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Dedicated to the memory of Professor Valter Šeda

PERIODICALLY FORCED DAMPED BEAMS RESTING ON NONLINEAR ELASTIC BEARINGS

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ABSTRACT. We show the existence of periodic solutions for certain damped linear beam equations with periodic perturbations resting on nonlinear elastic bearings.

1. Introduction

We consider the equation

$$\begin{gathered} u_{tt} + u_{xxxx} + \delta u_t + h(x,t) = 0\,, \\ u_{xx}(0,\cdot) = u_{xx}(\pi/4,\cdot) = 0\,, \\ u_{xxx}(0,\cdot) = -f\big(u(0,\cdot)\big)\,, \qquad u_{xxx}(\pi/4,\cdot) = g\big(u(\pi/4,\cdot)\big)\,, \end{gathered} \tag{1}$$

where $\delta > 0$ is a constant, f, g are analytic and h is a forcing term T-periodic in t. Equation (1) describes vibrations of a beam resting on two different bearings with purely elastic responses which are determined by f and g. The length of the beam is $\pi/4$. We are interested in forced periodic vibrations of (1).

The existence of periodic, homoclinic and chaotic solutions is shown in the papers [1] [4] for several types of nonlinearities of (1).

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2. Setting of the problem

By a weak T-periodic solution of (1), we mean any $u(x,t) \in C([0,\pi/4] \times S^T)$ satisfying the identity

$$\begin{aligned}\int_0^T \int_0^{\pi/4} \Big[u(x,t)\{v_{tt}(x,t) + v_{xxxx}(x,t) - \delta v_t(x,t)\} + h(x,t)v(x,t)\Big]\,\mathrm{d}x\,\mathrm{d}t \\ + \int_0^T \Big(f\big(u(0,t)\big)v(0,t) + g\big(u(\pi/4,t)\big)v(\pi/4,t)\Big)\,\mathrm{d}t = 0\end{aligned} \tag{2}$$

for any $v(x,t) \in C^\infty([0,\pi/4] \times S^T)$ such that the following boundary value conditions hold

$$v_{xx}(0,\cdot) = v_{xx}(\pi/4,\cdot) = v_{xxx}(0,\cdot) = v_{xxx}(\pi/4,\cdot) = 0\,. \tag{3}$$

Here $S^T = \mathbb{R}/\{T\mathbb{Z}\}$ is the circle. The eigenvalue problem

$$w_{xxxx}(x) = \mu^4 w(x)\,,$$
$$w_{xx}(0) = w_{xx}(\pi/4) = 0\,, \qquad w_{xxx}(0) = w_{xxx}(\pi/4) = 0$$

is known ([4]) to possesses a sequence of eigenvalues μ_k, $k = -1, 0, 1, \dots$, with

$$\mu_{-1} = \mu_0 = 0$$

and

$$\cos(\mu_k\pi/4)\cosh(\mu_k\pi/4) = 1\,, \qquad k = 1, 2, \dots\,. \tag{4}$$

The corresponding orthonormal in $L^2(0,\pi/4)$ system of eigenvectors is

$$w_{-1}(x) = \frac{2}{\sqrt{\pi}}\,, \qquad w_0(x) = \frac{16}{\pi}\left(x - \frac{\pi}{8}\right)\sqrt{\frac{3}{\pi}}\,,$$

$$\begin{aligned}w_k(x) = \frac{4}{\sqrt{\pi}W_k}\Big[&\cosh(\mu_k x) + \cos(\mu_k x) \\ &- \frac{\cosh\xi_k - \cos\xi_k}{\sinh\xi_k - \sin\xi_k}\big(\sinh(\mu_k x) + \sin(\mu_k x)\big)\Big]\end{aligned}$$

where the constants W_k are given by the formulas

$$W_k = \cosh\xi_k + \cos\xi_k - \frac{\cosh\xi_k - \cos\xi_k}{\sinh\xi_k - \sin\xi_k}\big(\sinh\xi_k + \sin\xi_k\big)$$

for $\xi_k = \mu_k\pi/4$, $k \in \mathbb{N}$. From (4) we get the asymptotic formulas

$$1 < \mu_k = 2(2k+1) + r(k) \qquad \text{for all} \quad k \ge 1$$

along with

$$|r(k)| \le \hat{c}_1\,\mathrm{e}^{-\hat{c}_2 k} \qquad \text{for all} \quad k \ge 1\,,$$

where $\hat{c}_1$, $\hat{c}_2$ are positive constants. Moreover, the eigenfunctions $\{w_i\}_{i=-1}^\infty$ are uniformly bounded in $C([0,\pi/4])$.

3. Preliminary results

Let $H_1(x,t) \in C\big([0,\pi/4] \times S^T\big)$, $H_2(t), H_3(t) \in C(S^T)$ be continuous T-periodic functions and consider the equation

$$\int_0^T \int_0^{\pi/4} \Big[z(x,t)\big\{v_{tt}(x,t) + v_{xxxx}(x,t) - \delta v_t(x,t)\big\} + H_1(x,t)v(x,t)\Big]\,\mathrm{d}x\,\mathrm{d}t + \int_0^T \big\{H_2(t)v(0,t) + H_3(t)v(\pi/4,t)\big\}\,\mathrm{d}t = 0 \tag{5}$$

for any $v(x,t) \in C^\infty\big([0,\pi/4] \times S^T\big)$ satisfying the boundary conditions (3) along with

$$\int_0^{\pi/4} v(x,t)\,\mathrm{d}x = \int_0^{\pi/4} xv(x,t)\,\mathrm{d}x = 0 \qquad \text{for all} \quad t \in S^T. \tag{6}$$

