Mathematica Slovaca

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Mathematica Slovaca, Vol. 34 (1984), No. 1, 73--88

Persistent URL: http://dml.cz/dmlcz/128714

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ON THE OSCILLATION OF NONLINEAR DIFFERENTIAL SYSTEMS WITH RETARDED ARGUMENTS

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1. Introduction

We consider systems of nonlinear differential inequalities with retarded arguments of the form

$$y_i'(t)-f_i(t,\, y_{i+1}(t),\, y_{i+1}(h_{i+1}(t)))=0, \quad i=1, 2, \dots, n-1, \tag{S}$$
$$\{y_n'(t)+f_n(t,\, y_1(t),\, y_1(h_1(t)))\}\ \operatorname{sgn} y_1(h_1(t))\leqslant 0.$$

where the following conditions are always assumed:

(a) $h_i\colon [a, \infty)\to R$ $(i=1, 2, \dots, n)$ are continuous and

$h_i(t)\leqq t$ for $t\geqq a$, $\lim_{t\to\infty} h_i(t)=\infty$, $(i=1, 2, \dots, n)$;

(b) f_i: $[a, \infty)\times R^2\to R$ $(i=1, 2, \dots, n)$ are continuous,
$vf_i(t, u, v)\geqq 0$ $(i=1, 2, \dots, n)$ for $uv>0$
and not identically zero on any subinterval of
$[a, \infty)$; $f_i(t, u, v)$ $(i=1, 2, \dots, n-1)$ are nondecreasing
in u and v for each fixed $t\in[a. \infty)$.

Denote by $\mathbf{W}$ the set of all solutions $y(t)=(y_1(t), \dots, y_n(t))$ of the system (S) which exist on some ray $[T_y, \infty)\subset[a, \infty)$ and satisfy $\sup\left\{\sum_{i=1}^{n}|y_i(t)|: t\geqq T\right\}>0$ for any $T\geqq T_y$.

Definition 1. *A solution $y\in\mathbf{W}$ is called oscillatory (resp. weakly oscillatory) if each component (resp. at least one component) has arbitrarily large zeros.*

A solution $y\in\mathbf{W}$ is called nonoscillatory (resp. weakly nonoscillatory) if each component (resp. at least one component) is eventually of a constant sign.

Definition 2. *We shall say that the system (S) has the property A, if every solution $y\in\mathbf{W}$ is oscillatory for n, even, while for n odd it is either oscillatory or y_i $(i=1, 2, \dots, n)$ tend monotonically to zero as $t\to\infty$.*

The oscillatory properties of solutions of two-dimensional differential systems with deviating arguments are studied in the following papers: Kitamura and Kusano [2, 3], Varech, Gritsai and Ševelo [4], Ševelo and Varech [5, 6]. The oscillation results for the system $x_k'(t)=f_k(t, x(g_1(t)), \dots, x(g_n(t)))$, $k=1, 2, \dots, n$ were studied, followd by Foltynska and Werbowski [1].

In the present paper we proceed further in this direction to extend the theory developed in [4—6] to the systems of the form (S). Our results in lude some of the results in [1, 5, 6] and they do not follow from Theorem 1 in [1].

2. O cill tion th orems

We introd ce the notation :

$$\gamma_i(t)=\sup\{s>0;\, h(s)<t\} \text{ for } t\geqq a,\ i-1, 2 \dots n,$$

$$\gamma(t)-\max\{\gamma_1(t), \dots, \gamma_n(t)\}.$$

Lemma 1. *Let* $y\ \ (y_1, \dots, y_n)\in\mathbf{W}$ *be a weakly nonosillatory solution of* (S), *then* y *is nonoscillatory.*

Proof. Suppose that y_k is a nono cillatory component of solution $y=(y\ , \dots, y_k, \dots, y_n)\in\mathbf{W}$ and $y_k(t)\neq 0$ for $t\geqq t_1\geqq a$.

i) *Let* $1<k\leqq n$. With the help of (a), (b), the system (S) implies that either

$$y_{k\ 1}'(t)\geqq 0 \text{ or } y_{k\ 1}'(t)\leqq 0 \text{ for } t\geqq\gamma(t_0)-t_1, \tag{1}$$

and not identically zero on any infinite subinterval of $[t_1, \infty)$. We remark that $y_{k\ 1}(t)\not\equiv 0$ for all $t\geqq t_2\geqq t_1$. If $y_{k\ 1}(t)\equiv 0$ for $t\geqq t_2$, then $y_{k\ 1}'(t)\equiv 0$ for $t\geqq t_2$ and the $(k-1)$-*st equation of* (S) *gives that* $f_{k\ 1}(t, y_k(t), y_k(h_k)))=0$ for all $t\geqq t$, which contradicts assumption (b). From (S) we get that $y_{k\ 1}(t)$ is the monotone function and thus there exists a $t_3\geqq t_1$ such that $y_{k\ 1}(t)\neq 0$ for $t\geqq t_3$. We have proved that $y_{k\ 1}$ is the nonoscillatory component of y. Analogously we can prove that $y_{k-2}(t), \dots, y_1(t)$ are also nono cillatory components of y.

ii) Let $k=1$. From the n-th inequality of (S) we obtain $y_n'(t)\operatorname{sgn} y_1(h_1(t))\leqq 0$ for $t>t_1$ and not identically zero on any subinterval of $[t_1, \infty)$. Thus there exists a $t_4\geqq t_1$ such that $y_n(t)\neq 0$ for $t\geqq t_4$. If we consider now the case i) for $k-n$, we get that all components of y are nonoscillatory.

