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INTEGRAL WITH RESPECT TO A PRE-MEASURE

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Introduction

In the classical definition of the Riemann integral the value of the integral is defined as the limit of the Riemann integral sums. The concept of a Riemann integral sum is based on a partition of the domain into sets of a comparatively simple shape.

Lebesgue introduced the concept of a measure and thus was able to suggest a new definition of the integral. The concept of the Lebesgue integral sum is based on a possibility of immensly "rich" partitions of the domain.

In cases of non additive set functions, however, Riemann's and Lebesgue's methods are of not much help, because they are essentially based on the possibility of forming partitions of the domain, and on the additivity of some set functions.

In this paper we propose to define a process of integration with respect to a pre-measure. The pre-measure is a natural generalization of a nonnegative additive measure. In fact it is a monotone set function vanishing on the empty set and defined on a family of subsets of some space which contains the empty set.

An important type of pre-measures, the so called subadditive measures, were studied in [1], [2], [3], [4] and [7]. The most important examples of pre-measures are, however, the nonnegative capacities vanishing in the empty set [6].

§ 1 is introductory. In § 2 we introduce a measurability of real functions defined on a pre-measurable space $(X, \mathscr{D})$ and investigate their properties. In § 3 we introduce the notion of the integral $\mathscr{I}_\mu$ with respect to a pre-measure μ and show that $\mathscr{I}_\mu$ is monotone, homogeneous and additive in a horizontal sense, i.e.

$$\mathscr{I}_\mu f = \mathscr{I}_\mu(f \wedge a) + \mathscr{I}_\mu(f - f \wedge a) \quad \text{if} \quad a \geqq 0.$$

It is shown further that if $\mathscr{D}$ is a σ-ring and μ is a σ-additive measure on $\mathscr{D}$, then our integral coincides with the Lebesgue integral. § 4 contains the limit theorems, namely the Beppo-Levi and the Lebesgue theorems and Fatou's lemma for a continuous pre-measure.

§ 1. Basic notation

We explain certain notions used throughout the present paper; specific terms will be explained when they appear for the first time. The terminology is essentially standard.

Definitions will be written as direct statements with the defined concept in italics.

We denote by R the set of all real numbers, $\bar{R}$ is the compactified real line, i.e. $\bar{R}=R\cup\{\infty, -\infty\}$. If $B\subset\bar{R}$, we put $B^+=B\cap\bar{R}^+$ and $B^-=B\cap\bar{R}^-$, (where $\bar{R}^+=\langle 0, \infty\rangle$ and $\bar{R}^-=\langle -\infty, 0\rangle$). Furthermore, $B+a=\{x+a\,;\ x\in B\}$ and $a.B=\{a\cdot x: x\in B\}$.

$\mathscr{F}$ denotes the family of all finite subsets of $\bar{R}$ which contain zero. Further

$$\mathscr{F}^+=\{F^+\,;\,F\in\mathscr{F}\}\quad\text{and}\quad\mathscr{F}^-=\{F^-\,;\,F\in\mathscr{F}\}.$$

Recall that the families $\mathscr{F}$, $\mathscr{F}^+$ and $\mathscr{F}^-$ ordered by the inclusion form directed sets.

For $F\in\mathscr{F}$ we write

$$\|F\|=\min\{|a-b|\,;\ a, b\in F, a\neq b\}.$$

A *real net* is a triple $(S, \geqq, D)$, where $(D, \geqq)$ is a directed set and S is a real function with the domain D. Throughout this paper we consider X to be a fixed set with respect to which we make definitions.

Further fixed symbols: For $A\subset X$ the symbol χ_A denotes the characteristic function of the set A.

We denote by $\vee$ and $\wedge$ the lattice operations on real functions, i.e.

$$(f\vee g)(x)=\max\{f(x), g(x)\},$$
$$(f\wedge g)(x)=\min\{f(x), g(x)\}.$$

and we put $f^+=f\vee 0$ and $f^-=-(f\wedge 0)$.

For an extended real valued function f on X we put

$$S_f=\{x\in X;\ f(x)\neq 0\}.$$

If $f: X\to\bar{R}$ and $F\in\mathscr{F}$, we put

$$f_F=\sum_{i=1}^{n}(a_i-a_{i-1})\chi_{A_i}+\sum_{j=1}^{m}(b_j-b_{j-1})\chi_{B_j},$$

where $F=\{b_m, \ldots, b_1, 0, a_1, \ldots, a_n\}$ with

$$b_m<\ldots<b_1<b_0=0=a_0<a_1<\ldots<a_n,$$
$$A_i=\{x\,;\,f(x)\geqq a_i\}\quad\text{and}\quad B_j=\{x\,;\,f(x)\leqq b_j\}.$$

We put $\inf\emptyset=\infty$ and $0.(\pm\infty)=0$.

A *pre-measurable* space is a pair $(X, \mathscr{D})$, where $\mathscr{D}$ is a family of subsets of X and $\emptyset\in\mathscr{D}$. The members of $\mathscr{D}$ are called measurable sets. A *pre-measure* μ is a monotone extended real valued set function defined on $\mathscr{D}$ with $\mu(\emptyset)=0$.

