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REGULAR MEASURES AND ENTROPY ON PSEUDO-COMPACT SPACES

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The notion of pseudo-compact spaces was introduced in [2]. The main purpose of this paper is to study regular measures and entropy on Hausdorff normal pseudo-compact spaces. We prove the validity of Riesz Representation Theorem for regular measures and Goodwyn's Comparing Theorem for entropy on those spaces.

1. Regular measures

Definition 1. (i) *Topological flow is a couple (X, T), where X is a topological space and T is a continuous endomorphism of X*

(ii) *Let (X_i, T_i), $i=1, 2$ be topological flows. A mapping φ: $X_1 \to X_2$ is called a morphism of flows, if it is continuous, surjective and satisfies the equalitiy*

$$\varphi \cdot T_1 = T_2 \cdot \varphi$$

(iii) *The system $\mathcal{B}(X)$ of Borel subsets of a topological space X is the smallest σ-algebra containing all open sets.*

(iv) *A probability measure γ defined on $\mathcal{B}(X)$ is called regular if for any $B \in \mathcal{B}(X)$ the following equality holds:*

$$\gamma(B) = \inf\{\gamma(U): B \subset B,\ U \text{ open}\}.$$

The system of all regular probabilities on $\mathcal{B}(X)$ is denoted by $\mathcal{M}(X)$.

(v) *Let (X, T) be a topological flow. A measure $\gamma \in \mathcal{M}(X)$ is called T-invariant if $\gamma = \gamma \cdot T^{-1}$. The system of all regular T-invariant measures is denoted by $\mathcal{M}_T(X)$.*

(vi) *Let $\gamma \in \mathcal{M}(X)$. We denote by $\mathcal{B}_\gamma(X)$ the system of all Borel subsets of X with the property:*

$\gamma(\partial B) = 0$, where $\partial B = \overline{B} \cap \overline{B^c}$ is the boundary of B.

It is easy to show the following facts.

Proposition 1. i) *$\mathcal{B}_\gamma(X)$ is a σ-algebra.*

ii) *The closure $\bar{B}$ of $B \in \mathcal{B}_\gamma(X)$ is an element of $\mathcal{B}_\gamma(X)$.*

iii) *If (X, T) is a topological flow and $B \in \mathcal{B}_\gamma(X)$ then $T^{-1}(B) \in \mathcal{B}_\gamma(X)$.*

Lemma 1. (cf. [5], [6]). *Let X be a normal topological space.*

i) *For any closed subset A and $\gamma \in \mathcal{M}(X)$ there exists a closed G_δ set $B \supset A$ such that*

$$\gamma(B-A)=0.$$

ii) *Let B be a closed G_δ subset of X. Then there exists a real continuous function φ defined on X such that*

$$0 \leq \varphi(x) \leq 1, \quad B=\varphi^{-1}(0).$$

Corollary 1. *Let X be a normal topological space, let $\gamma \in \mathcal{M}(X)$ and $B \in \mathcal{B}_\gamma(X)$. Then there exists a closed G_δ set $A \in \mathcal{B}_\gamma(X)$ such that $B \subset A$, $\gamma(A-B)=0$.*

Definition 2. *A topological space is called*

i) *pseudo-compact if every real continuous function on X is bounded.*

ii) *countable-compact if every countable open cover of X has a finite subcover.*

Lemma 2. (cf. [2]). *A Hausdorff normal topological space X is pseudo-compact if and only if it is countable — compact.*

Corollary 2. i) *A continuous image of a pseudo-compact space is pseudo-compact.*

ii) *A pseudo-compact subspace of compact metric space is compact metrisable.*

iii) *A closed subset of pseudo-compact space is pseudo-compact.*

Further we suppose that the considered topological spaces are Hausdorff, normal and pseudo-compact.

Theorem 1. *The expression*

$$E(f)=\int f \, d\gamma$$

gives one-to-one correspondence between $\mathcal{M}(X)$ and the set of all linear nonnegative (i. e. $E(f) \geq 0$ for $f \geq 0$) functionals on $\mathcal{C}(X)$ which the property $E(1_x)=1$, (where 1_x denote the function on X with only one real value 1).