The proof of Lemma 1 is complete

Lemma 2. *Suppose that*

$$y-(y\ , \dots, y_n)\in\mathbf{W} \tag{2}$$

is a nonoscillatory solution of (S) *in the interval* $[a, \infty)$ *If*

$$\int_T^\infty |f_k(t, c, c)|\,\mathrm{d}t=\infty \text{ for all } c\neq 0,\ k=1, 2, \dots, n\quad 1, \tag{3}$$

then there exist an integer $l \in \{1, 2, \ldots, n\}$, $n+l$ even, and a $t_0 \geqq a$ such that

$$y_i(t)y_1(t) > 0 \quad on \quad [t_0, \infty) \quad for \quad i = 1, 2, \ldots, l, \tag{4}$$

$$(-1)^{n+i} y_i(t) y_1(t) > 0 \quad \text{on} \quad [t_0, \infty) \quad \text{for} \quad i = l+1, \ldots, n \tag{5}$$

hold.

Proof. Without loss of generality we may suppose that $y_1(t) > 0$ for $t \geqslant a$. Similar arguments hold if $y_1(t) < 0$. According to (a) there exists a $T_1 \geqq \gamma(a)$ such that $y_1(h_1(t)) > 0$ for $t \geqslant T_1$. Then the n-th inequality of (S) implies that $y_n(t')$ is nonicreasing on $[T_1, \infty)$ and not identically zero on any infinite subinterval of $[T_1, \infty)$. We shall show that $y_n(t) > 0$ for $t \geqslant T_2 \geqslant T_1$. If $y_n(t) < 0$ for some $t_1 \geqslant T_2$, then $y_n(t) \leqq y_n(t_1) = c_n < 0$ for $t \geqslant t_1$. Taking this into account and then integrating the $(n-1)st$ equation of (S) from $t_2 = \gamma(t_1)$ to t, we have

$$y_{n-1}(t) = y_{n-1}(t_2) + \int_{t_2}^{t} f_{n-1}(s, y_n(s), y_n(h_n(s)))\, ds \leqq$$

$$\leqq y_{n-1}(t_2) + \int_{t_2}^{t} f_{n-1}(s, c_n, c_n)\, ds \to -\infty \quad \text{as} \quad t \to \infty.$$

Then there exists a $t_3 \geqq \gamma(t_2)$ such that $y_{n-1}(t) \leqq c_{n-1} < 0$, $y_{n-1}(h_{n-1}(t)) \leqq c_{n-1}$ for $t \geqslant t_3$. Integrating again the $(n-2)nd$ equation of (S) we prove that $y_{n-2}(t) \to -\infty$ as $t \to \infty$. Similarly we shall prove that $y_i(t) \to -\infty$ as $t \to \infty$ $(i = n-3, \ldots, 2, 1)$, which contradicts $y_1(t) > 0$ for $t \geqslant a$. Therefore $y_n(t) > 0$ on $[T_2, \infty)$. Thus with the help of the $(n-1)st$ equation we obtain that $y_{n-1}(t)$ is a nondecreasing function for $t \geqq T_3 = \gamma(T_2)$ and that it is eventually of one sign. a_1) Let $y_{n-1}(t) \geqq c_{n-1} > 0$ for $t \geqq T_4 \geqslant T_3$. Taking this into account and integrating the $(n-2)$nd equation of (S) from T_4 to t, we obtain

$$y_{n-2}(t) \geqq y_{n-2}(T_4) + \int_{T_4}^{t} f_{n-2}(s, c_{n-1}, c_{n-1})\, ds \to \infty$$

as $t \to \infty$. Repeating this method, we prove that $y_i(t) > 0$ $(i = 1, 2, \ldots, n-1)$ for $t \geqslant T_5 \geqslant T_4$. Therefore (4) is true for $l = n$.

b$_1$) Let $y_{n-1}(t) < 0$ on $[T_3, \infty)$. Then the $(n-2)nd$ equation of (S) implies that $y_{n-2}(t)$ is nonincreasing for $t \geqq T_6 = \gamma(T_3)$ and that it is eventually of one sign. We show that $y_{n-2}(t) > 0$ for $t \geqslant T_7 \geqq T_6$. If $y_{n-2}(t) < 0$ for some $t_4 \geqq T_7$; then $y_{n-2}(t) \leqq y_{n-2}(t_4) = c_{n-1} < 0$. Similarly as in the assumption $y_n(t_1) < 0$ we can prove that $y_1(t) \to -\infty$ as $t \to \infty$, which contradicts the assumption $y_1(t) > 0$ on $[a, \infty)$. Therefore $y_{n-2}(t) > 0$ on $[T_7, \infty)$. According to the $(n-3)rd$ equation of (S) we obtain that $y_{n-3}(t)$ is nondecreasing for $t \geqq T_8 = \gamma(T_7)$ and $y_{n-3}(t)$ is either positive for $t \geqq T_9 \geqq T_8$ or $y_{n-3}(t) < 0$ for $t \geqq T_8$. a$_2$) If $y_{n-3}(t) > 0$ for $t \geqslant T_9$, we can prove that $y_i(t) > 0$ $(i = 1, 2, \ldots, n-3)$ for $t \geqslant T_{10} \geqslant T_9$. Then (4) is true for $l = n-2$. b$_2$) If $y_{n-3}(t) < 0$ for $t \geqslant T_8$, we can proceed as in the case of b$_1$), only instead of $\overline{n-1}$ we have $\overline{n-3}$. So we get that either $y_i(t)>0$ $(i=1, 2, \dots, n-4=l)$ or $y_{n-4}(t)>0$ and $y_{n-5}(t)<0$ for sufficiently large t. Proceeding further similarly to the case of b_1), b_2) we prove (4) and (5) for $l=n-4, \dots, 4, 2$ $(l=n-4, \dots, 3, 1)$ if n is even (odd). This completes the proof.