§ 2. Measurability

In this paragraph we shall deal with the pre-measurable space $(X, \mathscr{D})$. We say that an extended real valued function $f: X \to \bar{R}$ is *$\mathscr{D}$-measurable* or only *measurable* iff the sets $\{x\,; f(x) \geqq a\}$ and $\{x\,; f(x) \leqq -a\}$ are in $\mathscr{D}$ for every positive element a in $\bar{R}^+$.

If $\mathscr{D}$ is a σ-ring, then this notion of measurability coincides with the ordinary one defined, e.g. in [5].

By $\mathscr{L}(\mathscr{D})$ we denote the family of all $\mathscr{D}$-measurable extended real valued functions on X.

Proposition 1. *If $f \in \mathscr{L}(\mathscr{D})$, then $f \wedge a$, $f - f \wedge a$, $f \vee (-a)$, $f - f \vee (-a)$, f^+, f^- and $c \cdot f$ are from $\mathscr{L}(\mathscr{D})$ for every positive a from R and for every real c.*

Proof. Let $b > 0$, then

$$\{x\,; f(x) \wedge a \geqq b\} = \begin{cases} \{x\,; f(x) \geqq b\} & \text{if } a \geqq b \\ \emptyset & \text{if } a < b \end{cases}$$

and

$$\{x\,; f(x) \wedge a \leqq -b\} = \{x\,; f(x) \leqq -b\}.$$

Hence $f \wedge a$ is measurable. The proofs of the other assertions are analogous.

Proposition 2. *Let $f \in \mathscr{L}(\mathscr{D})$ and $F \in \mathscr{F}$; then f_F is in $\mathscr{L}(\mathscr{D})$.*

Proof. Let $a > 0$. If $\{x\,; f_F(x) \geqq a\} \neq \emptyset$, put $b = \min\{c \in F;\ c \geqq a\}$. Then $\{x\,; f_F(x) \geqq a\} = \{x\,; f(x) \geqq b\}$.

A *simple function* is a $\mathscr{D}$-measurable function with a finite range.

Proposition 3. *If $f \in \mathscr{L}(\mathscr{D})$, then there exists a sequence of simple functions $\{f_n\}$ in $\mathscr{L}(\mathscr{D})$ such that f_n converges pointwise to f on X.*

Proof. Let f be a $\mathscr{D}$-measurable function. Put

$$F_n = \{i/2^n\,; i = 0, \pm 1, \pm 2, \ldots, \pm n \cdot 2^n\}$$

and $f_n = f_{F_n}$. If $|f(x)| < n$ then $|f(x) - f_n(x)| \leqq 1/2^n$. If $|f(x)| = \infty$, then $|f_n(x)| \geqq n$ for every n and hence

$$f(x) = \lim_n f_n(x).$$

For our considerations we shall need some properties of measurable functions in the case when $\mathscr{D} = \mathscr{C}$ is a lattice or σ-lattice of the subsets of X. In this case $\mathscr{L}(\mathscr{C})$ has some further interesting properties.

Proposition 4. *$\mathscr{L}(\mathscr{C})$ is a lattice. If $\mathscr{C}$ is a σ-lattice, then $f, g \in \mathscr{L}(\mathscr{C})$ and $f, g \geqq 0$ implies $f + g \in \mathscr{L}(\mathscr{C})$. Moreover $\mathscr{L}(\mathscr{C})$ is a σ-lattice.*

Proof. Since $\{x;\ (f \wedge g)(x) \leqq -a\} = \{x;\ f(x) \leqq -a\} \cup \{x;\ g(x) \leqq -a\}$ and

$$\{x\,; (f \wedge g)(x) \geqq a\} = \{x\,; f(x) \geqq a\} \cap \{x\,; g(x) \geqq a\},$$

the first assertion is trivial. Let now f, g be non-negative $\mathscr{C}$-measurable functions, $\mathscr{C}$ be a σ-lattice and let $a>0$. Then

$$\{x\,;f(x)+g(x)\geqq a\}=\bigcap_{\substack{0<t<a\\ t\text{-rational}}}\{x\,;f(x)+g(x)>t\},$$

but

$$\{x\,;f(x)+g(x)>t\}=\bigcup_{\substack{0<r,\,0<s,\,r+s>t\\ r,\,s\text{ rational}}}[\{x\,;f(x)\geqq r\}\cap$$

$$\{x\,;g(x)\geqq s\}]\cup\bigcup_{\substack{r>t\\ r\text{-rational}}}[\{x\,;f(x)\geqq r\}\cup\{x\,;g(x)\geqq r\}]$$

and so $f+g$ is $\mathscr{C}$-measurable.

Let $\{f_n\}$ be a sequence of $\mathscr{C}$-measurable functions. Denote $f=\vee_n f_n$; then for $a>0$ we have

$$\{x\,;f(x)\geqq a\}=\bigcap_{\substack{0<r<a\\ r\text{-rational}}}\cup_n\{x\,;f_n(x)\geqq r\}\in\mathscr{C}$$

and similarly $\{x\,;f(x)\leqq -a\}\in\mathscr{C}$.

The proof of the measurability of $f\vee g$ and $\wedge_n f_n$ is similar.

