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A NOTE ON NONNEGATIVE MATRICES

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1. Introduction. The main purpose of this note is to prove that the set of all points the coordinates of which are eigenvalues of a nonnegative n by n matrix with a given Perron root is closed.

2. Results. The reader is referred to the book [1] for the necessary definitions and theorems. We shall prove first:

Theorem 1. *Let* $\mathbf{A}$ *be a nonnegative matrix which has positive Perron root* $p(\mathbf{A})$. *Then there exists a diagonal matrix* $\mathbf{D}$ *with positive diagonal entries such that the matrix* $\mathbf{DAD}^{-1}=\mathbf{B}=(b_{ik})$ *satisfies*

$$b_{ik} \leqq p(\mathbf{A})$$

for all i, k.

Proof. Let us assume first that $\mathbf{A}$ is irreducible. By the Perron—Frobenius theorem, there exist positive column vectors $\boldsymbol{u}=(u_i)$ and $\boldsymbol{v}=(v_i)$ such that

$$\mathbf{A}\boldsymbol{u}=p(\mathbf{A})\boldsymbol{u} ,$$

$$\mathbf{A}^{\mathrm{T}}\boldsymbol{v}=p(\mathbf{A})\boldsymbol{v}$$

$\mathbf{A}^{\mathrm{T}}$ being the transpose matrix to $\mathbf{A}$. Define $\mathbf{D}=\operatorname{diag}\{d_i\}$, where $d_i=v_i^{1/2}u_i^{-(1/2)}$. It is easily seen that then $\mathbf{B}=\mathbf{DAD}^{-1}$ satisfies

$$\mathbf{B}\boldsymbol{w}=p(\mathbf{A})\boldsymbol{w} , \tag{1}$$

$$\mathbf{B}^{\mathrm{T}}\boldsymbol{w}=p(\mathbf{A})\boldsymbol{w} \tag{2}$$

where $\boldsymbol{w}=(w_i)$ with $w_i=u_i^{1/2}v_i^{1/2}$. Without loss of generality, we can assume that

$$w_1 \geqq w_2 \geqq \ldots \geqq w_n .$$

Let i, k be two indices; if $i \leqq k$, we have by (2),

$$p(\mathbf{A})w_k=\sum_j b_{jk}w_j \geqq b_{ik}w_i \geqq b_{ik}w_k$$

so that

$$b_{ik} \leqq p(\mathbf{A}) . \tag{3}$$

If $i>k$, (1) yields

$$p(\mathbf{A})w_i=\sum_j b_{ij}w_j \geqq b_{ik}w_k \geqq b_{ik}w_i$$

and (3) is fulfilled as well.

Let now $\mathbf{A}$ be reducible. As is well known [1], there exists a permutation matrix $\mathbf{P}$ such that

(4)
$$\mathbf{P}^{\mathrm{T}}\mathbf{A}\mathbf{P}=\begin{pmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} & \dots & \mathbf{A}_{1r} \\ \mathbf{0} & \mathbf{A}_{22} & \dots & \mathbf{A}_{2r} \\ - & - & - & - \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{A}_{rr} \end{pmatrix}$$

where $r>1$ and $\mathbf{A}_{11}, \mathbf{A}_{22}, \dots, \mathbf{A}_{rr}$ are square irreducible matrices of order at least one. Let $\mathbf{D}_i$, $i=1, \dots, r$ be diagonal matrices such that no entry of the matrix

$$\mathbf{B}_{ii}=\mathbf{D}_i\mathbf{A}_{ii}\mathbf{D}_i^{-1}$$

exceeds the corresponding Perron root $p(\mathbf{A}_{ii})$, $i=1, \dots, r$. Let m be the maximum of all the entries of all the matrices $\mathbf{D}_i\mathbf{A}_{ik}\mathbf{D}_k^{-1}$, $i<k$. Define $\omega=1$ if $m\leqq p(\mathbf{A})$, $\omega=m/p(\mathbf{A})$ if $m>p(\mathbf{A})$. As $p(\mathbf{A}_{ii})\leqq p(\mathbf{A})$, $i=1, \dots, r$, it is easily checked that if

$$\mathbf{D}=\mathbf{P}\begin{pmatrix} \mathbf{D}_1 & & & & \\ & \omega\mathbf{D}_2 & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \omega^{r-1}\mathbf{D}_r \end{pmatrix}\mathbf{P}^{\mathrm{T}},$$

the matrix

$$\mathbf{B}=\mathbf{D}\mathbf{A}\mathbf{D}^{-1}$$

has all entries less than or equal to $p(\mathbf{A})$. The proof is complete.