Lemma 3. *Suppose that the assumptions of Lemma 2 hold. If a component y_k $(k\in\{1, 2, \dots, n\})$ of a solution $y=(y_1, \dots, y_n)\in\mathbf{W}$* has the property

$$\liminf_{t\to\infty} |y_k(t)| = L_k ,$$

then

a) $\lim_{t\to\infty} y_i(t) = +\infty(-\infty)$, $(i=1, 2, \dots, k-1)$ *when* $L_k>0$, $k>1$;

b) $\liminf_{t\to\infty} |y_i(t)| = 0$, $(i=k+1, \dots, n)$ *when* $L_k<\infty$, $k<n$.

Proof. Lemma 3 may be proved in the same way as Lemma 2 [1] and therefore we omit here the proof.

Theorem 1. *Suppose that*

$$f_n(t, x, y) \text{ is nondecreasing in } x \text{ and } y \text{ for each fixed } t\geqq a . \tag{6}$$

If, in addition,

$$\int_T^\infty |f_k(t, c, c)|\, dt = \infty \quad \text{for } k=1, 2, \dots, n \tag{7}$$

for every $c\neq 0$, then the system (S) *has the property A.*

Proof. Suppose that the system (S) has a nonoscillatory solution $y= =(y_1, \dots, y_n)\in\mathbf{W}$. Without loss of generality we may suppose that $y_1(t)>0$ for $t\geqq t_0\geqq a$. According to (a), $y_1(h_1(t))>0$ for $t\geqq t_1=\gamma(t_0)$. Then the n-th inequality of (S) implies $y_n'(t)\leqq 0$ for $t\geqq t_1$ and it is not identically zero on any subinterval of $[t_1, \infty)$. As $y_1(t)>0$, $y_n'(t)\leqq 0$ for $t\geqq t_1$, by Lemma 2 there exists an integer $l\in\{1, \dots, n\}$, $n+l$ is even and a $T_0\geqq t_1$ such that

$$\begin{gathered} y_i(t)>0 \text{ or } [T_0, \infty) \text{ for } i=1, 2, \dots, l, \\ (-1)^{n+i} y_i(t)>0 \text{ on } [T_0, \infty) \text{ for } i=l+1, \dots, n \end{gathered} \tag{8}$$

hold.

I. Let $l\geqq 2$. In view of (8) and (a) we have $y_1(t)>0$, $y_2(t)>0$ for $t\geqq T$. Then by the 1*st* equation of (S), in view of (b) we get $y_1'(t)\geqq 0$ for $t\geqq t_2=\gamma(T_0)$ and not identically zero on any subinterval of $[t_2, \infty)$. The function $y_1(t)$ is nondecreasing and therefore $y_1(t)\geqq d_1>0$ for $t\geqq t_2$. From the n-th inequality of (S), we have, with the help of (b) and (6),

$$y_n'(t) \leqq -f_n(t, y_1(t), y_1(h_1(t))) \leqq -f_n(t, d_1, d_1) \text{ for } t \geqq t_3 = \gamma(t_2).$$

Integrating the last inequality from t_3 to t, we obtain

$$\int_{t_3}^{t} f_n(s, d_1, d_1)\,\mathrm{d}s \leqq y_n(t_3) - y_n(t) \leqq y_n(t_3),$$

which contradicts (7) for $k = n$, as $t \to \infty$.

II. Let $l = 1$ (n is odd). According to (8) and (b) we have $y_2(t) < 0$, $y_2(h_2(t)) < 0$ for $t \geqq t_1 = \gamma(t_0)$. Then the 1st equation of (S) gives that $y_1(t)$ is nonincreasing and therefore $\lim_{t \to \infty} y_1(t) = \delta \geqq 0$. We suppose that $\delta > 0$. Proceeding analogously as in the proof of I, we obtain a contradiction to (7). Therefore $\delta = 0$. Then applying Lemma 3 we get $\lim_{t \to \infty} y_i(t) = 0$ for $i = 1, 2, \ldots, n$.

The proof of Theorem 1 is complete.

Theorem 1 generalizes the results in [5, Theorem 1] and in [1, Remark 1].

Theorem 2. *Suppose that* (3) *holds and in addition*

$$f_n(t, x, y) = p_n(t) g_n(x, y), \tag{9}$$

where p_n: $[a, \infty) \to [0, \infty)$, g_n: $R^2 \to R$ *are continuous functions with* p_n *not identically zero on any subinterval of* $[a, \infty)$, $y g_n(x, y) > 0$ *for* $xy > 0$ *and* $\liminf_{|y| \to \infty} |g_n(x, y)| > 0$ *for all* $x \neq 0$.