§ 3. The integral

In this paragraph μ will be a pre-measure. Let $F\in\mathscr{F}$ and $F=\{b_m,\ldots,b_1,0,a_1,\ldots,a_n\}$, where

$$b_m<\ldots<b_1<b_0=0=a_0<a_1<\ldots<a_n.$$

The *integral sum* $S(f,F)$ of the measurable function f with respect to the set F is defined as follows

$$S(f,F)=\sum_{i=1}^{n}(a_i-a_{i-1})\mu(\{x\,;f(x)\geqq a_i\})+$$

$$+\sum_{j=1}^{m}(b_j-b_{j-1})\mu(\{x\,;f(x)\leqq b_j\})$$

whenever the right-hand side contains no expression of the type $\infty-\infty$.

A $\mathscr{D}$-measurable function f is *integrable* whenever the net $(S(f,F),\supset,\mathscr{F})$ is convergent.

The *integral* of a measurable (not necessarily integrable) function f, in symbol $\mathscr{I}f$, $\mathscr{I}_\mu f$ or $\int f\,\mathrm{d}\mu$ is defined by

$$\mathscr{I}f=\lim_{F\in\mathscr{F}}S(f,F)$$

if the limit exists.

Theorem 5. *Let f be a $\mathscr{D}$-measurable function on X.*
(i) *If $f \geqq 0$, then $\mathscr{I}f$ exists and $\mathscr{I}f \geqq 0$. Moreover in this case we have*

$$\mathscr{I}f = \sup_{F \in \mathscr{F}} S(f, F).$$

(ii) *$\mathscr{I}$ is a monotone functional.*
(iii) *If $\mathscr{I}f$ exists, then for every real c*

$$\mathscr{I}(c \cdot f) = c \cdot \mathscr{I}f$$

(iv) *If $a \geqq 0$, then*

$$\mathscr{I}f = \mathscr{I}(f \wedge a) + \mathscr{I}(f - f \wedge a)$$

if one of the right-hand side expression is finite.
(v) *If $\mathscr{I}f^+$ or $\mathscr{I}f^-$ is finite, then $\mathscr{I}f$ exists and $\mathscr{I}f = \mathscr{I}f^+ - \mathscr{I}f^-$. Moreover if f is integrable, then the last equality holds too, and f^+, f^- are integrable too.*

For proving Theorem 5 we need some properties of the integral sum $S(f, F)$.

Lemma 6. *Let f be an integrable function and let $F \in \mathscr{F}$, then we have*
(i) *$S(f, F) = S(f, F^+) + S(f, F^-)$.*
(ii) *If $F_1, F_2 \in \mathscr{F}$ with $F_1 \subset F_2$, then $S(f, F_1^+) \leqq S(f, F_2^+)$ and $S(f, F_1^-) \geqq S(f, F_2^-)$.*

Proof. The first part is a trivial consequence of the definition of the integral sum.

Let $F = \{0, a_1, a_2, \ldots, a_n\}$ and let $F^* = F \cup \{a\}$, where

$$0 = a_0 < a_1 < \ldots < a_{j-1} < a < a_j < \ldots < a_n.$$

We prove first that

$$S(f, F) \leqq S(f, F^*).$$

Denote

$$y = \sum_{\substack{i=1 \\ i \neq j}}^{n} (a_i - a_{i-1}) \mu(\{x\,;\, f(x) \geqq a_i\})\,;$$

then

$$\begin{aligned} S(f, F) = y + (a_j - a_{j-1}) \mu(\{x\,;\, f(x) \geqq a_j\}) \leqq \\ \leqq y + (a_j - a) \mu(\{x\,;\, f(x) \geqq a_j\}) + \\ + (a - a_{j-1}) \mu(\{x\,;\, f(x) \geqq a\}) = S(f, F^*). \end{aligned}$$

In the foregoing reasoning we have used the monotonicity of μ and the fact that $\{x\,;\, f(x) \geqq a_j\} \subset \{x\,;\, f(x) \geqq a\}$. If now $F_1 \subset F_2$, then $F_1^+ \subset F_2^+$. Let $c_1, c_2, \ldots, c_n$ be such real numbers that $F_2^+ = F_1^+ \cup \{c_1, c_2, \ldots, c_n\}$. Then from the first part of this proof we have

$$S(f, F_1^+) \leqq S(f, F_1^+ \cup \{c_1\}) \leqq \ldots \leqq S(f, F_2^+).$$

The proof of $S(f, F_1^-) \geqq S(f, F_2^-)$ is similar.

Lemma 7.

(i) $S(c\cdot f, F)=c\cdot S(f, (1/c)\cdot F)$ *for* $c\neq 0$.

(ii) $S(f, F)=S(f^+, F)-S(f^-, -F)$.

(iii) $S(f, F)=S(f\wedge a, F)\ +\ S(f-f\wedge a, F-a)$ *if* $a\in F$ *and* $a\geqq 0$.

Proof. (i) follows from the equalities

$$\{x\,;\,c\cdot f(x)\geqq a\}=\{x\,;\,f(x)\geqq a/c\}$$

and

$$\{x\,;\,c\cdot f(x)\leqq b\}=\{x\,;\,f(x)\leqq b/c\}$$

for $c>0$ and from the similar equalities for $c<0$.