Note that conditions (6) correspond to the orthogonality of $v(x,t)$ to $w_{-1}(x)$ and $w_0(x)$ for any $t \in S^T$. We look for $z(x,t)$ in the form

$$z(x,t) = \sum_{i=1}^{\infty} z_i(t)w_i(x). \tag{7}$$

We formally put (7) into (5) to get a system of ordinary differential equations

$$\ddot{z}_i(t) + \delta\dot{z}_i(t) + \mu_i^4 z_i(t) = h_i(t), \tag{8}$$

where

$$h_i(t) = -\left(\int_0^{\pi/4} H_1(x,t)w_i(x)\,\mathrm{d}x + H_2(t)w_i(0) + H_3(t)w_i(\pi/4)\right). \tag{9}$$

Let us put

$$M_1 = \sup_{\substack{i\geq 1,\\ x\in[0,\pi/4]}} |w_i(x)|.$$

Let $\omega = 2\pi/T$. We consider Banach spaces X_ω and Y_ω defined as follows

$$X_\omega := \Big\{u(x,t) \in C\big([0,\pi/4] \times S^T\big) :\ u(x,t) = \sum_{k\in\mathbb{Z}} u_k(x)\,\mathrm{e}^{\imath\omega kt},\quad u_k \in C\big([0,\pi/4],\mathbb{C}\big),\ \sum_{k\in\mathbb{Z}} \|u_k\|_\infty < \infty,\ u_{-k}(x) = \overline{u_k(x)}\Big\}$$

$$Y_\omega := \Big\{v(t) \in C(S^T) :\ v(t) = \sum_{k\in\mathbb{Z}} v_k\,\mathrm{e}^{\imath\omega kt},\ v_k \in \mathbb{C},\ \sum_{k\in\mathbb{Z}} |v_k| < \infty,\ v_{-k} = \overline{v_k}\Big\}$$

with the norms

$$|u| := \sum_{k\in\mathbb{Z}} \|u_k\|_\infty\,, \qquad |v| := \sum_{k\in\mathbb{Z}} |v_k|\,,$$

respectively, where $\|\cdot\|_\infty$ is the maximum norm. Clearly, $\|u\|_\infty \le |u|$ and $\|v\|_\infty \le |v|$.

We also consider the Banach spaces $X_{\omega,0}$ defined as follows

$$X_{\omega,0} = \bigl\{v(x,t) \in X_\omega : \text{ conditions of (6) hold}\bigr\}$$

with the same norm $|\cdot|$ as for X_ω.

If $H_1(x,t) \in X_{\omega,0}$ and $H_2(t), H_3(t) \in Y_\omega$, then

$$H_1(x,t) = \sum_{k\in\mathbb{Z}} h_{1,k}(x)\,\mathrm{e}^{\imath\omega kt},$$

$$\int_0^{\pi/4} h_{1,k}(x)\,\mathrm{d}x = \int_0^{\pi/4} x h_{1,k}(x)\,\mathrm{d}x = 0\,, \qquad h_{1,-k}(x) = \overline{h_{1,k}(x)}\,,$$

$$H_2(t) = \sum_{k\in\mathbb{Z}} h_{2,k}\,\mathrm{e}^{\imath\omega kt}, \qquad h_{2,-k} = \overline{h_{2,k}}\,,$$

$$H_3(t) = \sum_{k\in\mathbb{Z}} h_{3,k}\,\mathrm{e}^{\imath\omega kt}, \qquad h_{3,-k} = \overline{h_{3,k}}\,.$$

Hence $h_i(t)$ from (9) has the form

$$h_i(t) = \sum_{k\in\mathbb{Z}} h_{i,k}\,\mathrm{e}^{\imath\omega kt} \tag{10}$$

with

$$h_{i,k} = -\left(\int_0^{\pi/4} h_{1,k}(x) w_i(x)\,\mathrm{d}x + h_{2,k} w_i(0) + h_{3,k} w_i(\pi/4)\right).$$

Clearly $h_{i,-k} = \overline{h_{i,k}}$. Consequently, we get

$$|h_i| = \sum_{k\in Z} |h_{i,k}| \le M_1\Bigl(\frac{\pi}{4}|H_1| + |H_2| + |H_3|\Bigr)\,. \tag{11}$$

Now we look for a solution $z_i \in Y_\omega$ of (8) with $h_i(t)$ of the form (10). Hence from $z_i(t) = \sum_{k\in\mathbb{Z}} z_{i,k}\,\mathrm{e}^{\imath\omega kt}$ we get

$$z_{i,k} = \frac{h_{i,k}}{\mu_i^4 - \omega^2 k^2 + \imath\delta\omega k}\,.$$

Clearly, $z_{i,-k} = \overline{z_{i,k}}$. For any $t \geq 0$, we have

$$(\mu_i^4 - t)^2 + \delta^2 t \geq \gamma_i^2$$

for the constants γ_i defined as follows

$$\gamma_i = \gamma(\mu_i, \delta, \omega) := \begin{cases} \mu_i^4 & \text{for } \delta^2 \geq 2\mu_i^4 , \\ \frac{\delta}{2}\sqrt{4\mu_i^4 - \delta^2} & \text{for } 0 < \delta^2 \leq 2\mu_i^4 . \end{cases}$$

Thus we get

$$|z_i| = \sum_{k\in\mathbb{Z}} |z_{i,k}| \leq |h_i|/\gamma_i .$$

Clearly such $z_i(t)$ satisfies (8). Now the series $\sum_{i=1}^{\infty} 1/\gamma_i$ converges, so the function (7) is well-defined and

$$\begin{aligned} z(x,t) = \sum_{i=1}^{\infty} z_i(t) w_i(x) &= \sum_{i=1}^{\infty}\sum_{k\in\mathbb{Z}} \frac{h_{i,k}}{\mu_i^4 - \omega^2 k^2 + \imath\delta\omega k}\, \mathrm{e}^{\imath\omega k t}\, w_i(x) \\ &= \sum_{k\in\mathbb{Z}} \left(\sum_{i=1}^{\infty} \frac{h_{i,k}}{\mu_i^4 - \omega^2 k^2 + \imath\delta\omega k} w_i(x) \right) \mathrm{e}^{\imath\omega k t} . \end{aligned}$$