If

$$\int_a^{\infty} p_n(t)\,\mathrm{d}t = \infty, \tag{10}$$

then the system (S) *has the property A.*

Proof. Arguing as in the proof of Theorem 1 we can show that (8) holds. a) In case I ($i \geqq 2$) we have proved that $y_1(t)$ is a nondecreasing function for which $y_1(t) \geqq d_1 > 0$ for $t \geqq t_2$ and $\lim_{t \to \infty} y_1(t) = d_2 \geqslant 0$, where either $d_2 < \infty$ or $d_2 = \infty$. Then in view of (9) there exists a $K > 0$ such that

$$g_n(y_1(t), y_1(h_1(t))) \geqq K \text{ for } t \geqq t_3 = \gamma(t_2).$$

From the n-th inequality of (S) with the help of the last inequality we have

$$y_n'(t) \leqq -f_n(t, y_1(t), y_1(h_1(t))) = -p_n(t) g_n(y_1(t), y_1(h_1(t))) \leqq$$
$$\leqq -K p_n(t), \quad \text{for} \quad t \geqq t_3.$$

Integrating the last inequality from t_3 to t, we obtain

$$K\int_{t_3}^{t} p_n(s)\,\mathrm{d}s \leqq y_n(t_3) - y_n(t) \leqq y_n(t_3),$$

which gives a contradiction to (10) as $t \to \infty$.

b) Let $l=1$. Analogously as in case II of the proof of Theorem 1 we can show that $\lim_{t\to\infty} y_1(t)=0$. Then by Lemma 3 we ge $\lim_{t\to\infty} y_i(t)=0$ for $i=1, 2, \ldots, n$.

The proof of Theorem 2 is complete. This Theorem generalizes Theorem 2 [6].

We turn now to the system (S), where

$$f_i(t, x, y) = p_i(t)x, \quad i=1, 2, \ldots, n-2 \tag{11}$$

$$f_k(t, x, y)\ \mathrm{sgn}\ y = p_k(t)|y|^{\alpha_k}, \quad \alpha_k > 0,\ k=n-1, n,$$

where

$$p_i\colon [a, \infty) \to [0, \infty), \quad i=1, 2, \ldots, n \tag{12}$$

are continuous functions and not identically zero on any subinterval of $[a, \infty)$,

$$\int_{p_i}^{\infty} (t)\,\mathrm{d}t = \infty, \quad i=1, 2, \ldots, n-1.$$

The system (S), in the particular case where (11), (12) hold and $p_i(t)>0$, $i=1, 2, \ldots, n-1$, $\alpha_{n-1}=1$, $h_n(t)=t$ on $[a, \infty)$, is equivalent to the n-th order scalar differential inequality

$$\left\{\left(\frac{1}{p_{n-1}(t)}\left(\cdots\left(\frac{1}{p_2(t)}\left(\frac{1}{p_1(t)}y'(t)\right)'\right)'\cdots\right)'\right)' + p_n(t)|y(h_1(t))|^{\alpha_n}\right\}\cdot$$

$$\cdot\ \mathrm{sgn}\ y(h_1(t)) \leqq 0.$$

We introduce the notation. $\alpha_{n-1}=\alpha$, $\alpha_n=\beta$;

$$\bar{p}_i(t) = \min\{p_i(s);\ t/4 \leqslant s \leqslant t\}, \quad t \geqslant a,\ i=1, \ldots, n-1$$

$$P_j^i(t) = \bar{p}_j(t)\bar{p}_{j-1}(t)\ldots\bar{p}_i(t) \quad \text{for} \quad i \leqq j,$$

$$P_j^i(t) = 1 \quad \text{for} \quad i > j, \quad P_j^1(t) = P_j(t).$$

Let $i_k \in \{1, 2, \ldots, n\}$ $1 \leqq k \leqq n-1$ and $t, s \in [a, \infty)$. We define $I_0 = 1 = J_0$, and

$$I_k(t, s; p_{i_k}, \ldots, p_{i_1}) = \int_s^t p_{i_k}(x) I_{k-1}(x, s; p_{i_{k-1}}, \ldots, p_{i_1})\,\mathrm{d}x,$$

$$J_k(t, s; p_{i_k}, \ldots, p_{i_1}) = \int_s^t p_{i_1}(x) J_{k-1}(t, x; p_{i_k}, \ldots, p_{i_2})\,\mathrm{d}x.$$

Lemma 4. *Suppose that (11), (12) hold. Let y be a solution of (S) on the interval $[a, \infty)$. Then the following relations hold:*

$$y_i(s)=\sum_{j=0}^{n-i-1}(-1)^j y_{i+j}(t)I_j(t, s; p_{i+j-1}, \ldots, p_i)+ \tag{13}$$

$$+(-1)^{n-i}\int_s^t p_{n-1}(x)|y_n(h_n(x))|^\alpha \operatorname{sgn} y_n(h_n(x))I_{n-i-1}(x, s; p_{n-2}, \ldots, p_i)\,\mathrm{d}x,$$

$$\text{for } a\leqq s\leqq t,\ i=1, 2, \ldots, n-1;$$

$$y_i(r)=\sum_{j=0}^{m} y_{i+j}(s)J_j(r, s; p_i, \ldots, p_{i+j-1})+ \tag{14}$$

$$+\int_s^r y_{i+m+1}(x)p_{i+m}(x)J_m(r, x; p_i, \ldots, p_{i+m-1})\,\mathrm{d}x,$$

$$\text{for } r\geqq s\geqq a,\ i<n-1,\ 0\leqslant m<n-i-1.$$

Proof. a) Let $a\leqq s\leqq t$. It is evident that

$$y_i(s)=y_i(t)-\int_s^t y_i'(x)\,\mathrm{d}x=y_i(t)-\int_s^t p_i(x)y_{i+1}(x)\,\mathrm{d}x, \tag{15}$$

$$\text{for } i\leqq n-2,$$

$$y_{n-1}(s)=y_{n-1}(t)-\int_s^t p_{n-1}(x)|y_n(h_n(x))|^\alpha \operatorname{sgn} y_n(h_n(x))\,\mathrm{d}x.$$

We calculate the second integral in (15) by parts. Denote:

$$v(x)=\int_s^x p_i(\tau)\,\mathrm{d}\tau=I_1(x, s; p_i),\ u(x)=y_{i+1}(x).$$

Then we obtain

$$y_i(s)=y_i(t)-y_{i+1}(t)I_1(t, s; p_i)+\int_s^t y_{i+1}'(x)I_1(x, s; p_i)\,\mathrm{d}x=$$

$$=y_i(t)-y_{i+1}(t)I_1(t, s; p_i)+\int_s^t y_{i+2}(x)p_{i+1}(x)I_1(x, s; p_i)\,\mathrm{d}x$$

$$\text{for } i<n-2.$$

If $i=n-2$, we get (13).

