [4]. In this paper Dobrakov deals with relations between Darboux property and nonatomicity in the case that μ is subadditively continuous, i.e., for any $A \in \mathcal{S}$ and any $\varepsilon > 0$ there exists $\delta > 0$ such that $B \in \mathcal{S}$ and $\mu(B) < \delta$ implies $\mu(A \cup B) \leq \mu(A) + \varepsilon$ and $\mu(A) \leq \mu(A - B) + \varepsilon$. Klimkin and Svistula in [7] solved this problem on F-algebras such that they replaced the nonatomicity by the Saks property, i.e., for any $\varepsilon > 0$ and any $A \in \mathcal{S}$ there exists ε-partition of A, i.e., there exist $A_1, A_2, \dots, A_n \in \mathcal{S}$, such that

$$\bigcup_{k=1}^{n} A_k = A\,, \qquad A_i \cap A_j = \emptyset\,,\ i \neq j\,,\quad \mu(A_k) < \varepsilon\,,\ k = 1, \dots, n\,.$$

Riečan in [9] considered the fuzzy sets, i.e., the functions $f\colon X \to [0,1]$ instead of crisp sets (see [10]) and proved that for any Dobrakov submeasure the Darboux, Saks and nonatomic property are equivalent.

In this paper we give the following generalization. We consider an MV-algebra instead of σ-algebra and prove that if μ is a Dobrakov submeasure on MV-algebra, then the nonatomic property is a sufficient condition for the Darboux property. The main ideas of the proof are taken from [7].

2. Notations and preliminaries

MV-algebras were originally introduced by Chang [3] as algebraic systems $\mathcal{M} = (M, \oplus, \odot, *, 0_{\mathcal{M}}, 1_{\mathcal{M}})$, consisting of a nonempty set M, two constant elements $0_{\mathcal{M}}, 1_{\mathcal{M}}$ in $\mathcal{M}$, two binary operations $\oplus, \odot$ and the unary operation $*$ satisfying the following axioms for all $x, y \in \mathcal{M}$:

$$\begin{gathered} x \oplus y = y \oplus x\,, \qquad x \oplus (y \oplus z) = (x \oplus y) \oplus z\,, \\ x \oplus 0_{\mathcal{M}} = x\,, \qquad x \oplus 1_{\mathcal{M}} = 1_{\mathcal{M}}\,, \\ (x^*)^* = x\,, \quad 0^*_{\mathcal{M}} = 1_{\mathcal{M}}\,, \quad x \oplus x^* = 1_{\mathcal{M}}\,, \\ (x^* \oplus y)^* \oplus y = (x \oplus y^*)^* \oplus x\,, \\ x \odot y = (x^* \oplus y^*)^*\,. \end{gathered}$$

We note that if $\mathcal{M}$ is an MV-algebra, then it is a distributive lattice with respect to the partial order $\leq$ defined by $x \leq y$ if and only if $x \odot y^* = 0_{\mathcal{M}}$, and with the least and greatest element $0_{\mathcal{M}}, 1_{\mathcal{M}}$, respectively. Lattice operations $\vee$ and $\wedge$ are defined by $a \vee b = (a \odot b^*) \oplus b$ and $a \wedge b = (a \oplus b^*) \odot b$.

Recall that $\mathcal{M}$ is a *σ-complete MV-algebra* if $\mathcal{M}$ is a σ-complete lattice.

Let $\mathcal{S}$ be a σ-algebra of all subsets of a nonempty set $\mathbb{X}$. Define

$$E \oplus F = E \cup F\,, \quad E \odot F = E \cap F\,, \quad E^* = \mathbb{X} - E\,; \qquad E, F \in \mathcal{S}\,.$$

Then $\mathcal{S}$ is a σ-complete MV-algebra. The converse is not true. In the sequel we will assume that $\mathcal{M}$ is a σ-complete MV-algebra.

DEFINITION 1. A mapping $\mu\colon \mathcal{M} \to [0,\infty)$ is called a *submeasure* on $\mathcal{M}$ if the following conditions hold:

(i) If $x, y \in \mathcal{M}$, $x \le y$, then $\mu(x) \le \mu(y)$;

(ii) To any $y \in \mathcal{M}$ and any $\varepsilon > 0$ there exists $\delta > 0$ such that $x \in \mathcal{M}$ and $\mu(x) < \delta$ implies $\mu(y \oplus x) \le \mu(y) + \varepsilon$;

(iii) If $(y_n)_{n=1}^{\infty} \subset \mathcal{M}$, $y_n \searrow 0_{\mathcal{M}}$, then $\mu(y_n) \searrow 0$.