We first prove a lemma.

Lemma 3. *Let E satisfy the above conditions. Then the function defined on the open sets by*

$$\gamma^0(U)=\sup \{E(f): f \leq \chi_U\}$$

has the following properties:

i) *For U_1, U_2 open*

$$\gamma^0(U_1 \cup U_2)+\gamma^0(U_1 \cap U2)=\gamma^0(U_1)+\gamma^0(U_2)$$

ii) *For $U_1 \subset U_2 \subset \ldots \subset U_n \subset \ldots$, all being open and $U=\bigcup_{n=1}^{\infty} U_n$*

$$\gamma^0(U)=\lim_{n\to\infty}\gamma^0(U_n)$$

iii) $\gamma^0(U)=1-\inf\{\gamma^0(V)\colon U\cup V=X,\ V\ \text{open}\}$.

Proof. For any closed subset A put

$$\gamma_0(A)=1-\gamma^0(A^c).$$

From the definition of γ_0 and from the properties of E we get

$$\gamma_0(A)=\inf\{E(g)\colon \chi_A\leq g\in\mathscr{C}(X)\}.$$

The condition iii) is then equivalent to the condition:

for any U open

$\gamma^0(U)=\sup\{\gamma_0(A)\colon A\ \text{closed},\ A\subset U\}$.

Let $\varphi\in\mathscr{C}(X)$, $\varphi\leq\chi_U$. For $\varepsilon>0$ we put

$$A_\varepsilon=\{x\in X\colon \varphi(x)\geqq\varepsilon\}\subset U.$$

Let $g\in\mathscr{C}(X)$, $\chi_{A_\varepsilon}\leq g$. Then $f-g\leq\varepsilon\cdot 1_x$ and $E(\varphi)-E(g)\leq\varepsilon$. Taking supremum over f and infimum over g we get

$$\gamma^0(U)<\gamma_0(A)+\varepsilon$$

thus

$$\gamma^0(U)\leq\sup\{\gamma_0(A)\colon A\subset U\}.$$

Let $A\subset U$. There exists a real continuous function φ such that $0\leq\varphi\leq 1$, $\varphi|_A=1$, $\varphi|_{U^c}=0$. Hence $\chi_A\leq\varphi\leq\chi_U$ and $\gamma_0(A)\leq E(\varphi)\leq\gamma^0(U)$. Now we prove ii). Let $U_1\subset\ldots\subset U_n\subset\ldots$, $U=\bigcup_{n=1}^{\infty}U_n$. Let $\varepsilon>0$. Take A closed such that $A\subset U$ and

$\gamma_0(A)>\gamma^0(U)-\varepsilon$. The system $\{U_n\}_{n=1}^{\infty}$

is a countable open cover of the pseudo-compact set A. Hence there exists $n_0\in N$ such that $A\subset U_{n_0}$ and

$$\gamma^0(U_{n0})\geqslant\gamma_0(A)>\gamma^0(U)-\varepsilon.$$

Finally we prove i).

Let U_1, U_2 be open. Let $\varepsilon>0$. Take $f_1, f_2\in\mathscr{C}(X)$,

$$f_i\leq\chi_{U_i}\quad\text{and}\quad E(f_i)\geqslant\gamma^0(U_i)-\varepsilon/2\quad\text{for}\quad i=1,2.$$

We have

$$\begin{aligned}\gamma^0(U_1)+\gamma^0(U_2)&\leq E(f_1)+E(f_2)+\varepsilon=E(f_1\vee f_2)+E(f_1\wedge f_2)+\varepsilon\leq\\&\leq\gamma^0(U_1\cup U_2)+\gamma^0(U_1\cap U_2)+\varepsilon.\end{aligned}$$

Similarly we get that for A_1, A_2 closed

$$\gamma_0(A_1 \cup A_2) + \gamma_0(A_1 \cap A_2) \leq \gamma_0(A_1) + \gamma_0(A_2).$$

Now we are ready to prove the inequality

$$\gamma^0(U_1 \cup U_2) + \gamma^0(U_1 \cap U_2) \leq \gamma^0(U_1) + \gamma^0(U_2).$$