Using further the method by parts $(n-2-i)$ times $(i<n-2)$ on the last integral, we obtain (13).

b) Let $a\leqq s\leqq t$ and let $i<n-1$. It is clear that

$$y_i(t)=y_i(s)+\int_s^t y_i'(x)\,\mathrm{d}x=y_i(s)+\int_s^t y_{i+1}(x)p_i(x)\,\mathrm{d}x. \tag{16}$$

For $i=n-2$ (14) is true. Let $i<n-2$. We calculate the last integral in (16) by parts. Denote $v(x)=-\int_x^t p_i(\tau)\,d\tau$, $u(x)=y_{i+1}(x)$. Then we have

$$y_i(t)=y_i(s)+y_{i+1}(s)\int_s^t p_i(x)\,dx+\int_s^t y'_{i+1}(x)J_1(t,x;p_i)\,dx=$$

$$=y_i(s)+y_{i+1}(s)J_1(t,s\colon p_i)+\int_s^t y_{i+2}(x)p_{i+1}(x)J_1(t,x;p_i)\,dx.$$

Using further the method by parts $m-1$ times on the last integral, we get (14).

Lemma 5. *Suppose that* (11), (12) *and the assumption* (i) *of Lemma* 2 *hold. Then there exist* $l\in\{1,2,\ldots,n\}$, $n+l$ *is even and a* $T\geqq a$ *such that* (4), (5) *hold and*

$$|y_i(t/2)|\geqslant C_i t^{n-i}P^i_{n-1}(t)|y_n(t)|^\alpha \quad \text{for} \quad t\geqslant T, \tag{17}$$

where

$$C_i=\frac{2^{-2(n-i)}}{(n-1)!\,(n-i)!},\quad i=1,2,\ldots,n-1.$$

Proof. The inequality (4), (5) follows from Lemma 2. Without loss of generality we suppose that $y_1(t)>0$ for $t\geqq t_0$. Then from (13) we obtain for $s=t/2$, in view of (5) and the monotonicity of $y_n(t)$

$$(-1)^{i+n}y_i(t/2)\geqq\int_{t/2}^t (y_n(x))^\alpha p_{n-1}(x)I_{n-i-1}(x,t/2;p_{n-2},\ldots,p_i)\,dx$$

$$\geqq(y_n(t))^\alpha\bar{p}_{n-1}(t)\int_{t/2}^t (t-x)p_{n-2}(x)I_{n-i-2}(x,t/2;p_{n-3},\ldots,p_i)\,dx\geqq\ldots\geqq$$

$$\geqq(y_n(t))^\alpha\bar{p}_{n-1}(t)\ldots\bar{p}_i(t)\int_{t/2}^t\frac{(t-x)^{n-i-1}}{(n-i-1)!}\,dx.$$

Calculating the last integral we get

$$(-1)^{i+n}y_i(t/2)\geqq\left(\frac{t}{2}\right)^{n-i}\frac{P^i_{n-1}(t)}{(n-i)!}(y_n(t))^\alpha \text{ for } t\geqq 2t_0 \tag{18}$$

$$\text{and } i=l,l+1,\ldots,n-1.$$

According to (4) and the monotonicity of $y_n(t)$ we have from (14) for $m=l-i-1$, $r=1/2$, $s=t/4$

$$y_i(t/2)\geqq y_l(t/2)\int_{t/4}^{t/2}p_{l-1}(x)J_{l-i-1}(t/2,x;p_i,\ldots,p_{l-2})\,dx\geqq$$

$$\geqq y_l(t/2)\bar{p}_{l-1}(t)\int_{t/4}^{t/2}(x-t/4)p_{l-2}(x)J_{l-i-2}(t/2, x; p_i, \dots, p_{l-3})\,\mathrm{d}x \geqq \dots \geqq$$

$$\geqq y_l(t/2)\bar{p}_{l-1}(t)\dots\bar{p}_i(t)\int_{t/4}^{t/2}\frac{(x-t/4)^{l-i-1}}{(l-i-1)!}\,\mathrm{d}x\,.$$

If we calculate the last integral we obtain

$$y_i(t/2)\geqq\left(\frac{t}{4}\right)^{l-i}\frac{P^i_{l-1}(t)}{(l-i)!}\,y_l(t/2) \quad \text{for} \quad t\geqq 4t_0=T, i=1, 2, \dots, l-1\,. \tag{19}$$

Combining (18) for $i=l$ and (19) we get (17).

Remark 1. a) The inequality (4) implies $|y_i(t)|\geqslant|y_i(t/2)|$ for $i=1, 2, \dots, l-1$. Then (17) can be written in the form

$$|y_i(t)|\geqslant C_i t^{n-i}P^i_{n-1}(t)|y_n(t)|^\alpha \quad \text{for} \quad t\geqslant T, i=1, \dots, l-1\,. \tag{17'}$$

b) If $0<\alpha\leqq 1$, then it is evident that (17) holds also for $i=n$.