DEFINITION 2. The submeasure μ is *nonatomic* if, for every $y \in \mathcal{M}$ such that $\mu(y) > 0$, there exists $x \in \mathcal{M}$, $x \le y$, such that $0 < \mu(x) < \mu(y)$.

DEFINITION 3. The submeasure μ has the *Darboux property* if, for all $y \in \mathcal{M}$ and any $t \in \mathbb{R}$, $0 < t < \mu(y)$, there exists $x \in \mathcal{M}$, $x \le y$, such that $\mu(x) = t$.

DEFINITION 4. The submeasure μ has the *Saks property* if, for any $\varepsilon > 0$ and any $y \in \mathcal{M}$ there exists ε-partition of y, i.e., there exist $y_1, \dots, y_n \in \mathcal{M}$ such that $y_i \le y_j^*$ for any $i \ne j$ and

$$\sum_{k=1}^{n} y_i = y_1 \oplus y_2 \oplus \cdots \oplus y_n = y\,, \qquad \text{where} \quad \mu(y_i) < \varepsilon\,, \ \ i = 1, \dots, n\,.$$

LEMMA 5. *Let μ be a nonatomic submeasure on $\mathcal{M}$ and y be an element of $\mathcal{M}$ such that $\mu(y) > 0$. Then to any $\varepsilon > 0$ there exists $x \in \mathcal{M}$ such that $x \le y$ and $0 < \mu(x) < \varepsilon$.*

Proof. Suppose the converse, i.e., there exists $\varepsilon > 0$ such that for any $x \in \mathcal{M}$, $x \le y$, either $\mu(x) \ge \varepsilon$ or $\mu(x) = 0$. Since μ is nonatomic, there is $x_1 \in \mathcal{M}$, $x_1 \le y$, such that $0 < \mu(x_1) < \mu(y)$. According to previous assumptions $\mu(x_1) \ge \varepsilon$.

Since $x_1 \le y$, it follows from the properties of MV-algebras that $y = x_1 \oplus (y \odot x_1^*)$. For more details see [5]. Denote $\varepsilon_1 = \frac{1}{2}\bigl(\mu(y) - \mu(x_1)\bigr)$. Clearly, $\varepsilon_1 > 0$ and according to property (ii) of a submeasure μ there exists $\delta > 0$ such that $\mu(x_1 \oplus z) \le \mu(x_1) + \varepsilon_1$, whenever $z \in \mathcal{M}$, $\mu(z) < \delta$. Put $z = y \odot x_1^*$. Evidently $\mu(y \odot x_1^*) \ge 0$. To prove $\mu(y \odot x_1^*) > 0$ assume that $\mu(y \odot x_1^*) = 0$. Then

$$\mu(y) = \mu\bigl(x_1 \oplus (y \odot x_1^*)\bigr) \le \mu(x_1) + \varepsilon_1 \le \mu(x_1) + \frac{1}{2}\bigl(\mu(y) - \mu(x_1)\bigr) < \mu(y)\,,$$

which is a contradiction and so $\mu(y \odot x_1^*) > 0$.

Hence, there exists $x_2 \le y \odot x_1^*$ such that $\mu(x_2) \ge \varepsilon$ and we can continue in the previous process. We obtain a sequence $(x_n)_{n=1}^{\infty} \subset \mathcal{M}$ such that

$$x_1 \oplus x_2 \oplus \dots = \sum_{n=1}^{\infty} x_n \le y \qquad \text{and} \qquad \mu(x_n) \ge \varepsilon\,, \quad n = 1, 2, \dots\,.$$

Denote $z_n = \sum\limits_{i=n}^{\infty} x_i$. Evidently $z_n \searrow 0_{\mathcal{M}}$, which gives $\mu(z_n) \searrow 0$. This shows that there exists a natural number n such that

$$\mu(x_n) \le \mu(z_n) < \varepsilon\,,$$

which contradicts that $\mu(x_n) \ge \varepsilon$ and this entails $0 < \mu(x) < \varepsilon$. □

3. Nonatomic submeasure and the Darboux and Saks property

PROPOSITION 6. *Any nonatomic submeasure μ on $\mathcal{M}$ has the Saks property.*

Proof. Let $y \in \mathcal{M}$, $\varepsilon > 0$ and μ be a nonatomic submeasure on $\mathcal{M}$. Put

$$a_1 = \sup\{\mu(x) :\ x \in \mathcal{M}\,,\ x \le y\,,\ \mu(x) < \varepsilon\}\,.$$

If $a_1 = 0$, the proof is finished, because in this case $\mu(y) = 0 < \varepsilon$.