Corollary. *If $\mathbf{A}=(a_{ik})$ is a nonnegative matrix with the Perron root $p(\mathbf{A})$ then for any indices $k_1, \dots, k_r$, $r\geqq 2$,*

(5)
$$a_{k_1k_2}a_{k_2k_3}\dots a_{k_rk_1}\leqq p^r(\mathbf{A}).$$

Proof. If $p(\mathbf{A})=0$, $\mathbf{A}$ is either of order one and there is nothing to prove, or $\mathbf{A}$ is reducible and in the corresponding form (4) all matrices $\mathbf{A}_{ii}$ are zero matrices of order one. It follows that all expressions on the left-hand sides of (5) are to zero. Thus the assertion is true in this case.

If $p(\mathbf{A})>0$, there exists by Thm. 1 a diagonal matrix $\mathbf{D}$ with positive diagonal entries such that $\mathbf{D}\mathbf{A}\mathbf{D}^{-1}=\mathbf{B}=(b_{ik})$ satisfies

$$b_{ik}\leqq p(\mathbf{A})$$

for all i, k. Since

$$a_{k_1k_2}a_{k_2k_3} \dots a_{k_rk_1} = b_{k_1k_2}b_{k_2k_3} \dots b_{k_rk_1} .$$

the estimate (5) follows.

Definition. *Let $p > 0$. We shall denote by $N(p)$ the set of all nonnegative matrices which have the Perron root p and whose entries do not exceed p.*

Theorem 2. *Let $\Sigma_n(p)$ denote the set of all points $(\lambda_1, \lambda_2, \dots, \lambda_{n-1})$ of a complex $(n-1)$-dimensional space C_{n-1} such that there exists an n by n nonnegative matrix* **A** *with the Perron root p and all the remaining eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_{n-1}$.* Then *$\Sigma_n(p)$ is a closed set.*

Remark. If $(\lambda_1, \lambda_2, \dots, \lambda_{n-1}) \in \Sigma_n(p)$ and P is a permutation of the indices $1, \dots, n-1$ then $(\lambda_{P1}, \lambda_{P2}, \dots, \lambda_{P(n-1)}) \in \Sigma_n(p)$ as well.

Proof. The theorem is true if $p = 0$. Let thus $p > 0$. Let $\{(\lambda_{1i}, \lambda_{2i}, \dots, \gamma_{n-1,i})\}$ be a sequence of points in $\Sigma_n(p)$ which converges to $(\lambda_1, \lambda_2, \dots, \lambda_{n-1})$. By the definition of $\Sigma_n(p)$ and by Theorem 1, there exist matrices $\mathbf{B}_i \in N(p)$, $i = 1, 2, \dots$ such that for each i, $\mathbf{B}_i$ has the Peron root p and the remaining eigenvalues $\lambda_{1i}, \lambda_{2i}, \dots, \lambda_{n-1,i}$. $N(p)$ being compact, there exists a subsequence $\{\mathbf{B}_{i_k}\}$ of $\{\mathbf{B}_i\}$ which is convergent:

$$\mathbf{B}_{i_k} \to \mathbf{B} .$$

As eigenvalues of a matrix depend continuously on its entries [2], it follows that **B** which also belongs to $N(p)$ has the Perron root p and all remaining eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_{n-1}$. Thus $(\lambda_1, \lambda_2, \dots, \lambda_{n-1}) \in \Sigma_n(p)$ and the proof is complete.

REFERENCES

[1] GANTMACHER, F. R.: Teorija matric. Gostechizdat. Moscow 1953. English translation: Theory of Matrices. Chelsea 1959.

[2] OSTROWSKI, A. M.: Solution of equations and systems of equations. Academic Press 1960.

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ЗАМЕТКА ПО НЕОТРИЦАТЕЛЬНЫМ МАТРИЦАМ

Мирослав Фидлер

Резюме

Доказывается, что множество всех точек n-мерного комплексного пространства, координаты которых являются собственным значением неотрицательной матрицы порядка n с заданным корнем Перрона — замкнуто.