Theorem 3. *Suppose that* (11), (12) *hold. If* $0<\alpha\beta<1$ and

$$\int_T^\infty (h_1(t))^{(n-1)\beta}p_n(t)(P_{n-1}(h_1(t)))^\beta\,\mathrm{d}t=\infty\,, \tag{20}$$

then the system (S) *has the property A.*

Proof. Suppose that the system (S) has a nonoscillatory solution $y=(y_1, \dots, y_n)\in\mathbf{W}$. Without loss of generality we may suppose that $y_1(t)>0$ for $t\geqslant t_0\geqslant a$. According to (a) we have $y_1(h_1(t))>0$ for $t\geqslant t_1=\gamma(t_0)$. Then the *n-th* inequality of (S) implies that $y_n'(t)\leqslant 0$ for $t\geqslant t_1$ and it is not identically zero on any subinterval of $[t_1, \infty)$. As $y_1(t)>0$ and $y_n'(t)\leqq 0$ for $t\geqslant t_1$, then by Lemma 5 we get (4), (5) and (17), resp. (17').

I. Let $l\geqq 2$. From (17') we have for $i=1$

$$y_1(t)\geqq C_1 t^{n-1}P_{n-1}(t)(y_n(t))^\alpha, \quad t\geqq t_2>t_1\,.$$

Then the *n-th* inequality of (S) implies

$$\begin{aligned} y_n'(t) &\leqq -C_1^\beta p_n(t)(h_1(t))^{(n-1)\beta}(P_{n-1}(h_1(t)))^\beta(y_n(h_n(t)))^{\alpha\beta}\leqslant \\ &\leqq -C_1^\beta p_n(t)(h_1(t))^{(n-1)\beta}(P_{n-1}(h_1(t)))^\beta(y_n(t))^{\alpha\beta} \\ &\quad \text{for } t\geqq t_3=\gamma(t_2)\,. \end{aligned} \tag{21}$$

In (21) we have used the fact that $y_n(t)$ is nonincreasing.

Dividing (21) by $(y_n(t))^{\alpha\beta}$ and then integrating from t_3 to t, we obtain

$$\frac{(y_n(t))^{1-\alpha\beta}-(y_n(t_3))^{1-\alpha\beta}}{1-\alpha\beta}\leqq -C_1^\beta\int_{t_3}^t p_n(s)(P_{n-1}(h_1(s)))^\beta(h_1(s))^{(n-1)\beta}\,\mathrm{d}s$$

From the last inequality we get

$$C_1^\beta \int_{t_3}^{\infty} p_n(s)(h_1(s))^{(n-1)\beta}(P_{n-1}(h_1(s)))^\beta \, ds \leqq \frac{(y_n(t_3))^{1-\alpha\beta}}{1-\alpha\beta} < \infty ,$$

which contradicts (20).

II. Let $l=1$ (m is odd). Then by (5) the function $y_1(t)$ is nonincreasing and with regard to $y_1(t)>0$ it follows that $\lim_{t\to\infty} y_1(t)=\delta\geqq 0$. We suppose that $\delta>0$. Therefore there exists a $K>0$ such that

$$\inf_{t\geqq 2t_0} \frac{y_1(t)}{y_1(t/2)} = K. \tag{22}$$

From (17) we get for $i=1$ with the help of (22)

$$y_1(t) = \frac{y_1(t)}{y_1(t/2)} y_1(t/2) \geqq KC_1 t^{n-1} P_{n-1}(t)(y_n(t))^\alpha$$

$$\text{for } t\geqq t_1' \geqslant 2t_1 .$$

Procceding further in the same way as in case I, we get a contradiction to (20). Then $\lim_{t\to\infty} y_1(t)=0$ and by Lemma 3 we have $\lim_{t\to\infty} y_k(t)=0$ for $k=1, 2, \ldots, n$.

Theorem 3 extends the results of Ševelo and Varech [5, Theorem 2].

Theorem 4. *Suppose that* (11) *and* (12) *hold. In addition there exists a differentiable function* $g\colon [a,\infty)\to R$ *such that*

$$g'(t)\geqq 0,\ 0\leqq g(t)\leqq h_1(t) \text{ for } t\geqq T\geqq a. \tag{23}$$

If $\alpha=1$, $\beta>1$ *and*

$$\int_T^\infty p_n(t)\int_T^t (g(s))^{n-2} P_{n-1}(g(s)) g'(s) \, ds \, dt = \infty , \tag{24}$$

then the system (S) *has the property A.*

Proof. Suppose that the system (S) has a nonoscillatory solution $y=(y_1, \ldots, y_n)\in \mathbf{W}$. We suppose that $y_1(t)>0$ for $t\geqq t_0$. Proceeding in the same way as in the proof of Theorem 2 we get (4), (5) and (17). With regard to $y_1(t)>0$, (4) and (5) we have either

$$y_2(t)>0 \text{ or } y_2(t)<0 \text{ for } t\geqq t_1>t_0 .$$

I. Let $y_2(t)>0$ for $t\geqq t_1$. Then the 1*st* equation of (S) implies that $y_1'(t)\geqq 0$ for $g\geqq t_2' = \bar\gamma(t_1)$, where $\bar\gamma(t)=\max(\gamma_n(t), \sup\{s;\, g(t)<t\})$ for $t\geqq a$.