Consider $a_1 > 0$. It implies that there exists $y_1 \in \mathcal{M}$, $y_1 \le y$, $\mu(y_1) < \varepsilon$ such that

$$\frac{a_1}{2} < \mu(y_1) \le a_1\,.$$

The proof is complete if $\mu(y \odot y_1^*) < \varepsilon$. If not, we will construct a sequence $(y_n)_n^{\infty} \subset \mathcal{M}$, $y_n \le y \odot y_1^* \odot \dots \odot y_{n-1}^*$, $\mu(y_n) < \varepsilon$ such that

$$\frac{a_n}{2} < \mu(y_n) \le a_n$$

and

$$a_n = \sup\{\mu(x) :\ x \in \mathcal{M}\,,\ x \le y \odot y_1^* \odot \dots \odot y_{n-1}^*\,,\ \mu(x) < \varepsilon\}\,.$$

Put now

$$z_n = \sum_{i=n}^{\infty} y_i = y_n \oplus y_{n+1} \oplus \cdots\,.$$

It is easy to see that

$$0 < \frac{a_n}{2} < \mu(y_n) \le \mu(z_n)$$

and since $z_n \searrow 0_{\mathcal{M}}$, we obtain $\mu(z_n) \searrow 0$, which gives that $\lim\limits_{n\to\infty} a_n = 0$.

Put $x = y \odot \left(\sum\limits_{i=1}^{\infty} y_i\right)^*$. To show $\mu(x) = 0$, suppose the contrary, i.e., $\mu(x) > 0$. Applying Lemma 5, there exists $z \in \mathcal{M}$, $z \le x$, such that $\mu(z) < \varepsilon$ and

$$z \le x = y \odot \left(\sum_{i=1}^{\infty} y_i\right)^* \le y \odot \left(\sum_{i=1}^{k} y_i\right)^* = y \odot y_1^* \odot \cdots \odot y_k^*, \qquad k = 1,2,\dots.$$

This implies $\mu(z) \le a_{k+1}$ for $k = 1,2,\dots$ and, because $\lim\limits_{n\to\infty} a_n = 0$, we obtain $\mu(x) \le 0$, which contradicts our assumption and so $\mu(x) = 0$.

Since $\sum\limits_{i=n}^{\infty} y_i \searrow 0_{\mathcal{M}}$, there exists n_0 such that

$$\mu\left(\sum_{i=n_0}^{\infty} y_i\right) < \varepsilon.$$

Moreover

$$y_1 \oplus y_2 \oplus \cdots \oplus y_{n_0-1} \oplus \sum_{i=n_0}^{\infty} y_i \oplus \left(y \odot \left(\sum_{i=n_0}^{\infty} y_i\right)^*\right) = y$$

and so we can conclude that

$$\mathcal{E} = \left\{y_1, y_2, \dots, y_{n_0-1}, \sum_{i=n_0}^{\infty} y_i, x\right\}$$

is an ε-partition of y. □

PROPOSITION 7. *Let μ be a submeasure with the Saks property on an MV-algebra $\mathcal{M}$. Then μ has the Darboux property.*

Proof. Consider $y \in \mathcal{M}$, $t \in \mathbb{R}$ such that $0 < t < \mu(y)$ and a sequence of real numbers $(\varepsilon_n)_{n=1}^{\infty}$ such that $\varepsilon_n \searrow 0$, $\varepsilon_n < t$. By the assumption, the submeasure μ has the Saks property, which gives an ε_1-partition of y, i.e., there exist $y_1, \dots, y_n \in \mathcal{M}$ such that $y_i \le y_j^*$, $i \ne j$, $\sum\limits_{i=1}^{n} y_i = y$ and $\mu(y_i) < \varepsilon_1 < t$ for all $i = 1,2,\dots,n$. Since $\mu(y_1 \oplus \cdots \oplus y_n) = \mu(y) > t$, there exists l such that

$$\mu(y_1 \oplus \cdots \oplus y_l) < t, \quad \mu(y_1 \oplus \cdots \oplus y_l \oplus y_{l+1}) \ge t.$$

Denote $x_1 = y_1 \oplus \cdots \oplus y_l$ and $z_1 = y_1 \oplus \cdots \oplus y_l \oplus y_{l+1}$. Then

$$x_1 \le z_1 \le y, \quad \mu(x_1) < t, \quad \mu(z_1) \ge t,$$
$$\mu(z_1 \odot x_1^*) = \mu(y_{l+1}) < \varepsilon_1.$$

Now we will apply the Saks property to the $z_1 \odot x_1^*$. There exists an ε_2-partition $\{v_1, \dots, v_k\}$ of $z_1 \odot x_1^*$ such that

$$z_1 \odot x_1^* = \sum_{i=1}^{k} v_i \qquad \text{and} \qquad \mu(v_i) < \varepsilon_2\,, \quad i = 1, 2, \dots, k\,.$$