(ii) If $f\geqq 0$ ($f\leqq 0$), then $S(f, F^-)=0$ ($S(f, F^+)=0$) for every $F\in\mathscr{F}$. From this, from the definition of f^+ and f^- and from the first part of this lemma it follows that

$$S(f, F)=S(f^+, F^+)-S(f^-, -F^-).$$

(iii) by Lemma 6 (i)

$$S(f, F)=S(f, F^+)+S(f, F^-).$$

Thus it is enough to prove the asserion if $F\in\mathscr{F}^+$. Let $F=\{0=a_0<a_1<\ \dots\ <a_n\}$. Then

$$S(f, F)=\sum_{i=1}^{n}(a_i-a_{i-1})\mu(\{x\,;\,f(x)\geqq a_i\})=\sum_{a_i\leqq a}+\sum_{a_i>a}$$

However,

$$\sum_{a_i\leqq a}=\sum_{a_i\leqq a}(a_i-a_{i-1})\mu(\{x\,;\,f(x)\geqq a_i\})+$$
$$+\sum_{a_i>a}(a_i-a_{i-1})\mu(\{x\,;\,(f\wedge a)(x)\geqq a_i\})=$$
$$=\sum_{i=1}^{n}(a_i-a_{i-1})\mu(\{x\,;\,(f\wedge a)(x)\geqq a_i\})=S(f\wedge a, F).$$

On the other hand

$$\sum_{a_i>a}=\sum_{a_i>a}(a_i-a-(a_{i-1}-a))\mu(\{x\,;\,(f-f\wedge a)(x)\geqq$$
$$\geqq a_i-a\})=S(f-f\wedge a, F).$$

And so we get

$$S(f, F)=\sum_{a_i\leqq a}+\sum_{a_i>a}=S(f\wedge a, f)+S(f-(f\wedge a), F).$$

Proof of Theorem 5. (i) is a simple conclusion of Lemma 6 (ii) and the fact that monotone real valued nets have always limits (finite or infinite). (ii) follows from the relations

$$\mu(\{x\,;f(x)\geqq a\})\leqq\mu(\{x\,;g(x)\geqq a\})$$

and

$$-\mu(\{x\,;f(x)\leqq -a\}\leqq -\mu(\{x\,;g(x)\leqq -a\})$$

if $f\leqq g$ and $a>0$.

If $c=0$, then (iii) is trivial. If $c\neq 0$, then the assertion follows from (i) of Lemma 7.

(iv) Let $\mathscr{I}(f\wedge a)$ be finite. Choose $F_1\in\mathscr{F}$ with $a\in F_1$ and such that for $F\supset F_1$ we have

$$|S(f\wedge a, F)-\mathscr{I}(f\wedge a)|<\varepsilon$$

$f-f\wedge a$ is a non-negative measurable function, hence by (i) of this theorem $\mathscr{I}(f-f\wedge a)$ exists. Choose $F_2\in\mathscr{F}^+$ such that for any $F\supset F_2$ we have

$$S(f-f\wedge a, F)\geqq n \quad \text{in the case} \quad \mathscr{I}(f-f\wedge a)=\infty$$

and

$$|\mathscr{I}(f-f\wedge a)-S(f-f\wedge a, F)|<\varepsilon \quad \text{if} \quad \mathscr{I}(f-f\wedge a)<\infty.$$

Let $F_0=F_1\cup(F_2+a)$ and let $F\supset F_0$. Then

$$\begin{aligned}S(f, F)=S(f\wedge a, F)+S(f-f\wedge a, F-a)\geqq\\ \geqq n+\mathscr{I}(f\wedge a)-\varepsilon\end{aligned}$$

if $\mathscr{I}(f-f\wedge a)=\infty$ and in the other case

$$\begin{aligned}&|S(f, F)-\mathscr{I}(f-f\wedge a)-\mathscr{I}(f\wedge a)|\leqq\\ &\leqq|S(f-f\wedge a, F-a)-\mathscr{I}(f-f\wedge a)|+\\ &+|S(f\wedge a, F)-\mathscr{I}(f\wedge a)|\leqq 2\varepsilon,\end{aligned}$$

since $F-a\supset F_2$. And so $\mathscr{I}f$ exists and

$$\mathscr{I}f=\mathscr{I}(f\wedge a)+\mathscr{I}(f-f\wedge a).$$

The proof for the infinite $\mathscr{I}(f\wedge a)$ is similar.

(v) Put $a=0$ in (iv); then

$$\mathscr{I}f=\mathscr{I}(f\wedge 0)+\mathscr{I}(f-f\wedge 0)=\mathscr{I}(-f^-)+\mathscr{I}f^+=\mathscr{I}f^+-\mathscr{I}f^-.$$

For the second part of (v) it is sufficient to prove that $\mathscr{I}f^+$ and $\mathscr{I}f^-$ are finite. Let $F_0\in\mathscr{F}$ be such that

$$|S(f, F)-\mathscr{I}f|<\varepsilon \quad \text{for} \quad F\supset F_0.$$

Choose F with $F^-=F_0^-$.

Then by (ii) of Lemma 7 we have $S(f, F) = S(f^+, F) - S(f^-, -F) = S(f^+, F) - S(f^-, -F_0)$ and so

$$S(f^+, F^+)=S(f^+, F)\leqq S(f^-, -F_0)+\mathscr{I}f+\varepsilon$$

for every $F\in\mathscr{F}$, $F\supset F_0$, $F^-=F_0^-$. Since $\mathscr{I}f^+=\sup_{F\in\mathscr{F}} S(f^+, F^+)$, we get $\mathscr{I}f^+<\infty$.