Take A, B closed,

$$A \subset U_1 \cup U_2,\ \gamma_0(A) > \gamma^0(U_1 \cup U_2) - \varepsilon/2$$
$$B \subset U_1 \cap U_2,\ \gamma_0(B) > \gamma^0(U_1 \cap U_2) - \varepsilon/2$$

The closed sets

$$\tilde{A}_1 = A - U_2 \quad \tilde{A}_2 = A - U_1$$

are disjoint. Hence there exist disjoint open sets $W_1 \supset \tilde{A}_1$, $\tilde{A}_2 \subset W_2$. Put

$$A_1 = A - W_2 \quad A_2 = A - W_1.$$

We have

$$A_1 \subset U_1 \ \ A_2 \subset U_2,\ \ A_1 \cup A_2 = A - (W_1 \cap W_2) = A.$$

Further we put

$$B_1 = A_1 \cup B \quad B_2 = A_2 \cup B.$$

Then

$B_i \subset U_1$, $i = 1, 2$, hence $\gamma^0(U_i) \geq \gamma(B_i)$
$B_1 \cup B_2 = A \cup B$, hence $\gamma_0(B_1 \cup B_2) \geq \gamma_0(A) \geq \gamma^0(U_1 \cup U_2) - \varepsilon/2$
$B_1 \cap B_2 \supset B$, hence $\gamma_0(B_1 \cap B_2) \geq \gamma_0(B) \geq \gamma^0(U_1 \cap U_2) - \varepsilon/2$.

Combining the above inequalities we have

$$\gamma^0(U_1) + \gamma^0(U_2) \geq \gamma_0(B_1) + \gamma_0(B_2) \geq$$
$$\geq \gamma_0(B_1 \cup B_2) + \gamma_0(B_1 \cap B_2) \geq \gamma^0(U_1 \cup U_2) + \gamma^0(U_1 \cap U_2) - \varepsilon.$$

Now we are able to prove Theorem 1.
We define the outer measure γ^* on the system of all subsets of X by

$$\gamma^*(M) = \inf\{\gamma^0(U) : M \subset U,\ U \text{ open}\}.$$

Arguing in the similar way as in [1] we can prove that the restriction of γ^* to the system

$$\mathcal{S} = \{B : B \subset X,\ \gamma^*(B) + \gamma^*(B^c) = 1\}$$

containing $\mathcal{B}(X)$, is a probability measure. Let us denote by γ the restriction of γ^* on $\mathcal{B}(X)$. The regularity of γ can be obtained from (iii) by obvious arguments. We show that the equation

$$\int f\, d\gamma = E(f)$$

is fulfilled for any $f \in \mathscr{C}(X)$.

Suppose that

$$0 \leqslant f \leqslant 1.$$

We can write

$$\int f\, d\gamma = \lim_{n\to\infty} 1/2^n \cdot \sum_{k=1}^{2^n} \gamma(U_{n,k})$$

where

$$U_{n,k} = \{x : f(x) > k/2^n\}.$$

For any given $\varepsilon > 0$ and $n \in N$ we can take functions $g_{n,k} \in \mathscr{C}(X)$ such that

$$g_{n,k} \leqslant \chi_{U_{n,k}} \quad \text{and} \quad E(g_{n,k}) > \gamma(U_{n,k}) - \varepsilon.$$

We have

$$1/2^n \cdot \sum_{k=1}^{2^n} g_{n,k} \leqslant 1/2^n \cdot \sum_{k=1}^{2^n} \chi_{U_{n,k}} \leqslant f$$

hence

$$E(f) \geqslant E\left(1/2^n \cdot \sum_{k=1}^{2^n} g_{n,k}\right) = 1/2^n \cdot \sum_{k=1}^{2^n} E(g_{n,k}) \geqslant 1/2^n \cdot \sum_{k=1}^{2^n} \gamma(U_{n,k}) - \varepsilon$$

for any ε and n.
Similarly

$$1 - E(f) = E(1-f) \geqslant \int (1-f)\, d\gamma = 1 - \int f\, d\gamma$$

thus

$$E(f) \leqslant \int f\, d\gamma \leqslant E(f).$$

Definition 3. *The topology induced on $\mathscr{M}(X)$ by the correspondence $\gamma \leftrightarrow E \in [\mathscr{C}(X)]^*$ and the weak topology on $[\mathscr{C}(X)]^*$ will be called weak topology on $\mathscr{M}(X)$.*

