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Leonard Carlitz; Josef Kaucký; Jaromír Vosmanský
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## ON TWO COMBINATORIAL IDENTITTIES

L. CARLITZ, J. KAUCKÝ, J. VOSMANSKÝ

## 1.

When investigating Bessel functions, the third author discovered a combinatorical identity which, after a small modification, can be written in the following form

$$
\begin{equation*}
\sum_{j=0}^{2 r}(-1)^{j}\binom{2 r}{j}\binom{k+j}{2 r}\binom{k+2 r-j}{2 r}=(-1)^{r}\binom{k}{r}\binom{k+r}{r} \tag{1}
\end{equation*}
$$

where $r$ and $k$ are arbitrary non-negative integers. Special cases of (1) can be found in an implicit form in the last part of [3].

The proof presented in part II is due to L. Carlitz. In the last part J. Kaucký uses Vosmanský's identity (1) to give a new simple proof of the well-known Dixon's combinatorial identity

$$
\begin{equation*}
\sum_{j=0}^{2 r}(-1)^{j}\binom{2 r}{j}^{3}=(-1)^{r} \frac{(3 r)!}{(r!)^{3}} \tag{2}
\end{equation*}
$$

The different proofs of (2) can be found, e.g., in [2] § 6.3 or in [1] § 5.4.

## II.

Put

$$
R(n, k, r)=\sum_{j=0}^{\prime}(-1)^{\prime}\binom{r}{j}\binom{n+j}{r}\binom{k+r-j}{r} .
$$

Then

$$
\begin{gathered}
\sum_{n, k=0}^{\infty} R(n, k, r) x^{n} y^{k}= \\
=\sum_{j=0}^{r}(-1)^{\prime}\binom{r}{j} x^{r-j}(1-x)^{-r-1} y^{\prime}(1-y)^{-r-1}= \\
=(1-x)^{-r-1}(1-y)^{-r-1} \sum_{j=0}^{r}(-1)^{j}\binom{r}{j} x^{r-j} y^{\prime}
\end{gathered}
$$

so that

$$
\begin{equation*}
\sum_{n}^{\infty} R(n, k, r) x^{n} y^{k}=(x-y)^{r}(1-x)^{r}(1-y)^{\prime} \tag{3}
\end{equation*}
$$

Clearly it follows from (3) that

$$
\begin{aligned}
& F(x, y, z) \equiv \sum_{r}^{\infty} z^{r} \sum_{n, k}^{\infty} R(n, k, r) x^{n} y^{k}= \\
= & (1-x)^{1}(1-y)^{1} \sum_{0}^{\infty} z^{r} \frac{(x-y)^{r}}{(1-x)^{r}(1-y)^{r}}= \\
= & (1-x)^{1}(1-y)^{1}\left\{1-\frac{(x-y) z}{(1-x)(1-y)}\right\}^{\prime}
\end{aligned}
$$

so that

$$
\begin{equation*}
F(x, y, z)=\{(1-x)(1-y)-(x-y) z\}^{1} \tag{4}
\end{equation*}
$$

In the right-hand side of (4) replace $y$ by $x{ }^{1} y$ and we get

$$
\begin{aligned}
& \left\{(1-x)\left(1-x^{1} y\right)-\left(x-x^{1} y\right) z\right\}^{1}= \\
= & \left\{(1+y)-(1+z) x-(1-z) x^{1} y\right\}^{1}= \\
= & (1+y)^{1}\left\{1-\frac{(1+z) x}{1+y}-\frac{(1-z) x^{1} y}{1+y}\right\}^{1}= \\
= & (1+y)^{1} \sum_{s+0}^{\infty}\binom{s+\eta}{s} \frac{(1+z)^{s}(1-z)^{t}}{(1+y)^{+t}} y^{\prime} x^{s} .
\end{aligned}
$$

We retain only those terms that are independent on $x$, that is, those in which $s=t$. This gives

$$
\begin{gathered}
(1+y)^{1} \sum_{s}^{\infty}\binom{2 s}{s} \frac{\left(1-z^{2}\right)^{s}}{(1+y)^{2 s}} y^{s}= \\
=(1+y)^{1}\left\{1-4 y \frac{1-z^{2}}{(1+y)^{2}}\right\}^{12}=\left\{(1-y)^{2}+4 y z^{2}\right\}^{12}= \\
=(1-y)^{1}\left\{1+\frac{4 y z^{2}}{(1-y)^{2}}\right\}^{12}= \\
=\sum_{r=}^{\infty}(-1)^{r}\binom{2 r}{r} \frac{y^{r} z^{2 r}}{(1-y)^{2 r+1}} .
\end{gathered}
$$