Since $\mu(x_1) < t$ and $\mu\big(x_1 \oplus (z_1 \odot x_1^*)\big) = \mu(x_1 \oplus v_1 \oplus \dots \oplus v_k) \geq t$, it is clear that there is a natural number m such that

$$\mu(x_1 \oplus v_1 \oplus \dots \oplus v_m) < t \qquad \text{and} \qquad \mu(x_1 \oplus v_1 \oplus \dots \oplus v_m \oplus v_{m+1}) \geq t\,.$$

Put $x_2 = x_1 \oplus v_1 \oplus \dots \oplus v_m$ and $z_2 = x_1 \oplus v_1 \oplus \dots \oplus v_m \oplus v_{m+1}$. Then

$$x_1 \leq x_2 \leq z_2 \leq z_1 \leq y\,,$$
$$\mu(x_2) < t\,, \quad \mu(z_2) \geq t\,, \quad \mu(z_2 \odot x_2^*) = \mu(v_{m+1}) < \varepsilon_2\,.$$

By this way we obtain two sequences $(x_n)_{n=1}^{\infty}$, $(z_n)_{n=1}^{\infty}$ of elements of $\mathcal{M}$ such that

$$x_1 \leq x_2 \leq \dots \leq x_n \leq z_n \leq \dots \leq z_2 \leq z_1 \leq y\,,$$
$$\mu(z_n \odot x_n^*) < \varepsilon_n\,, \qquad n = 1, 2 \dots\,.$$

Put $x = \bigvee_{n=1}^{\infty} x_n$. It is evident that $x \in \mathcal{M}$ and $x \leq y$. The proof will be complete by showing that $\mu(x) = t$. Conversely suppose that $\mu(x) < t$. Then we can put $\varepsilon = \frac{1}{2}\big(t - \mu(x)\big) > 0$. By the property (ii) of the submeasure there exists $\delta > 0$ such that for any $w \in \mathcal{M}$ with $\mu(w) < \delta$, $\mu(x \oplus w) \leq \mu(x) + \varepsilon$. Since $\varepsilon_n \searrow 0$, there exists n_0 such that $\mu\big(z_{n_0} \odot x_{n_0}^*\big) \leq \varepsilon_{n_0} < \delta$. Then

$$\begin{aligned}\mu\big(z_{n_0}\big) &= \mu\big(x_{n_0} \oplus (z_{n_0} \odot x_{n_0}^*)\big) \leq \mu\big(x \oplus (z_{n_0} \odot x_{n_0}^*)\big)\\ &\leq \mu(x) + \varepsilon = \mu(x) + \frac{1}{2}\big(t - \mu(x)\big) < t\,,\end{aligned}$$

contrary to $\mu(z_i) \geq t$ for all $i = 1, 2, \dots$. This entails that $\mu(x) \geq t$.

Take now $\varepsilon > 0$. By the property (ii) of Definition 1 there exists $\delta > 0$ such that $\mu\big(x_{n_0} \oplus w\big) \leq \mu\big(x_{n_0}\big) + \varepsilon$, whenever $w \in \mathcal{M}$ and $\mu(w) < \delta$. Since

$$\mu\big(x \odot x_{n_0}^*\big) \leq \mu\big(z_{n_0} \odot x_{n_0}^*\big) \leq \varepsilon_{n_0} < \delta\,,$$

we have

$$\mu(x) = \mu\big(x_{n_0} \oplus (x \odot x_{n_0}^*)\big) \leq \mu\big(x_{n_0}\big) + \varepsilon\,.$$

But $\mu\big(x_{n_0}\big) < t$, which implies that $\mu(x) < t + \varepsilon$ for any $\varepsilon > 0$, and so we can conclude that $\mu(x) = t$. □

PROPOSITION 8. *Any nonatomic submeasure on $\mathcal{M}$ has the Darboux property.*

P r o o f . This follows directly from Propositions 6 and 7. □

It is evident that if μ has the Darboux property, then μ is nonatomic. Combining this fact with Proposition 8 we can conclude our assertions with the following theorem.

THEOREM 9. *Let μ be a submeasure on an MV-algebra. Then the nonatomicity, the Saks property and the Darboux property are equivalent properties of a submeasure μ.*

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Department of Mathematics
Military Academy
SK-031 19 Liptovský Mikuláš
SLOVAK REPUBLIC

Mathematical Institute
Slovak Academy of Sciences
Štefánikova 49
SK-814 73 Bratislava
SLOVAK REPUBLIC

E-mail: jureckova@valm.sk
chovanec@valm.sk