We define the function z as follows

$$z(t) = -y_n(t) \int_{t_2}^{t} \frac{(g(s))^{n-2} g'(s) P_{n-1}(g(s))}{(y_1(g(s)))^{\beta}} \, ds \tag{25}$$

$$\text{for } t \geqq t_2 = \max\{T, \bar{\gamma}(t_1)\}.$$

It is evident that

$$z(t) < 0 \quad \text{for} \quad t > t_2. \tag{26}$$

In view of the n-th intequality of (S), (23) and the monotonicity of y_1 we get from (25) the following

$$z'(t) \geqq p_n(t)(y_1(h_1(t)))^{\beta} \int_{t_2}^{t} \frac{(g(s))^{n-2} g'(s) P_{n-1}(g(s))}{(y_1(g(s)))^{\beta}} \, ds -$$

$$- y_n(t) \frac{(g(t))^{n-2} g'(t) P_{n-1}(g(t))}{(y_1(g(t)))^{\beta}} \geqq$$

$$\geqq p_n(t) \int_{t_2}^{t} (g(s))^{n-2} g'(s) P_{n-1}(g(s)) \, ds -$$

$$- \frac{y_n(g(t))}{(y_1(g(t)/2))^{\beta}} (g(t))^{n-2} g'(t) P_{n-1}^{2}(g(t)) p_1(g(t)/2).$$

If we use (17) for $i=2$, $\alpha=1$ and we substitute $g(t)$ for t, then from the last inequality we obtain

$$z'(t) \geqq p_n(t) \int_{t_2}^{t} (g(s))^{n-2} g'(s) P_{n-1}(g(s)) \, ds - \tag{27}$$

$$- \frac{y_2(g(t)/2) g'(t) p_1(g(t)/2)}{C_2 (y_1(g(t)/2))^{\beta}}.$$

Using the 1st equation of (S) and then integrating (27) from t_2 to t, we obtain

$$z(t) \geqq z(t_2) + \int_{t_2}^{t} p_n(x) \int_{t_2}^{x} (g(s))^{n-2} g'(s) P_{n-1}(g(s)) \, ds \, dx -$$

$$- \frac{2 y_1(g(t_2/2))^{1-\beta}}{C_2(\beta - 1)}.$$

In view of (24) the last inequality implies $\lim_{t\to\infty} z(t) = \infty$, which contradicts (26).

II. Let $y_2(t) < 0$ for $t \geqq t_1$. The first equation of (S) implies that $y_1(t)$ is a nonincreasing function. Then in view of $y_1(t) > 0$ it follows that $\lim_{t\to\infty} y_1(t) = \delta \geqq 0$. We suppose that $\delta > 0$.

We now define the function w as follows:

$$w(t)=-y_n(t)\int_{t_2}^{t}(g(s))^{n-2}g'(s)P_{n-1}(g(s))\,\mathrm{d}s,\quad t\geqq t_2. \tag{28}$$

It is clear that $w(t)<0$ for $t\geqq t_2$.

Using the n-th inequality of (S), the monotonicity of y_1 and (17) for $i=2$, we obtain from (28):

$$\begin{aligned} w'(t)\geqq\ & p_n(t)(y_1(h_1(t)))^{\beta}\int_{t_2}^{t}(g(s))^{n-2}g'(s)P_{n-1}(g(s))\,\mathrm{d}s- \\ & -y_n(t)(g(t))^{n-2}g'(t)P_{n-1}(g(t))\geqq \\ & \geqq\delta^{\beta}p_n(t)\int_{t_2}^{t}(g(s))^{n-2}g'(s)P_{n-1}(g(s))\,\mathrm{d}s+ \\ & +\frac{1}{C_2}\,y_2(g(t)/2)g'(t)p_1(g(t)/2). \end{aligned} \tag{29}$$

Integrating (29) from t_2 to t, we get

$$\begin{aligned} w(t)\geqq w(t_2)+\delta^{\delta}\int_{t_2}^{t}p_n(x)\int_{t_2}^{x}(g(s))^{n-2}g'(s)P_{n-1}(g(s))\,\mathrm{d}s\,\mathrm{d}x- \\ -\frac{2}{C_2}\,y_1(g(t_2)/2). \end{aligned}$$

In view of (24) the last inequality implies $\lim\limits_{t\to\infty}w(t)=\infty$, which contradicts $w(t)<0$ for $t\leqq t_2$. Therefore $\delta=0$, i.e. $\lim\limits_{t\to\infty}y_1(t)=0$. Then by Lemma 3 we have $\lim\limits_{t\to\infty}y_k(t)=0$ for $k=1,2,\ldots,n$.

Remark 2. Consider now the scalar equation

$$y^{(n)}(t)+p_n(t)\,|y(h_1(t))|^{\beta}\operatorname{sgn}y(h_1(t))=0,\quad n\geqq 2,\ \beta>1, \tag{E}$$

which is a special case of the system (S).

It is easy to prove that

$$\int_T^{\infty}p_n(t)\int_T^{t}(g(s))^{n-2}g'(s)\,\mathrm{d}s\,\mathrm{d}t=\infty$$

iff

$$\int_T^{\infty}p_n(t)(g(t))^{n-1}\,\mathrm{d}t=\infty.$$

Then from Theorem 3 we get the following very wel-known

Corollary. *Suppose that* (12), (23) *hold. If*

$$\int_T^\infty p_n(t)(g(t))^{n-1}\,\mathrm{d}t=\infty,$$

then every solution of (E) *is oscillatory if n is even while for n odd it is either oscillatory or tends monotonically to zero as* $t\to\infty$.