Hence $z(x,t) \in X_{\omega,0}$ and by (11), it satisfies

$$\begin{aligned} |z| = \sum_{k\in\mathbb{Z}} \left\| \sum_{i=1}^{\infty} \frac{h_{i,k}}{\mu_i^4 - \omega^2 k^2 + \imath\delta\omega k} w_i(x) \right\|_{\infty} &\leq M_1 \sum_{i=1}^{\infty} \frac{1}{\gamma_i} \sum_{k\in\mathbb{Z}} |h_{i,k}| \\ &= M_1 \sum_{i=1}^{\infty} \frac{1}{\gamma_i} |h_i| \leq M_2 \Big(\frac{\pi}{4} |H_1| + |H_2| + |H_3| \Big) , \end{aligned}$$

where

$$M_2 := M_1 \sum_{i=1}^{\infty} \frac{1}{\gamma_i} < \infty .$$

Summarizing, we get the next result.

PROPOSITION 1. *For any given functions $H_1(x,t) \in X_{\omega,0}$, $H_2(t), H_3(t) \in Y_\omega$ equation* (5) *has a unique solution $z(x,t) \in X_{\omega,0}$ of the form*

$$z(x,t) = \sum_{i=1}^{\infty} z_i(t) w_i(x)$$

with $z_i(t) \in Y_\omega$ for any $i \geq 1$. Such a solution satisfies:

(a) $|z| \leq M_2\big(\frac{\pi}{4}|H_1| + |H_2| + |H_3|\big)$.

(b) *The mapping $L_1 \colon X_{\omega,0} \times Y_\omega \times Y_\omega \to X_{\omega,0}$ defined by $L_1(H_1, H_2, H_3) := z(x,t)$ is compact.*

Proof. It remains to prove the compactness of L_1. For this reason, let us put

$$\gamma_{i,k} := \sqrt{(\mu_i^4 - \omega^2 k^2)^2 + \delta^2\omega^2 k^2}\,.$$

Clearly, $\gamma_{i,k} \geq \gamma_i$ and $\gamma_{i,k} \geq \delta\omega|k|$ for any $i \geq 1$ and $k \in \mathbb{Z}$. Now let us take a bounded sequence $\{(H_{1,n}, H_{2,n}, H_{3,n})\}_{n\in\mathbb{N}} \subset X_{\omega,0} \times Y_\omega \times Y_\omega$. Hence

$$H_{1,n}(x,t) = \sum_{k\in\mathbb{Z}} h_{1,k,n}(x)\,\mathrm{e}^{\imath\omega kt},$$

$$\int_0^{\pi/4} h_{1,k,n}(x)\,\mathrm{d}x = \int_0^{\pi/4} x h_{1,k,n}(x)\,\mathrm{d}x = 0\,, \qquad h_{1,-k,n}(x) = \overline{h_{1,k,n}(x)}\,,$$

$$H_{2,n}(t) = \sum_{k\in\mathbb{Z}} h_{2,k,n}\,\mathrm{e}^{\imath\omega kt}, \qquad h_{2,-k,n} = \overline{h_{2,k,n}}\,,$$

$$H_{3,n}(t) = \sum_{k\in\mathbb{Z}} h_{3,k,n}\,\mathrm{e}^{\imath\omega kt}, \qquad h_{3,-k,n} = \overline{h_{3,k,n}}\,.$$

Then we get

$$z_n(x,t) = \sum_{k\in\mathbb{Z}}\left(\sum_{i=1}^{\infty}\frac{h_{i,k,n}}{\mu_i^4 - \omega^2k^2 + \imath\delta\omega k}w_i(x)\right)\mathrm{e}^{\imath\omega kt},$$

where

$$h_{i,k,n} = -\left(\int_0^{\pi/4} h_{1,k,n}(x)w_i(x)\,\mathrm{d}x + h_{2,k,n}w_i(0) + h_{3,k,n}w_i(\pi/4)\right).$$

We note that there is a constant $\tilde{K}_1 > 0$ such that

$$\sum_{k\in\mathbb{Z}} |h_{i,k,n}| \leq \tilde{K}_1$$

for any $i, n \in \mathbb{N}$. By using the Cantor diagonal procedure, we can suppose that

$$h_{i,k,n} \to h_{i,k,0}$$

as $n \to \infty$ for any $i \in \mathbb{N}$ and $k \in \mathbb{Z}$. Then clearly

$$\sum_{k\in\mathbb{Z}} |h_{i,k,0}| \leq \tilde{K}_1$$

for any $i \in \mathbb{N}$. So the function

$$z_0(x,t) = \sum_{k\in\mathbb{Z}}\left(\sum_{i=1}^{\infty}\frac{h_{i,k,0}}{\mu_i^4 - \omega^2k^2 + \imath\delta\omega k}w_i(x)\right)\mathrm{e}^{\imath\omega kt}$$

belongs to $X_{\omega,0}$. We prove that $z_n(x,t) \to z_0(x,t)$ in $X_{\omega,0}$. So let $\varepsilon > 0$ be given. We take so large $i_0, k_0 \in \mathbb{N}$ that

$$\sum_{i=i_0+1}^{\infty} \frac{1}{\gamma_i} \le \frac{\varepsilon}{6\tilde{K}_1 M_1}, \qquad 6\tilde{K}_1 M_1 i_0 \le \varepsilon\delta\omega(k_0+1).$$