Proposition 2. a) *$\mathscr{M}(X)$ is a sequentially compact topological space, i. e. any sequence in $\mathscr{M}(X)$ has a cluster point.*

b) *Let $\varphi: X \to Y$ be a continuous surjective mapping. Then α) for any $\nu \in \mathscr{M}(Y)$ there exists $\gamma \in \mathscr{M}(X)$ such that $\nu = \gamma \cdot \varphi^{-1} \in \mathscr{M}(Y)$.*

β) for any $\gamma \in \mathscr{M}(X)$ the measure $\nu = \gamma \cdot \varphi^{-1} \in \mathscr{M}(Y)$.

γ) Let $T: X \to X$ be a continuous mapping. Then there exists $\gamma \in \mathscr{M}_T(X)$.

Proof. a) The set of all functionals E which correspond to some measures from $\mathcal{M}(X)$ is sequentially compact in $[\mathcal{C}(X)]^*$ (cf. [7]).

b) The part β can be obtained from the fact that the measure $\gamma \cdot \varphi^{-1}$ corresponds to the functional $\varphi^*(E)$ defined on $\mathcal{C}(Y)$ by

$$\varphi^*(E)(f) = E(g \cdot \varphi).$$

Suppose that $\nu \in \mathcal{M}(Y)$. Let us define functional E_φ on the closed subspace

$$C_\varphi = \{g \cdot \varphi : g \in \mathcal{C}(Y)\} \subset \mathcal{C}(X)$$

by

$$E_\varphi(g \cdot \varphi) = \int g \, d\nu .$$

The modification of Hahn—Banach theorem enables us to extend E into the continuous nonnegative linear functional E defined on $\mathcal{C}(X)$.

2. Entropy — general concept

Definition 4. *Base of entropy (BE) is defined as a triple* $(\mathcal{P}, T, H)$ *where*

$\mathcal{P}$ *is a set quasi-ordered by a reflexive and transitive relation "$<$" such that for any two elements* $P, Q \in \mathcal{P}$ *there exists a join* $P \vee Q$ *with the properties*

i) $P, Q < P \vee Q$
ii) $P, Q < R \in \mathcal{P} \Rightarrow P \vee Q < R$

$T: \mathcal{P} \to \mathcal{P}$ *and* $H: \mathcal{P} \to R^+$

are mapping with following properties

i) $Q < P \Rightarrow T(Q) < T(P),\ H(Q) \leq H(P)$
ii) $T(Q \vee P) = T(Q) \vee T(P),\ \ H(Q \vee P) \leq H(Q) + H(P)$
iii) $H(T(P)) \leq H(P)$.

Definition 5. *Let* $(\mathcal{P}, T, H)$ *be a BE. We say that* $P, Q \in \mathcal{P}$ *are equivalent* $(P \sim Q)$ *if* $P < Q$ *and* $Q < P$.

Proposition 3. *Let* $(\mathcal{P}, T, H)$ *is a BE. Then for* $P, Q, R \in \mathcal{P}$ *we have*
i) $P \sim P \vee P$
ii) $(P \vee Q) \vee R \sim P \vee (Q \vee R)$
iii) $P \sim Q \Rightarrow H(P) = H(Q)$.

Notation. *Let* $P_1, \ldots, P_n \in \mathcal{P}$. *Then we can define their common join* $\bigvee_{i=1}^{n} P_i$ *(up totthe equivalence) independently of the ordering.*

The number $H\left(\bigvee_{i=1}^{n} P_i\right)$ *does not depend on the ordering of elements* $P_1, \ldots, P_n$.