Thus we have proved that

$$
\sum_{r}^{\infty} z^{r} \sum_{k}^{\infty} R(k, k, r) y^{k}=\sum_{r}^{\infty}(-1)^{r}\binom{2 r}{r} \frac{y^{\prime} z^{2 r}}{(1-y)^{2 r+1}}
$$

Hence

$$
R(k, k, 2 r+1)=0
$$

(which of course is clear from the definition) and

$$
\begin{gathered}
\sum_{k=0}^{\infty} R(k, k, 2 r) y^{k}=(-1)^{r}\binom{2 r}{r} \frac{y^{r}}{(1-y)^{2 r+1}}= \\
=(-1)^{r}\binom{2 r}{r} y^{r} \sum_{s=0}^{\infty}\binom{2 r+s}{s} y^{s}=(-1)^{r}\binom{2 r}{r} \sum_{k}^{\infty}\binom{k+r}{2 r} y^{k} .
\end{gathered}
$$

Hence finally

$$
R(k, k, 2 r)=(-1)^{r}\binom{2 r}{r}\binom{k+r}{2 r}=(-1)^{r}\binom{k}{r}\binom{k+r}{r}
$$

in agreement with the asserted result.
III.

In well-known formula (see e.g. [2] chapter 6)

$$
\begin{gathered}
\sum_{j}^{p}\binom{p}{j}\binom{q}{j} \alpha^{p-i} \beta^{i}= \\
=\sum_{i=0}^{p}\binom{p}{j}\binom{q+j}{j}(\alpha-\beta)^{p} \beta^{\prime}, \quad p \leqslant q
\end{gathered}
$$

we replace $p, q, \alpha, \beta$ by $p=q=2 r, \alpha=\Delta, \beta=-1$. We obtain

$$
\begin{gathered}
\sum_{0}^{2 r}(-1)^{j}\binom{2 r}{j}^{2} \Delta^{2 r-j}= \\
=\sum_{j}^{2 r}(-1)^{j}\binom{2 r}{j}\binom{2 r+j}{j} E^{2 r},
\end{gathered}
$$

where $E=\alpha-\beta=\Delta+1$. Now $E$ and $\Delta$ will be considered as operators. Then for the function

$$
f(n)=\binom{2 r+n}{2 r}
$$

we have

$$
\Delta f(n)=\binom{2 r+n+1}{2 r}-\binom{2 r+n}{2 r}=\binom{2 r+n}{2 r-1}
$$

so that

$$
\Delta^{2 r{ }^{j}} f(n)=\binom{2 r+n}{j}
$$

and because

$$
E^{2 r} f(n)-\left(\begin{array}{cc}
4 r & J+n \\
& 2 r
\end{array}\right)
$$

we have

$$
\begin{gathered}
\sum_{j}^{2 r}(-1)^{\prime}\binom{2 r}{j}^{2}\binom{2 r+n}{j}- \\
-\sum_{j}^{r}(1)^{\prime}\binom{2 r}{j}\binom{2 r+j}{2 r}\left(\begin{array}{c}
4 r \\
\hline+n \\
2 r
\end{array}\right) .
\end{gathered}
$$

Finally for $n \quad 0$ and using Vosmanský's identity (1) with $k-2 r$, we have

$$
\sum_{1}^{2 r}(1)^{\prime}\binom{2 r}{j}^{3} \quad(\quad 1)^{r}\binom{2 r}{r}\binom{3 r}{r}=(-1)^{r} \frac{(3 r)!}{(r!)^{3}},
$$

the stated Dixon formula.

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Leonard Carlitz<br>Duke University, Durham, North Carolina $27^{7} 06$ U.S A<br>Josef Kaucky<br>Tabor 25<br>61600 Brno<br>Jaromir Vosmansky<br>Katedra matematıcke analyzy<br>Přrodovědecke fakulty UJEP<br>Janačkovo nam. 2a<br>66295 Brno