Then

$$\begin{aligned}\sum_{k\in\mathbb{Z}} \sum_{i=i_0+1}^{\infty} \frac{|h_{i,k,n} - h_{i,k,0}|}{\gamma_{i,k}} &\le \sum_{k\in\mathbb{Z}} \sum_{i=i_0+1}^{\infty} \frac{|h_{i,k,n} - h_{i,k,0}|}{\gamma_i} \\ &= \sum_{i=i_0+1}^{\infty} \frac{1}{\gamma_i} \sum_{k\in\mathbb{Z}} |h_{i,k,n} - h_{i,k,0}| \\ &\le 2\tilde{K}_1 \sum_{i=i_0+1}^{\infty} \frac{1}{\gamma_i} \le \frac{\varepsilon}{3M_1},\end{aligned}$$

and

$$\begin{aligned}\sum_{i=1}^{i_0} \sum_{|k|\ge k_0+1} \frac{|h_{i,k,n} - h_{i,k,0}|}{\gamma_{i,k}} &\le \sum_{i=1}^{i_0} \sum_{|k|\ge k_0+1} \frac{|h_{i,k,n} - h_{i,k,0}|}{\delta\omega|k|} \\ &\le \frac{1}{\delta\omega(k_0+1)} \sum_{i=1}^{i_0} \sum_{|k|\ge k_0+1} |h_{i,k,n} - h_{i,k,0}| \\ &\le \frac{2\tilde{K}_1 i_0}{\delta\omega(k_0+1)} \le \frac{\varepsilon}{3M_1}.\end{aligned}$$

By using the above estimates, we obtain

$$\begin{aligned}|z_0 - z_n| &\le M_1 \sum_{k\in\mathbb{Z}} \sum_{i=1}^{\infty} \frac{|h_{i,k,n} - h_{i,k,0}|}{\gamma_{i,k}} \\ &= M_1 \sum_{|k|\le k_0} \sum_{i=1}^{i_0} \frac{|h_{i,k,n} - h_{i,k,0}|}{\gamma_{i,k}} + M_1 \sum_{|k|\ge k_0+1} \sum_{i=1}^{i_0} \frac{|h_{i,k,n} - h_{i,k,0}|}{\gamma_{i,k}} \\ &\qquad + M_1 \sum_{k\in\mathbb{Z}} \sum_{i=i_0+1}^{\infty} \frac{|h_{i,k,n} - h_{i,k,0}|}{\gamma_{i,k}} \\ &\le M_1 \sum_{|k|\le k_0} \sum_{i=1}^{i_0} \frac{|h_{i,k,n} - h_{i,k,0}|}{\gamma_i} + \frac{2\varepsilon}{3}.\end{aligned}$$

Since $h_{i,k,n} \to h_{i,k,0}$ as $n \to \infty$ for any $i \in \mathbb{N}$ and $k \in \mathbb{Z}$, there is an $n_0 \in \mathbb{N}$ such that for any $n \ge n_0$, it holds that

$$M_1 \sum_{|k|\le k_0} \sum_{i=1}^{i_0} \frac{|h_{i,k,n} - h_{i,k,0}|}{\gamma_i} \le \frac{\varepsilon}{3}.$$

Summarizing, we get $|z_0 - z_n| \le \varepsilon$ for any $n \ge n_0$. Since $\varepsilon > 0$ is arbitrary, the compactness of L_1 is proved. The proof is finished. □

Let us put

$$Y_{\omega,0} := \Big\{h = \sum_{k\in\mathbb{Z}} h_k\, \mathrm{e}^{\imath\omega kt} \in Y_\omega :\ h_0 = 0\Big\}$$

with the same norm $|\cdot|$ as for Y_ω. We introduce the projection $Q\colon Y_\omega \to Y_\omega$ given by

$$Qy = \frac{1}{T}\int_0^T y(s)\,\mathrm{d}s\,,$$

and the projection $P\colon Y_\omega \to Y_{\omega,0}$, $P = I - Q$. Note that $T = 2\pi/\omega$. Clearly $\|P\| = \|Q\| = 1$.

Now we consider the equation

$$\ddot y + \delta\dot y = h(t) \tag{12}$$

for $y, w, h \in Y_\omega$. We need the following simple result.

PROPOSITION 2. *Equation* (12) *has a solution* $y \in Y_\omega$ *if and only if* $h \in Y_{\omega,0}$ *and this solution is unique for* $y := L_2 h \in Y_{\omega,0}$ *satisfying*

$$|y| \le \frac{1}{\omega\sqrt{\delta^2+\omega^2}}|h|\,.$$

Moreover, the linear mapping $L_2\colon Y_{\omega,0} \to Y_{\omega,0}$ *is compact.*

Proof. If equation (12) has a solution $y \in Y_\omega$ then clearly $\int_0^T h(t)\,\mathrm{d}t = 0$, so $h \in Y_{\omega,0}$. On the other hand, if $h \in Y_{\omega,0}$, then

$$h(t) = \sum_{k\in\mathbb{Z}\setminus\{0\}} h_k\,\mathrm{e}^{\imath\omega kt}\,.$$

Let

$$y(t) := \sum_{k\in\mathbb{Z}\setminus\{0\}} \frac{h_k}{-\omega^2k^2 + \imath\delta\omega k}\,\mathrm{e}^{\imath\omega kt}\,.$$

Hence $y \in Y_{\omega,0}$ and

$$|y| = \sum_{k\in\mathbb{Z}\setminus\{0\}} \frac{|h_k|}{\sqrt{\omega^4k^4+\delta^2\omega^2k^2}} \le \sum_{k\in\mathbb{Z}\setminus\{0\}} \frac{|h_k|}{\sqrt{\omega^4+\delta^2\omega^2}} = \frac{|h|}{\omega\sqrt{\omega^2+\delta^2}}\,.$$

Similarly we can show that $\dot y, \ddot y \in Y_{\omega,0}$ and thus $y(t)$ solves (12). This proves the first part of Proposition 2.