Definition 6. *Let* $(\mathcal{P}, T, H)$ *be a BE. For any* $P \in \mathcal{P}$ *and* $n \in N$ *we define* $P^n = \bigvee_{i=1}^{n} T^i(P)$ *and* $H(T, P) = \lim_n \sup (1/n \cdot H(P^n)$. *The entropy of the base* $(\mathcal{P}, T, H)$ *is defined by*

$$h(T) = \sup_{P \in \mathcal{P}} H(T, P).$$

Definition 7. *Let* $(\mathcal{P}_i, T_i, H_i)$, $i = 1, 2$ *be BE's. The mapping*

$$f\colon \mathcal{P}_1 \to \mathcal{P}_2$$

is called a BE-morphism if for any $P, Q \in \mathcal{P}_1$ *the following conditions are satisfied:*

i) $P < Q \Rightarrow f(P) < f(Q)$
ii) $f(P \vee Q) = f(P) \vee f(Q)$
iii) $f \cdot T_1(P) = T_2 \cdot f(P)$
iv) $H_2 \cdot f(P) = H_1(P)$

Definition 8. *We say that BE's* $(\mathcal{P}_1, T_1, H_1)$ *and* $(\mathcal{P}_2, T_2, H_2)$ *are*
a) weakly isomorphic if there exist BE-morphisms

$$f_1\colon \mathcal{P}_1 \to \mathcal{P}_2 \text{ and } f_2\colon \mathcal{P}_2 \to \mathcal{P}_1$$

b) isomorphic if there exists a bijective mapping $f\colon \mathcal{P}_1 \to \mathcal{P}_2$ *such that* f *and* f^{-1} *are BE-morphisms.*

Proposition 4. *Let* $(\mathcal{P}_i, T_i, H_i)$, $i = 1, 2$ *be BE's and* $f\colon \mathcal{P}_1 \to \mathcal{P}_2$ *BE-morphism. Then we have*

a) $P \in \mathcal{P}_1\colon f(P^n) = [f(P)]^n$, $H(T_1, P) = H(T_2, f(P))$
b) $h(T_1) \leq h(T_2)$.

Proof. The following equalities can be easily proved by induction

$$f(P^n) = f\Big(\bigvee_{i=1}^{n} T_1^i(P)\Big) = \bigvee_{i=1}^{n} T_2^i(f(P)) = [f(P)]^n.$$

Thus we have

$$H_1(T_1, P) = \lim_n \sup 1/n \cdot H_1(P^n) = \lim_n \sup 1/n \cdot H_2 \cdot f(P^n) = H_2(T_2, f(P)).$$

Hence

$$h_1(T_1) = \sup_{P \in \mathcal{P}_1} H_1(T_1, P) = \sup_{P \in \mathcal{P}_1} H_2(T_2, f(P)) \leq \sup_{Q \in \mathcal{P}_2} H_2(T_2, Q) = h_2(T_2).$$

Corollary. i) *Weakly isomorphic BE'a have the same entropy.*
ii) *Isomorphic BE's have the same entropy.*

Proposition 5. *Let* $(\mathcal{P}, T, H)$ *be BE and* $P < Q \in \mathcal{P}$. *Then* $H(T, P) \leq H(T, Q)$.
Proof. We can show by induction with respect to n that $P^n < Q^n$ thus $H(P^n) \leq H(Q^n)$ and $H(T, P) \leq H(T, Q)$.

3. Topological and measure theoretic entropy

Definition 9. *Let* (X, T) *be a topological flow. a) Let* $\mathcal{P}$ *be the system of all finite open covers of* X *quasiordered by the relation*

$$P < Q \Leftrightarrow \forall V \in Q\ \exists U \in P\colon V \subset U.$$

For $P, Q \in \mathcal{P}$ *we put*

$$P \vee Q = \{U \cap V\colon U \in P, V \in Q\},$$
$$\bar{T}(P) = \{T^{-1}(U)\colon U \in P\}$$
$$N(P) = \min\{\text{card}(Q)\colon Q \subset P, Q \in \mathcal{P}\}$$
$$H(P) = \log N(P).$$

b) Let $\varphi\colon (X', T') \to (X, T)$ *be a morphism of flows. Put*

$$\bar{\varphi}\colon \mathcal{P} \to \mathcal{P}', \quad \bar{\varphi}(P) = \{\varphi^{-1}(U)\colon U \in P\}.$$

Notation. *Using the standard methods we can prove that* $(\mathcal{P}, \bar{T}, H)$ *is a BE and that* $\bar{\varphi}$ *is a BE-morphism. The entropy* $h(\bar{T})$ *is called the topological entropy of the flow* (X, T).