Since $\mathscr{I}(-f)=-\mathscr{I}f$ and $f^-=(-f)^+$, we get that $\mathscr{I}f^-$ is finite too.

Proposition 8. *Let f be an integrable function. Then*

$$\mu(\{x\,;f(x)=\infty\})=\mu(\{x\,;f(x)=-\infty\})=0.$$

Proof. We prove only $\mu(\{x\,;f(x)=\infty\})=0$. If f is integrable, then by (v) of Theorem 5 f^+ is integrable. Let $F_n=\{0, n\}$. Then, since

$$\{x\,;f^+(x)\geqq n\}\supset\{x\,;f(x)=\infty\},$$

we get

$$n\cdot\mu(\{x\,;f(x)\geqq\infty\})\leqq S(f^+, F_n)\leqq\mathscr{I}f^+<\infty,$$

hence

$$\mu(\{x\,;f(x)=\infty\})=0.$$

For our latter considerations we shall need the following notion. An *essential supremum* of f is

$$\operatorname{ess\,sup} f=\inf\{a\geqq 0\,;\mu(\{x\,;f(x)\geqq a\})=0\}.$$

Proposition 9. *Let f be a measurable function. If $\mathscr{I}f$ exists and $a=\operatorname{ess\,sup} f$, then*

$$\mathscr{I}f=\mathscr{I}(f\wedge a).$$

Proof. If $a=\infty$, then the proof is trivial. Let a be non-negative. The proof follows by

$$\{x\,;f(x)\geqq b\}=\{x\,;(f\wedge a)(x)\geqq b\}\quad\text{if}\quad a\geqq b$$

and

$$\mu(\{x\,;f(x)\geqq b\})=0\quad\text{if}\quad b>a.$$

Proposition 10. *Let f be a measurable function with $S_f\in\mathscr{D}$, $|f|\leqq c$ and $\mu(S_f)<\infty$. Then f is integrable and $|\mathscr{I}f|\leqq c\cdot\mu(S_f)$.*

Proof. Let $|f|\leqq c$. Then $f^+, f^-\leqq c\cdot\chi_{S_f}$. By the monotonicity of $\mathscr{I}$ and by (i) of Theorem 5 f^+, f^- are integrable and $\mathscr{I}f^+, \mathscr{I}f^-\leqq c\cdot\mu(S_f)$. The integrability of f follows now by (iv) of Theorem 5. The last assertion of the proposition is a conclusion of the mononicity of $\mathscr{I}$ and the following inequality

$$-c\cdot\chi_{S_f}\leqq f\leqq c\cdot\chi_{S_f}.$$

Proposition 11. *Let f and $|f|$ be measurable functions and let $\mathscr{I}f$ exist. Then*

$$|\mathscr{I}f|\leqq\mathscr{I}|f|.$$

If $|f|$ is integrable, then f is integrable too.

Proof. Let $\mathscr{I}f$ exists. Since $-f,\ f \leqq |f|$, by the monotonicity of $\mathscr{I}$ we get $\pm\mathscr{I}f \leqq \mathscr{I}|f|$.

The following example is an illustration of the fact that the integrability of f does not involve the integrability of $|f|$ even if μ is bounded.

Example 12. Let $X=(-1, 0)\cup(0, 1)$, $\mathscr{D}=2^X$; let further $\mu(A)=1$ if $A\cap(-1, 0)$ and $A\cap(0, 1)$ are nonempty and $\mu(A)=0$ otherwise. Then μ is a pre-measure on $\mathscr{D}$.

Take f on X defined by $f(x)=1/x$. Then $\mathscr{I}_\mu f=0$ and $\mathscr{I}_\mu|f|=\infty$.

Proposition 13. *Let f and $|f|$ be measurable functions. Let $|f|\leqq g$, where g is an integrable function. Then f is integrable.*

Proof. The proof is a conclusion of Proposition 11 and the fact that $|f|$ is integrable.

We give other properties of the integral in connection with the simple functions.

Proposition 14. *If f is a simple function with the range F and $\mathscr{I}f$ exists, then*

$$\mathscr{I}f=S(f, F).$$

If $\mathscr{D}$ is a σ-ring and μ is a σ-additive measure on $\mathscr{D}$ and f is a simple function, then

$$\mathscr{I}_\mu f=\int f\,\mathrm{d}\mu,$$

where $\int f\,\mathrm{d}\mu$ is the Lebesgue integral of the function f.

Proof. Suppose first that $f\geqq 0$. Let $F_1\in\mathscr{F}$ with $F_1\supset F$.

Since $\{x\,;f(x)\geqq c\} = \{x\,;f(x)\geqq a\}$ if $a=\min\{x\in F;x\geqq c\}$, in the case $\{x\in F; x\geqq c\}\neq\emptyset$ and $a=\max F$ in the case $\{x\in F;x\geqq c\}=\emptyset$, we have $S(f, F_1) = S(f, F)$ for $F_1\supset F$ and so

$$\mathscr{I}f=S(f, F).$$

The proof for a not necessarily positive function f follows by applying the result just proved separately to f^+ and f^- and (v) of Theorem 5. The proof of the second part is trivial.