To show the compactness of $L_2\colon Y_{\omega,0} \to Y_{\omega,0}$, we take a bounded sequence $\{h_n(t)\}_{n\in\mathbb{N}} \subset Y_{\omega,0}$ with

$$h_n(t) = \sum_{k\in\mathbb{Z}\setminus\{0\}} h_{k,n}\,\mathrm{e}^{\imath\omega kt}, \qquad h_{k,n} = \overline{h_{-k,n}},$$

and there is a constant $\tilde{K}_2 > 0$ such that

$$\sum_{k\in\mathbb{Z}\setminus\{0\}} |h_{k,n}| \le \tilde{K}_2$$

for any $n \in \mathbb{N}$. Again by using the Cantor diagonal procedure, we can suppose that $h_{k,n} \to h_{k,0}$ as $n \to \infty$ for any $k \in \mathbb{Z}\setminus\{0\}$. Then

$$\sum_{k\in\mathbb{Z}\setminus\{0\}} |h_{k,0}| \le \tilde{K}_2$$

and the function

$$y_0(t) := \sum_{k\in\mathbb{Z}\setminus\{0\}} \frac{h_{k,0}}{-\omega^2k^2 + \imath\delta\omega k}\,\mathrm{e}^{\imath\omega kt}$$

belongs to $Y_{\omega,0}$. Let $\varepsilon > 0$ be given. We take $k_0 \in \mathbb{N}$ so large that

$$\sqrt{\omega^4k_0^4 + \delta^2\omega^2k_0^2} \ge \frac{4\tilde{K}_2}{\varepsilon}$$

and put $y_n = L_2h_n$. Then

$$\begin{aligned}
|y_0 - y_n| &= \sum_{k\in\mathbb{Z}\setminus\{0\}} \frac{|h_{k,n} - h_{k,0}|}{\sqrt{\omega^4k^4 + \delta^2\omega^2k^2}} \\
&= \sum_{0<|k|\le k_0} \frac{|h_{k,n} - h_{k,0}|}{\sqrt{\omega^4k^4 + \delta^2\omega^2k^2}} + \sum_{|k|\ge k_0+1} \frac{|h_{k,n} - h_{k,0}|}{\sqrt{\omega^4k^4 + \delta^2\omega^2k^2}} \\
&\le \sum_{0<|k|\le k_0} \frac{|h_{k,n} - h_{k,0}|}{\sqrt{\omega^4k^4 + \delta^2\omega^2k^2}} + \frac{\varepsilon}{4\tilde{K}_2} \sum_{|k|\ge k_0+1} |h_{k,n} - h_{k,0}| \\
&\le \sum_{0<|k|\le k_0} \frac{|h_{k,n} - h_{k,0}|}{\sqrt{\omega^4k^4 + \delta^2\omega^2k^2}} + \frac{\varepsilon}{2}\,.
\end{aligned}$$

Since $h_{k,n} \to h_{k,0}$ as $n \to \infty$ for any $k \in \mathbb{Z}\setminus\{0\}$, there is an $n_0 \in \mathbb{N}$ such that for any $n \ge n_0$, it holds that

$$\sum_{0<|k|\le k_0} \frac{|h_{k,n} - h_{k,0}|}{\sqrt{\omega^4k^4 + \delta^2\omega^2k^2}} \le \frac{\varepsilon}{2}\,.$$

Summarizing, we get $|y_0 - y_n| \le \varepsilon$ for any $n \ge n_0$. Since $\varepsilon > 0$ is arbitrary, the compactness of L_2 is proved. The proof is finished. □

4. Periodic solutions

In this section, we present the main results concerning equation (1). We seek a solution $u(x,t)$ of (2) in the form

$$u(x,t) = y_1(t)w_{-1}(x) + y_2(t)w_0(x) + z(x,t)$$

where $y_1(t), y_2(t) \in Y_\omega$ and $z(x,t) \in X_{\omega,0}$ belongs to the infinite dimensional space spanned by $\{w_i\}_{i=1}^{\infty}$. To get the equations for $y_1(t)$, $y_2(t)$, and $z(x,t)$ we take $v(x,t) = \phi_1(t)w_{-1}(x) + \phi_2(t)w_0(x) + v_0(x,t)$ in (2) with $\phi_i \in C^\infty(S^T)$, $v_0(x,t) \in C^\infty([0,\pi/4] \times S^T)$ satisfying besides (3) also (6). Plugging the above expression for $v(x,t)$ into (2) and using the orthonormality, we arrive at the system of equations

$$\begin{aligned} &\ddot{y}_1(t) + \delta\dot{y}_1(t) + \frac{2}{\sqrt{\pi}}\int_0^{\pi/4} h(x,t)\,\mathrm{d}x \\ &\quad + \frac{2}{\sqrt{\pi}} f\Big(\frac{2}{\sqrt{\pi}}y_1(t) - 2\sqrt{\frac{3}{\pi}}y_2(t) + z(0,t)\Big) \\ &\quad + \frac{2}{\sqrt{\pi}} g\Big(\frac{2}{\sqrt{\pi}}y_1(t) + 2\sqrt{\frac{3}{\pi}}y_2(t) + z(\pi/4,t)\Big) = 0\,, \end{aligned} \tag{13}$$