Proposition 6. *Let* $\mathcal{P}_0$ *be the system of all those elements of* $\mathcal{P}$ *which consist of open* F_σ *sets. Then*

$$h(T) = \sup_{P \in \mathcal{P}_0} H(T, P)$$

Proof. Let $P = \{U_1 \cdot, U_n\} \subset \mathcal{P}$. We can assume that for $k = 1, \ldots, n$ $\bigcup_{1 \neq k} U_i \neq X$. Put $V_0 = \emptyset = U_{n+1}$.

Suppose that for $k \in \{1, \ldots, n\}$ we have constructed open F_σ sets $V_0, \ldots, V_{k-1}$ such that $V_i \subset U_i$, for $i = 0, 1, \ldots, k-1$, where $V_0 = U_0 = 0 = \emptyset = U_{n+1}$.

$$\bigcup_{i=0}^{k-1} V_i \cup \bigcup_{j=k}^{n+1} U_j = X.$$

Put

$$A_k = X - \bigcup_{i=0}^{k=1} V_i - \bigcup_{j=k+1}^{n+1} U_j = X$$

Then $0 \neq A_k \subset U_k$, A_k and U_k^c are disjoint closet sets. Let φ_k be a continuous real function defined on X such that

$$0 \leq \varphi_k \leq 1, \quad \varphi_k|_{A_k} = 0, \quad \varphi_k|_{U_k^c} = 1.$$

Put

$$V=\varphi^{-1}(\langle 0, 1/2)).$$

Then we have

$$V_k \text{ is open } F_\sigma \text{ and } A_k \subset V_k \subset U_k$$

thus

$$\left(\bigcup_{i=0}^{k} V_i\right) \cup \left(\bigcup_{j=k+1}^{n+1} U_j\right) = X.$$

Put $Q=\{V_1, \ldots, V_n\}$. Then $Q \subset \mathcal{P}_0$ and $P<Q$ thus

$$H(T, P) \leq H(T, Q).$$

Definition 10. *Let* (X, T) *be a topological flow and* $\gamma \in \mathcal{M}(X)$. a) *The finite set* $P=\{B_1, \ldots, B_n\}$ *is called a* γ*-covering of* X *if* $B_i \in \mathcal{B}(X)$ *for* $i=1, \ldots, n$ *and* $\gamma(B_i \cap B_j)=0$ *for* $i \neq j$. *The system of all* γ*-coverings will be denoted by* $\mathcal{P}_\gamma$.

For $P=\{A_1, \ldots, A_n\}$, $Q=\{B_1, \ldots, B_m\}$ *put*

$$d(P, Q)=\sup_{i=1, \ldots, n}\left\{\inf_{j=1, \ldots, m}\{\gamma(A_i-B_j)\}\right\}.$$

Now we define the quasi-ordering on $\mathcal{P}_\gamma$

$$P<Q \Leftrightarrow d(Q, P)=0.$$

For $P, Q \in \mathcal{P}$ *we put*

$$P \vee Q=\{A \cap B: A \in P, B \in Q\}.$$

Finally we put

$$H(P)=-\sum_{A \in P} \gamma(A) \cdot \log \gamma(A)$$

and

$$\bar{T}(P)=\{T^{-1}(A): A \in P\}.$$

b) *Let* $\varphi: (X', T') \rightarrow (X, T)$ *be a morphism of flows. Take* $g' \in \mathcal{M}(X')$ *such that* $\gamma=\gamma' \cdot \varphi^{-1}$. *Put* $\bar{\varphi}: \mathcal{P}_\gamma \rightarrow \mathcal{P}_{\gamma'}$

$$\bar{\varphi}(P)=\{\varphi^{-1}(A): A \in P\}.$$

Notation. *We can show by obvious arguments that* $(\mathcal{P}_\gamma, \bar{T}, H)$ *is a BE and* $\bar{\varphi}$ *is a BE-morphism.*