Theorem 5. *Suppose that* (11), (12) *and* (23) *hold. In addition we assume that* $\alpha\beta>1$. *If*

$$\int_T^\infty p_n(t)\,\mathrm{d}t<\infty$$

and

$$\int_T^\infty (g(t))^{n-2}g'(t)P_{n-1}(g(t))\Big(\int_t^\infty p_n(s)\,\mathrm{d}s\Big)^\alpha\,\mathrm{d}t=\infty, \tag{30}$$

then the system (S) *has the property A.*

Proof. Let $y=(y_1,\ldots,y_n)\in\mathbf{W}$ be a nonoscillatory solution of (S). Proceeding in the same way as in the proof of Theorem 4 we get (4), (5) and (17). We may suppose that $y_1(t)>0$ for $t\geqq t_1$. Integrating the n-th inequality of (S) from $t(\geqq t_2=\gamma(t_1))$ to τ, we get

$$y_n(\tau)-y_n(t)\leqq-\int_t^\tau p_n(s)(y_1(h_1(s)))^\beta\,\mathrm{d}s,$$

and then we have for $\tau\to\infty$

$$y_n(t)\geqq\int_t^\infty p_n(s)(y_1(h_1(s)))^\beta\,\mathrm{d}s,\quad t\geqslant t_2. \tag{31}$$

I. Let $l\geqq 2$. Since y_1 is nondecreasing and y_n is nonincreasing, (31) implies

$$(y_n(g(t)))^\alpha\geqq(y_1(g(t)))^{\alpha\beta}\Big(\int_t^\infty p_n(s)\,\mathrm{d}s\Big)^\alpha,\quad t\geqq t_3=\bar\gamma(t_2).$$

From the last inequality we obtain in view of (17) for $i=2$ and the monotonicity of y_1

$$y_2(g(t)/2)\geqq C_2(g(t))^{n-2}P_{n-1}^2(g(t))(y_1(g(t)/2))^{\alpha\beta}\Big(\int_t^\infty p_n(s)\,\mathrm{d}s\Big)^\alpha. \tag{32}$$

Multiplying (32) by $g'(t)p_1(g(t)/2)(y_1(g(t)/2))^{-\alpha\beta}$ and using the 1st equation of (S), we get

$$\frac{y_1'(g(t)/2)g'(t)}{(y_1(g(t)/2))^{\alpha\beta}}\geqq C_2(g(t))^{n-2}g'(t)P_{n-1}(g(t))\Big(\int_t^\infty p_n(s)\,\mathrm{d}s\Big)^\alpha.$$

Integrating the last inequality from t_3 to u, we obtain

$$\frac{2}{\alpha\beta-1}(y_1(g(t_2)/2))^{1-\alpha\beta}\geqq$$

$$\geqq C_2\int_{t_2}^{u}(g(t))^{n-2}g'(t)P_{n-1}(g(t))\left(\int_t^{\infty}p_n(s)\,\mathrm{d}s\right)^{\alpha}\mathrm{d}t,$$

which contradicts (30) as $u\to\infty$.

II. Let $l=1$. According to Lemma 5, $y_1(t)>0$ for $t\geqq t_1$, we get from the 1st equation of (S) $y_2(t)<0$, $y_1'(t)\leqq 0$ for $t\geqq t_1$. Therefore $\lim_{t\to\infty} y_1(t)=\delta\geqq 0$. We suppose that $\delta>0$. Then, in view of the monotonicity of y_n, y_1 we obtain from (31):

$$(y_n(g(t)))^{\alpha}\geqq\delta^{\alpha\beta}\left(\int_t^{\infty}p_n(s)\,\mathrm{d}s\right)^{\alpha},\quad t\geqq t_4=\max\{T,t_3\}.$$

If we use (17) for $i=2$, we get from the last inequality

$$-y_2(g(t)/2)\geqq C_2\delta^{\alpha\beta}(g(t))^{n-2}P_{n-1}^2(g(t))\left(\int_t^{\infty}p_n(s)\,\mathrm{d}s\right)^{\alpha}\tag{33}$$

for $t\geqq t_4$.

Multiplying (33) by $p_1(g(t)/2)g'(t)$ and using the 1st equation of (S), we obtain

$$-y_1'(g(t)/2)g'(t)\geqq C_2\delta^{\alpha\beta}(g(t))^{n-2}g'(t)P_{n-1}(g(t))\left(\int_t^{\infty}p_n(s)\,\mathrm{d}s\right)^{\alpha}.\tag{34}$$

Integrating (34) from t_4 to u, we obtain

$$2y_1(g(t_4)/2)\geqq$$

$$\geqq C_2\delta^{\alpha\beta}\int_{t_4}^{u}(g(t))^{n-2}g'(t)P_{n-1}(g(t))\left(\int_t^{\infty}p_n(s)\,\mathrm{d}s\right)^{\alpha}\mathrm{d}t,$$

which contradicts (30) as $u\to\infty$.

Therefore $\delta=0$, i.e. $\lim_{t\to\infty} y_1(t)=0$. Then in view of Lemma 3 we have $\lim_{t\to\infty} y_k(t)=0$ for $k=1,2,\ldots,n$.

The proof of Theorem 5 is complete.

This Theorem generalizes Theorem 5 [5].

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Received December 4, 1981

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О КОЛЕБЛЕМОСТИ РЕШЕНИЙ НЕЛИНЕЙНЫХ СИСТЕМ С ЗАПАЗДЫВАНИЕМ

Pavol Marušiak

Резюме

В статье приведены достаточные условия колеблемости решений системы (S) и системы

$$y_i'(t) = p_i(t) y_{i+1}(t), \qquad i = 1, 2, \ldots, n-2,$$

$$y_{n-1}'(t) = P_{n-1}(t) |y_n(h_n(t))|^{\alpha} \operatorname{sgn} y_n(h_n(t)),$$

$$y_n'(t) \operatorname{sgn} y_1(h_1(t)) \leq -p_n(t) |y_1(h_1(t))|^{\beta}, \qquad 0 < \alpha, 0 < \beta.$$