$$\begin{aligned} &\ddot{y}_2(t) + \delta\dot{y}_2(t) + \frac{16}{\pi}\sqrt{\frac{3}{\pi}}\int_0^{\pi/4} h(x,t)\Big(x - \frac{\pi}{8}\Big)\,\mathrm{d}x \\ &\quad - 2\sqrt{\frac{3}{\pi}} f\Big(\frac{2}{\sqrt{\pi}}y_1(t) - 2\sqrt{\frac{3}{\pi}}y_2(t) + z(0,t)\Big) \\ &\quad + 2\sqrt{\frac{3}{\pi}} g\Big(\frac{2}{\sqrt{\pi}}y_1(t) + 2\sqrt{\frac{3}{\pi}}y_2(t) + z(\pi/4,t)\Big) = 0\,, \end{aligned} \tag{14}$$

$$\begin{aligned} &\int_0^T\int_0^{\pi/4} \big[z(x,t)\{v_{tt}(x,t) + v_{xxxx}(x,t) - \delta v_t(x,t)\} + h(x,t)v(x,t)\big]\,\mathrm{d}x\,\mathrm{d}t \\ &\qquad + \int_0^T \big\{f\big(u(0,t)\big)v(0,t) + g\big(u(\pi/4,t)\big)v(\pi/4,t)\big\}\,\mathrm{d}t = 0 \end{aligned} \tag{15}$$

where we wrote $v(x,t)$ instead $v_0(x,t)$. Thus, in equation (15), $v(x,t)$ is any function in $C^\infty([0,\pi/4] \times S^T)$ such that the conditions (3) and (6) hold. We remark that in this way we have split the original equation in two parts: to the resonant finite-dimensional part represented by (13)–(14) and to the non-resonant infinite-dimensional part represented by (15).

Now we take in (13)–(15) the decomposition $y_i(t) \leftrightarrow c_i + y_i(t)$ for $i = 1,2$ and $c_i \in \mathbb{R}$, $y_i(t) \in Y_{\omega,0}$, and then we also plug this system to a homotopy with a parameter $\lambda \in [0,1]$. So we get the system

$$\ddot{y}_1(t) + \delta\dot{y}_1(t) + P\left\{ \frac{2}{\sqrt{\pi}} \int_0^{\pi/4} h(x,t)\,\mathrm{d}x \right.$$
$$+ \frac{2}{\sqrt{\pi}} f\left(\frac{2}{\sqrt{\pi}}c_1 + \lambda\frac{2}{\sqrt{\pi}}y_1(t) - 2\sqrt{\frac{3}{\pi}}c_2 - \lambda 2\sqrt{\frac{3}{\pi}}y_2(t) + \lambda z(0,t)\right)$$
$$\left. + \frac{2}{\sqrt{\pi}} g\left(\frac{2}{\sqrt{\pi}}c_1 + \lambda\frac{2}{\sqrt{\pi}}y_1(t) + 2\sqrt{\frac{3}{\pi}}c_2 + \lambda 2\sqrt{\frac{3}{\pi}}y_2(t) + \lambda z(\pi/4,t)\right)\right\} = 0\,, \tag{16}$$

$$\ddot{y}_2(t) + \delta\dot{y}_2(t) + P\left\{ \frac{16}{\pi}\sqrt{\frac{3}{\pi}} \int_0^{\pi/4} h(x,t)\left(x - \frac{\pi}{8}\right)\mathrm{d}x \right.$$
$$- 2\sqrt{\frac{3}{\pi}} f\left(\frac{2}{\sqrt{\pi}}c_1 + \lambda\frac{2}{\sqrt{\pi}}y_1(t) - 2\sqrt{\frac{3}{\pi}}c_2 - \lambda 2\sqrt{\frac{3}{\pi}}y_2(t) + \lambda z(0,t)\right)$$
$$\left. + 2\sqrt{\frac{3}{\pi}} g\left(\frac{2}{\sqrt{\pi}}c_1 + \lambda\frac{2}{\sqrt{\pi}}y_1(t) + 2\sqrt{\frac{3}{\pi}}c_2 + \lambda 2\sqrt{\frac{3}{\pi}}y_2(t) + \lambda z(\pi/4,t)\right)\right\} = 0\,, \tag{17}$$

$$\int_0^T \int_0^{\pi/4} \left[z(x,t)\left\{v_{tt}(x,t) + v_{xxxx}(x,t) - \delta v_t(x,t)\right\} + \lambda h(x,t)v(x,t)\right] \mathrm{d}x\,\mathrm{d}t$$
$$+ \lambda \int_0^T \left\{ f\left(\frac{2}{\sqrt{\pi}}c_1 + \lambda\frac{2}{\sqrt{\pi}}y_1(t) - 2\sqrt{\frac{3}{\pi}}c_2 - \lambda 2\sqrt{\frac{3}{\pi}}y_2(t) + \lambda z(0,t)\right)v(0,t) \right.$$
$$\left. + g\left(\frac{2}{\sqrt{\pi}}c_1 + \lambda\frac{2}{\sqrt{\pi}}y_1(t) + 2\sqrt{\frac{3}{\pi}}c_2 + \lambda 2\sqrt{\frac{3}{\pi}}y_2(t) + \lambda z(\pi/4,t)\right)v(\pi/4,t)\right\}\mathrm{d}t = 0 \tag{18}$$

$$\int_0^T f\left(\frac{2}{\sqrt{\pi}}c_1 + \lambda\frac{2}{\sqrt{\pi}}y_1(t) - 2\sqrt{\frac{3}{\pi}}c_2 - \lambda 2\sqrt{\frac{3}{\pi}}y_2(t) + \lambda z(0,t)\right)\mathrm{d}t = \theta_1\,,$$

$$\int_0^T g\left(\frac{2}{\sqrt{\pi}}c_1 + \lambda\frac{2}{\sqrt{\pi}}y_1(t) + 2\sqrt{\frac{3}{\pi}}c_2 + \lambda 2\sqrt{\frac{3}{\pi}}y_2(t) + \lambda z(\pi/4,t)\right)\mathrm{d}t = \theta_2\,, \tag{19}$$

$$\theta_1 = \frac{4}{\pi}\int_0^T\int_0^{\pi/4} h(x,t)\left(x - \frac{\pi}{4}\right)\mathrm{d}x\,\mathrm{d}t\,, \qquad \theta_2 = -\frac{4}{\pi}\int_0^T\int_0^{\pi/4} h(x,t)x\,\mathrm{d}x\,\mathrm{d}t\,.$$