The entropy $h\gamma(T)$ *is called measure-theoretic entropy.*

f ı i ion 12. *Denote by $\mathcal{P}^0_\gamma$ the subsystem of $\mathcal{P}_\gamma$ containing decompositions of , n o elements of $\mathcal{B}_\gamma(X)$. Denote by $\mathcal{P}^1_\gamma$ the subsystem of $\mathcal{P}_\gamma$ containing γ-covers fo med by closed elements of $\mathcal{B}_\gamma(X)$. Further we denote by $\mathcal{P}^2_\gamma$ the subsystem of $\mathcal{P}^1_\gamma$ co ta'ni ıg γ-covers formed by G_σ sets. The following proposition can be proved in t ,e st ndard way* (cf [3]).

roposition 7. *For any given $n\in N$ and $\varepsilon>0$ there exists $\delta>0$ such that for P $Q\in\mathcal{P}_\gamma$,* ca d (P), card $(Q)\leq n$ *and $d(Q, P)<\delta$ we have*

$$H(T, Q)<H(T, P)+\varepsilon .$$

opos't'on 8. a) *For any $B\in\mathcal{B}(X)$ and $\varepsilon>0$ there exists $A\in\mathcal{B}_\gamma(X)$ such that $\gamma(B$ $A)=0$*

b) *For any $P\in\mathcal{P}_\gamma$ and $\varepsilon>0$ there exists $Q\in\mathcal{P}^0_\gamma$ such that $d(Q, P)\leq\varepsilon$.*

roof. a) Take C closed and U open such that

$$C\subset B\subset U \quad \text{and} \quad \gamma(U-C)<\varepsilon/2.$$

Th re exists a continuous real function φ: $0\leq\varphi\leq1$ and $\varphi|_C=0$, $\varphi|_{U^c}=1$. The system of disjoint sets $\{D_t\}_{t\in(0,1)}$ where $D_t=\{\varphi^{-1}(t)\}$ is uncountable hence there x t, $t\in(0, 1)$ such that $\gamma(D_t)=0$. Put $A=\varphi^{-1}(\langle 0, t\rangle)$. Then $\partial A\subset D_t$ hence $A\in\mathcal{B}_\gamma(X)$.

b) Thi part follows from the fact that $\mathcal{B}_\gamma(X)$ is a σ-algebra.

Proposi 'on 9. *Let $P\in\mathcal{P}_\gamma$. Then there exist $P'\in\mathcal{P}^1_\gamma$ and $P''\in\mathcal{P}^2_\gamma$ such that $P\sim P'\sim P'$.*

Proof. Take $P\in\mathcal{P}_\gamma$. Put $P'=\{\bar{B}: B\in P\}$.

Ac ord'ng to Lemma 1 (i) every element $\bar{B}$ of P is contained in a closed G_δ set C ıth the same measure. The collection of those sets C forms the γ-covering P''.

Corollary.

$$h_\gamma(T)=\sup_{P\in\mathcal{P}_\gamma^2} H(T, P).$$

4. Goodwyn's theorem on pseudo-compact spaces

Th orem. *Let* (X, T) *be topological flow* (X is Hausdorff normal and pseudo-compact). For topological entropy we have

$$h(T)=\sup\{h_\gamma(T): \gamma\in\mathcal{M}_T(X)\}.$$

Proof. We shall make use of the fact that the theorem holds on compact metrizable spaces (cf. [8]).

Let $\alpha<h(T)$ There exists $P\in\mathcal{P}_0$ such that $H(T, P)>\alpha$. Suppose $P=\{U$., $U_m\}$ where U_i, $i=1, \ldots, m$ are open F_σ sets. For $i=1, \ldots, m$ there exist real continuous functions φ_i on X such that $0 \leqslant \varphi_i \leqslant 1$, $U_i = \varphi^{-1}((0, 1\rangle)$. For $n = 0, 1, \ldots$ put $Y_n = \langle 0, 1 \rangle^m$.

The product space $Y = \prod_{n=0}^{\infty} Y_n$ is compact metrizable. Now we define the continuous mapping $\Phi \colon X \to Y$

$$[\Phi(x)]_{n,i} = \varphi_i \cdot T^n(x)$$

for $n = 0, 1, \ldots, i = 1, \ldots, m$.