Corollary 15. *Let $A_1\supset A_2\supset\ldots\supset A_n$ be measurable sets. Let c_i be positive real numbers and let $f_i=c_i\cdot\chi_{A_i}$ $(i=1, 2, \ldots, n)$. Then we have*

$$\mathscr{I}\left(\sum_{i=1}^n f_i\right)=\sum_{i=1}^n\mathscr{I}f_i.$$

Proposition 16. *Let $\mathscr{I}f^+$ or $\mathscr{I}f^-$ be finite. Then f is a limit of a sequence $\{f_n\}_{n=1}^\infty$ of simple functions and*

$$\mathscr{I}f=\lim_n\mathscr{I}f_n.$$

If f is non-negative, then f_n may be taken non-negative and the sequence $\{f_n\}$ may be assumed increasing.

Proof. Suppose first that $f \geqq 0$. It follows from (i) of Theorem 5 that $\mathscr{I}f$ exists. Let $\{G_n\}$ be a sequence of sets from $\mathscr{F}$ with

$$\lim_n S(f, G_n) = \mathscr{I}f.$$

We write

$$F_n = \bigcup_{j=1}^{n} G_j \cup \{i/2^n\,;\ i = 1, 2, \ldots, n \cdot 2^n\}$$

and put

$$f_n = f_{F_n}.$$

Clearly f_n is a non-negative simple function and the sequence $\{f_n\}$ is increasing. If $f(x) < n$, then

$$0 \leqq f(x) - f_n(x) \leqq 1/2^n.$$

If $f(x) = \infty$, then $f_n(x) \geqq n$ and so

$$\lim_n f_n(x) = f(x) \quad \text{for every} \quad x \in X.$$

Moreover

$$\mathscr{I}f = \lim_n S(f, G_n) = \lim_n S(f, F_n) = \lim_n \mathscr{I}f_n \leqq \mathscr{I}f.$$

Hence

$$\lim_n \mathscr{I}f_n = \mathscr{I}f.$$

The proof for a not necessarily positive function f follows if we apply the result just proved separately to f^+, and f^- and from (v) of Theorem 5.

Corollary 17. *If f is a non negative $\mathscr{D}$-measurable function, then*

$$\mathscr{I}f = \sup\{\mathscr{I}g\,;\ g \leqq f,\ g \ \text{is a simple function}\}.$$

The simple conclusion of the second part of Proposition 14 and the last corollary is the following:

Corollary 18. *If $\mathscr{D}$ is a σ-ring and μ is a σ-additive measure on $\mathscr{D}$, then*

$$\mathscr{I}_\mu f = \int f \, \mathrm{d}\mu$$

for every Lebesgue $\mu - \mathscr{D}$ — integrable function f.

§ 4. Limit theorems

A pre-measure μ on $\mathscr{D}$ is called *continuous* iff it has the following two properties:

(i) $A_n \nearrow A \supset B$ $(A_n, B \in \mathscr{D})$ implies $\lim_n \mu(A_n) \geqq \mu(B)$.

(ii) $A_n \searrow A \subset B$ $(A_n, B \in \mathscr{D})$, $\mu(A_1) < \infty$ implies $\lim_n \mu(A_n) \leqq \mu(B)$.

It is easy to see that if μ is a continuous pre-measure on $\mathscr{D}$, then $A_n, A \in \mathscr{D}$, $A_n \nearrow A$ or $A_n \searrow A$ $(\mu(A_1) < \infty)$ implies $\mu(A_n) \to \mu(A)$.

In this paragraph we shall assume that μ is a continuous pre-measure.

Proposition 19. *Let $\{f_n\}$ be the sequence of non negative integrable functions with $\mathcal{I}f_n \leqq c$ and let $f_n \nearrow f$. Then f is integrable $\mathcal{I}f \leqq c$ and*

$$\mathcal{I}f = \lim_n \mathcal{I}f_n.$$

Proof. Let $\varepsilon > 0$. Let $F = \{0 = a_0 < a_1 < \ldots < a_k\} \in \mathcal{F}$. Choose δ with

$$\delta c/(a_1 - \delta) < \varepsilon$$

and

$$0 < 2\delta < \min\{a_j - a_{j-1};\, j = 1, 2, \ldots, k\}.$$

Denote $b_0 = 0$, $b_i = a_i - \delta$ $(i = 1, 2, \ldots, k)$. From

$$\cup_n \{x\,;\, f_n(x) \geqq a_i - \delta\} \supset \{x\,;\, f(x) \geqq a_i\}$$

and from the continuity of μ we get

$$\begin{aligned} S(f, F) &= \sum_{i=1}^{k} (a_i - a_{i-1})\mu(\{x\,;\, f(x) \geqq a_i\}) \leqq \\ &\leqq \lim_n \sum_{i=1}^{k} (a_i - a_{i-1})\mu(\{x\,;\, f_n(x) \geqq a_i - \delta\}) = \\ &= \lim_n \Big[\sum_{i=1}^{k} (b_i - b_{i-1})\mu(\{x\,;\, f_n(x) \geqq b_i\}) + \\ &\qquad + \delta \cdot \mu(\{x\,;\, f_n(x) \geqq a_1 - \delta\})\Big] \leqq \\ &\leqq \lim_n \mathcal{I}f_n + \frac{\delta}{a_1 - \delta} \lim_n \mathcal{I}f_n \leqq \lim_n \mathcal{I}f_n + \varepsilon. \end{aligned}$$

From this we have

$$\mathcal{I}f = \sup_{F \in \mathcal{F}^+} S(f, F) \leqq \lim_n \mathcal{I}f_n + \varepsilon.$$

Since ε was arbitrary, we get

$$\mathcal{I}f \leqq \lim_n \mathcal{I}f_n.$$

The opposite inequality follows by the monotonicity of $\mathcal{I}$.