We note that system (19) is derived from the system

$$
\begin{aligned}
(I-P)\Bigg\{&\int_0^{\pi/4} h(x,t)\,\mathrm{d}x \\
&+ f\bigg(\frac{2}{\sqrt{\pi}}c_1 + \lambda\frac{2}{\sqrt{\pi}}y_1(t) - 2\sqrt{\frac{3}{\pi}}c_2 - \lambda 2\sqrt{\frac{3}{\pi}}y_2(t) + \lambda z(0,t)\bigg) \\
&+ g\bigg(\frac{2}{\sqrt{\pi}}c_1 + \lambda\frac{2}{\sqrt{\pi}}y_1(t) + 2\sqrt{\frac{3}{\pi}}c_2 + \lambda 2\sqrt{\frac{3}{\pi}}y_2(t) + \lambda z(\pi/4,t)\bigg)\Bigg\} = 0\,, \\
(I-P)\Bigg\{&\frac{8}{\pi}\int_0^{\pi/4} h(x,t)\Big(x-\frac{\pi}{8}\Big)\,\mathrm{d}x \\
&- f\bigg(\frac{2}{\sqrt{\pi}}c_1 + \lambda\frac{2}{\sqrt{\pi}}y_1(t) - 2\sqrt{\frac{3}{\pi}}c_2 - \lambda 2\sqrt{\frac{3}{\pi}}y_2(t) + \lambda z(0,t)\bigg) \\
&+ g\bigg(\frac{2}{\sqrt{\pi}}c_1 + \lambda\frac{2}{\sqrt{\pi}}y_1(t) + 2\sqrt{\frac{3}{\pi}}c_2 + \lambda 2\sqrt{\frac{3}{\pi}}y_2(t) + \lambda z(\pi/4,t)\bigg)\Bigg\} = 0\,.
\end{aligned}
\tag{20}
$$

Since $(I-P)y = Qy = \frac{1}{T}\int_0^T y(s)\,\mathrm{d}s$, system (20) is equivalent to the system

$$
\begin{aligned}
&\int_0^T f\bigg(\frac{2}{\sqrt{\pi}}c_1 + \lambda\frac{2}{\sqrt{\pi}}y_1(t) - 2\sqrt{\frac{3}{\pi}}c_2 - \lambda 2\sqrt{\frac{3}{\pi}}y_2(t) + \lambda z(0,t)\bigg)\,\mathrm{d}t \\
&\qquad + \int_0^T g\bigg(\frac{2}{\sqrt{\pi}}c_1 + \lambda\frac{2}{\sqrt{\pi}}y_1(t) + 2\sqrt{\frac{3}{\pi}}c_2 + \lambda 2\sqrt{\frac{3}{\pi}}y_2(t) + \lambda z(\pi/4,t)\bigg)\,\mathrm{d}t \\
&= -\int_0^T\int_0^{\pi/4} h(x,t)\,\mathrm{d}x\,\mathrm{d}t\,, \\
&\quad -\int_0^T f\bigg(\frac{2}{\sqrt{\pi}}c_1 + \lambda\frac{2}{\sqrt{\pi}}y_1(t) - 2\sqrt{\frac{3}{\pi}}c_2 - \lambda 2\sqrt{\frac{3}{\pi}}y_2(t) + \lambda z(0,t)\bigg)\,\mathrm{d}t \\
&\qquad + \int_0^T g\bigg(\frac{2}{\sqrt{\pi}}c_1 + \lambda\frac{2}{\sqrt{\pi}}y_1(t) + 2\sqrt{\frac{3}{\pi}}c_2 + \lambda 2\sqrt{\frac{3}{\pi}}y_2(t) + \lambda z(\pi/4,t)\bigg)\,\mathrm{d}t \\
&= \frac{8}{\pi}\int_0^T\int_0^{\pi/4} h(x,t)\Big(\frac{\pi}{8}-x\Big)\,\mathrm{d}x\,\mathrm{d}t\,.
\end{aligned}
\tag{21}
$$

It is clear that system (21) implies system (19). Now, we take

$$d_1 = \frac{2}{\sqrt{\pi}} c_1 - 2\sqrt{\frac{3}{\pi}} c_2 , \qquad \zeta_1(t) = \frac{2}{\sqrt{\pi}} y_1(t) ,$$
$$d_2 = \frac{2}{\sqrt{\pi}} c_1 + 2\sqrt{\frac{3}{\pi}} c_2 , \qquad \zeta_2(t) = 2\sqrt{\frac{3}{\pi}} y_2(t) ,$$

and we split

$$h(x,t) = 8\frac{\theta_2 - 2\theta_1}{T\pi} + 96\frac{\theta_1 - \theta_2}{T\pi^2} x + p(x,t) \tag{22}$$

such that

$$\int_0^T \int_0^{\pi/4} p(x,t)\ \mathrm{d}x\,\mathrm{d}t = \int_0^T \int_0^{\pi/4} xp(x,t)\ \mathrm{d}x\,\mathrm{d}t = 0 . \tag{23}$$