Put $K = \Phi(X)$.

K is a pseudo-compact subspace of the compact metrisable space Y thus K is a compact metrisable.

We have

$$[\Phi(T(x))]_{n,i} = \varphi_i \cdot T^n(T(x)) = \varphi_i \cdot T^{n+1}(x) = [\Phi(x)]_{n+1,i} = [\tau \cdot \Phi(x)]_{n,i}$$
$$\text{for } n = 0, 1, \ldots, i = 1, \ldots, m$$

where τ is the shift on K defined by

$$[\tau(y)]_{n,i} = [y]_{n+1,i}.$$

Hence

$$\Phi \colon (X, T) \to (K, \tau)$$

is the morphism of flows. For $i = 1, \ldots, m$ we put

$$V_i = \{y \in K \colon [y]_{0,i} \neq 0\}.$$

For $i = 1, \ldots, m$ we have

$$U_i = \Phi^{-1}(V_i).$$

Put

$$Q = \{V_1, \ldots, V_m\}.$$

Then we have $P = \Phi(Q)$. According to Proposition 4a)

$$H(T, P) = H(\tau, Q) \leqslant h(\tau) = \sup \{h_\nu(\tau) \colon \nu \in \mathcal{M}_\tau(K)\}.$$

There exists $\nu \in \mathcal{M}_\tau(K)$ such that

$$h_\nu(\tau) \geqslant H(T, P) - 1/2 \cdot [H(T, P) - \alpha] > \alpha.$$

According to Proposition 2b), there exists $\gamma \in \mathcal{M}_T(X)$ such that $\nu = \gamma \cdot \Phi^{-1}$. According to Proposition 4b) and Notation 2 we have $h\gamma(T) \geqslant h_\nu(\tau) > \alpha$.

Hence

$$h(T) \leqslant \sup \{h_\gamma(T) \colon \gamma \in \mathcal{M}_T(X)\}.$$

To prove the conv rs inequality we use a similar construction Take $\gamma \in \mathcal{M}_T(X)$ and $\alpha < h_\gamma(T)$. There exists $P \in \mathcal{P}_\gamma^2$ such that $H(T, P) > \alpha$. Suppose $P = \{A_1, \ldots, A_p\}$, where A_i, $i = 1, \ldots, p$ are closed G_δ sets.

There exist continuous real functions φ_i, $i = 1, \ldots, p$ such that $0 \leq \varphi_i \leq 1$ and $A_i = \varphi_i^{-1}(0)$.

For $n = 0, 1, \ldots$, put $Y_n = \langle 0\ 1 \rangle^p$.

Define the continuous fun tion

$$\Phi : X \to Y = \prod_{n=0}^{\infty} Y_n$$
$$[\Phi(x)]_{n i} = \varphi_i \cdot T^n(x)$$

Put

$$K = \Phi(X),\ B_i = \{y \in K : [y]_{0 i} = 0\},\ i = 1, \ldots, p,\ Q = \{B_1, \ldots, B_p\}.$$

Then

$$A_i = \Phi^{-1}(B_i),\ i = 1, \ldots m, \text{ thus } P = \bar{\Phi}(Q).$$

According to Propo ition 4b), $H_\gamma(T, P) = H_{\gamma \Phi^{-1}}(\tau, Q)$, where τ is the shift on K.

Φ: $(X, T) \to (K, \tau)$ is a homomorphism of flows, thus

$$h(T) \geq h(\tau) \geq \gamma\ \Phi^{-1}(\tau) \geq H_{\gamma \Phi^{-1}}(\tau, Q) = H_\gamma(T, P) > \alpha$$

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РЕГУЛЯРНЫЕ МЕРЫ И ЭНТРОПИЯ НА ПСЕВДОКОМПАКТНЫХ ПРОСТРАНСТВАХ

Магда Коморникова, Йозеф Коморник

Резюме

В работе доказывается, что теорема Рисса о представлении регулярных вероятностных мер и теорема Гудвина о сравнении вероятностной и топологической энтропии верны и для нормальных псевдокомпактных пространств Гаусдорфа.