For the proof of the Lebesque-like theorem we shall need some lemmas.

Lemma 20. *Let f be a nonnegative integrable function, then*

$$\lim_{A \to \infty} \mathcal{I}(f - f \wedge A) = 0.$$

Proof. Let $F \in \mathcal{F}$, $\varepsilon > 0$ and let $\mathcal{I}f - S(f, F) < \varepsilon$. Clearly $S(f, F) < \infty$, then $f_F \leqq f \wedge a \leqq f$. By the monotonicity of $\mathcal{I}$ we get

$$S(f, F) = \mathcal{I}f_F \leqq \mathcal{I}(f \wedge A) \leqq \mathcal{I}f,$$

and so

$$\mathscr{I}(f-f\wedge A)=\mathscr{I}f-\mathscr{I}(f\wedge A)\leqq\mathscr{I}f-S(f,F)<\varepsilon,$$

which finishes our proof.

Lemma 21. *Let f be a nonnegative integrable function; then*

$$\lim_{a\to 0}\mathscr{I}(f\wedge a)=0.$$

Proof. It is enough to show that $\lim_n \mathscr{I}(f\wedge a_n)=0$ if $a_n\searrow 0$.

Let $a_n\searrow 0$, then the sequence $0\leqq f_n=f-f\wedge a_n$ is increasing, $f_n\nearrow f$ and $\mathscr{I}f_n\leqq\mathscr{I}f<\infty$, by the integrability of f.

From Proposition 19 it follows that $\lim_n \mathscr{I}f_n=\mathscr{I}f$. Since

$$\mathscr{I}(f\wedge a_n)=\mathscr{I}f-\mathscr{I}f_n,$$

we get

$$\mathscr{I}(f\wedge a_n)\to 0.$$

Proposition 22. *Let $\{f_n\}$ be a sequence of nonnegative integrable functions. Let f be a measurable function and let $f_n\searrow f$.*

Then f is integrable and

$$\lim_n \mathscr{I}f_n=\mathscr{I}f.$$

Proof. Since $0\leqq f\leqq f_1$, we get that f is integrable (see Proposition 13).

Since

$$\begin{aligned}|\mathscr{I}f_n-\mathscr{I}f|&=|\mathscr{I}(f_n\wedge a)+\\&+\mathscr{I}(f_n-f_n\wedge A)-\mathscr{I}(f\wedge A)-\mathscr{I}(f-f\wedge A)|\leqq\\&\leqq|\mathscr{I}(f_n\wedge A)-\mathscr{I}(f\wedge A)|+\mathscr{I}(f_n-f_n\wedge A)+\mathscr{I}(f-f\wedge A)=\\&=|\mathscr{I}(f_n\wedge A)-\mathscr{I}(f\wedge a)|+2\cdot\mathscr{I}(f_1-f_1\wedge A),\end{aligned}$$

from the fact that $\lim_{A\to\infty}\mathscr{I}(f_1-f_1\wedge A)=0$ it follows that we may assume that the functions f_n and f are bounded with a number A.

Since

$$|\mathscr{I}f_n-\mathscr{I}f|\leqq|\mathscr{I}(f_n-f_n\wedge a)-\mathscr{I}(f-f\wedge a)|+2\mathscr{I}(f_1\wedge a),$$

$$\lim_{a\to 0}\mathscr{I}(f_1\wedge a)=0$$

and since

$$\begin{aligned}\mu(\{x\,;(f_1-f_1\wedge a)(x)\geqq t\})&=\mu(\{x\,;f_1(x)\geqq a+t\})=\\&=(a+t)^{-1}(a+t)\mu(\{x\,;f_1(x)\geqq a+t\})\leqq a^{-1}\mathscr{I}f_1\end{aligned}$$

for every $t>0$, we may and do assume that there exists a real K such that

$$\mu(\{x\,;f_1(x)\geqq t\})\leqq K<\infty\quad\text{for}\quad t>0.$$

Denote

$$g_n(t) = \mu(\{x\,;\, f_n(x) \geqq t\})$$

and put

$$g(t) = \mu(\{x\,;\, f(x) \geqq t\}.$$

Then g_n and g are real non-negative monotone functions defined on $\langle 0, \infty)$, they vanish on $\langle A, \infty)$ and they are bounded with K. By the continuity of μ and by the fact that $f_n \searrow f$ we get $g_n \searrow g$. Let $\varepsilon > 0$. Choose $F_0 = \{0 = a_0 < a_1 < \ldots < a_k\}$ with $a_k \leqq A$ and such that for $F \supset F_0$ there holds: $|\mathscr{I}_\mu f - S(f, F)| < \varepsilon/2$. Choose a $\delta > 0$ such that for every partition Δ of the interval $\langle 0, A\rangle$ with the norm less then δ there holds $\left|\int_0^A g(t)\,\mathrm{d}t \;-\; \sum(d, \Delta)\right| < \varepsilon/2$, where $\Sigma(g, \Delta)$ is a Riemann integral sum of g with respect to the partition Δ. Let $F \supset F_0$ be such that $F \cap \langle 0, A\rangle$ is a partition of $\langle 0, A\rangle$ with the norm less then δ. Then

$$\left|\mathscr{I}_\mu f - \int_0^A g(t)\,\mathrm{d}t\right| = \left|\mathscr{I}_\mu f - S(f, F) + S(f, F) - \int_0^A g(t)\,\mathrm{d}t\right| =$$

$$= \left|\mathscr{I}_\mu f - S(f, F) + \Sigma(g, F \cap \langle 0, A\rangle) - \int_0^A g(t)\,\mathrm{d}t\right| < \varepsilon$$

because $S(f, F) = \Sigma(g, F \cap \langle 0, A\rangle)$. Thus we get

$$\mathscr{I}_\mu f = \int_0^A g(t)\,\mathrm{d}t.$$

Similarly

$$\mathscr{I}_\mu f_n = \int_0^A g_n(t)\,\mathrm{d}t.$$

Since $g_n \searrow g$, we get

$$\mathscr{I}_\mu f_n = \int_0^A g_n(t)\,\mathrm{d}t \searrow \int_0^A g(t)\,\mathrm{d}t = \mathscr{I}_\mu f.$$

And so

$$\lim_n \mathscr{I}_\mu f_n = \mathscr{I}_\mu f.$$

Theorem 23. *Let $\{f_n\}$ be a sequence of integrable functions.*

Let $f_n \nearrow f$, where f is a measurable function and $\mathscr{I} f_n \leqq c < \infty$ for every n. Then f is integrable and

$$\mathscr{I} f_n \nearrow \mathscr{I} f.$$

Proof. $f_n \nearrow f$ implies $f_n^+ \nearrow f^+$ and $f_n^- \searrow f^-$. Since

$$\mathscr{I} f_n^+ = \mathscr{I} f_n + \mathscr{I} f_n^- \leqq \mathscr{I} f_n + \mathscr{I} f_1^- \leqq c + \mathscr{I} f_1^- < \infty,$$

from Proposition 19 $\mathscr{I} f_n^+ \nearrow \mathscr{I} f^+$. Similarly by the last proposition $\mathscr{I} f_n^- \searrow \mathscr{I} f^-$, and so

$$\lim_n \mathscr{I} f_n = \lim_n \mathscr{I} f_n^+ - \lim_n \mathscr{I} f_n^- = \mathscr{I} f^+ - \mathscr{I} f^- = \mathscr{I} f.$$

In the last two theorems of this paragraph we shall assume that $\mathscr{D}$ is a σ-lattice.

Theorem 24. *If $\{f_n\}$ is a sequence of measurable functions which converges pointwise to a measurable function f and if g is an integrable function with $|f_n| \leqq g$ for $n = 1, 2, \ldots$, then f is integrable and*

$$\lim_n \mathscr{I}f_n = \mathscr{I}f.$$

Proof. It follows from Proposition 13 that the functions f_n and f are integrable. We put

$$h_n = \bigwedge_{i \geqq n} f_i \quad \text{and} \quad g_n = \bigvee_{i \geqq n} f_i.$$

From Proposition 4 we get that h_n and g_n are measurable. Clearly $h_n \leqq f_n \leqq g_n$, and

$$\lim_n h_n = \liminf_n f_n = f$$

and similarly

$$\lim_n g_n = f.$$

The functions h_n and g_n are integrable and $h_n \nearrow f \leftarrow g_n$. By the last theorem

$$\mathscr{I}h_n \nearrow \mathscr{I}f \leftarrow \mathscr{I}g_n.$$

From this and from the relation

$$\mathscr{I}h_n \leqq \mathscr{I}f_n \leqq \mathscr{I}g_n.$$

It follows that

$$\lim_n \mathscr{I}f_n = \mathscr{I}f.$$

Theorem 25. *If $\{f_n\}$ is a sequence of integrable functions with $f_n \geqq g$, where g is an integrable function, for which*

$$\liminf_n \mathscr{I}f_n \leqq c,$$

then the function f defined by

$$f(x) = \liminf_n f_n(x)$$

is integrable and

$$\mathscr{I}f \leqq \liminf_n \mathscr{I}f_n.$$

Proof. We let $g_k = f_k \wedge f_{k+1} \wedge \ldots$; then $f_k \geqq g_k \geqq g$ and so $g_k \to f$ and $\mathscr{I}g \leqq \mathscr{I}g_k \leqq \mathscr{I}f_n$ for $n \geqq k$. Hence $g_k \nearrow f$ and $\{\mathscr{I}g_k\}$ is upper bounded. Thus by Theorem 23

$$\mathscr{I}f = \lim_n \mathscr{I}g_n \leqq \liminf_n \mathscr{I}f_n.$$

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ИНТЕГРАЛ ДЛЯ ПРЕДМЕРЫ

Ян Шипош

Резюме

В работе определен интеграль при помощи предмеры. Приведено обобщение некоторых результатов известных в случае интеграла Лебега.