By using these notations along with Propositions 1 and 2 we can rewrite system (16) (19) as the following semi-fixed point problem

$$\zeta_1(t) = -\frac{4}{\pi} L_2 P \Bigg\{ \int_0^{\pi/4} p(x,t)\ \mathrm{d}x + f\big(d_1 + \lambda\zeta_1(t) - \lambda\zeta_2(t) + \lambda z(0,t)\big) + g\big(d_2 + \lambda\zeta_1(t) + \lambda\zeta_2(t) + \lambda z(\pi/4,t)\big) \Bigg\}, \tag{24}$$

$$\zeta_2(t) = -\frac{12}{\pi} L_2 P \Bigg\{ \frac{8}{\pi} \int_0^{\pi/4} p(x,t)\Big(x - \frac{\pi}{8}\Big)\ \mathrm{d}x - f\big(d_1 + \lambda\zeta_1(t) - \lambda\zeta_2(t) + \lambda z(0,t)\big) + g\big(d_2 + \lambda\zeta_1(t) + \lambda\zeta_2(t) + \lambda z(\pi/4,t)\big) \Bigg\}, \tag{25}$$

$$\begin{aligned} z(x,t) = \lambda L_1 \Big(p(x,t), f\big(d_1 + \lambda\zeta_1(t) - \lambda\zeta_2(t) + \lambda z(0,t)\big),& \\ g\big(d_2 + \lambda\zeta_1(t) + \lambda\zeta_2(t) + \lambda z(\pi/4,t)\big)\Big)& , \end{aligned} \tag{26}$$

$$\begin{aligned} \int_0^T f\big(d_1 + \lambda\zeta_1(t) - \lambda\zeta_2(t) + \lambda z(0,t)\big)\ \mathrm{d}t &= \theta_1 , \\ \int_0^T g\big(d_2 + \lambda\zeta_1(t) + \lambda\zeta_2(t) + \lambda z(\pi/4,t)\big)\ \mathrm{d}t &= \theta_2 . \end{aligned} \tag{27}$$

Since f and g are analytic, we have expansions $f(x) = \sum_{n=0}^{\infty} a_n x^n$ and $g(x) = \sum_{n=0}^{\infty} b_n x^n$. Then we put

$$F(x) = \sum_{n=0}^{\infty} |a_n| x^n \,, \qquad G(x) = \sum_{n=0}^{\infty} |b_n| x^n \,.$$

We note that X_ω, $X_{\omega,0}$, Y_ω and $Y_{\omega,0}$ are all Banach algebras [6]. Now again by using Propositions 1 and 2, from (24)–(26) we get

$$\begin{aligned}
|\zeta_1| &\le \frac{4}{\pi\omega\sqrt{\omega^2+\delta^2}}\Big\{\frac{\pi}{4}|p| + \big(F(A+|d_1|) + G(A+|d_2|)\big)\Big\}\,,\\
|\zeta_2| &\le \frac{12}{\pi\omega\sqrt{\omega^2+\delta^2}}\Big\{\frac{\pi}{8}|p| + \big(F(A+|d_1|) + G(A+|d_2|)\big)\Big\}\,,\\
|z| &\le M_2\Big(\frac{\pi}{4}|p| + \big(F(A+|d_1|) + G(A+|d_2|)\big)\Big)\,,
\end{aligned}$$

where $A = |\zeta_1| + |\zeta_2| + |z|$. By summing up the above inequalities, we obtain the following:

PROPOSITION 3. *Let $h \in X_\omega$. If system (24)–(27) has a solution $\zeta_1(t), \zeta_2(t) \in Y_{\omega,0}$ and $z(x,t) \in X_{\omega,0}$, then it holds that*

$$A \le \left(\frac{5}{2\omega\sqrt{\omega^2+\delta^2}} + M_2\frac{\pi}{4}\right)|p| + \big(F(A+|d_1|) + G(A+|d_2|)\big)\left(\frac{16}{\pi\omega\sqrt{\omega^2+\delta^2}} + M_2\right),$$

where $A = |\zeta_1| + |\zeta_2| + |z|$.

Now we can prove the main results of this note.

THEOREM 1. *Let $h \in X_\omega$ and $\theta_1, \theta_2 \in \mathbb{R}$. Let $\bar{c}_1$, $\bar{c}_2$ be simple roots of the equations $f(\bar{c}_1) = \theta_1/T$ and $g(\bar{c}_2) = \theta_2/T$, respectively. We assume the existence of positive constants r_1, r_2, k_1, k_2, K_1, K_2 and A such that*

$$k_1 \le |f'(s_1)| \le K_1 \,, \qquad k_2 \le |g'(s_2)| \le K_2$$

for any $|s_i - \bar{c}_i| \le r_i + A$, $i = 1, 2$, and

$$\begin{aligned}
A > &\left(\frac{5}{2\omega\sqrt{\omega^2+\delta^2}} + M_2\frac{\pi}{4}\right)|p|\\
&+ \big(F(A+|\bar{c}_1|+r_1) + G(A+|\bar{c}_2|+r_2)\big)\left(\frac{16}{\pi\omega\sqrt{\omega^2+\delta^2}} + M_2\right).
\end{aligned}$$

If

$$k_i r_i > AK_i\,, \qquad i = 1,2\,,$$

then equation (1) *has a solution* $u(x,t) \in X_\omega$.

Proof. We solve the system (24)–(27) on the ball B in the Banach space $X = \mathbb{R}^2 \times Y^2_{\omega,0} \times X_{\omega,0}$ given by

$$B := \big\{(d_1, d_2, \zeta_1, \zeta_2, z) \in X : \ |d_1 - \bar{c}_1| \le r_1\,,\ |d_2 - \bar{c}_2| \le r_2\,,\ |\zeta_1| + |\zeta_2| + |z| \le A\big\}\,.$$

The norm on X is given by

$$|(d_1, d_2, \zeta_1, \zeta_2, z)| = |d_1| + |d_2| + |\zeta_1| + |\zeta_2| + |z|\,.$$

We show that the system (24)–(27) has no solutions on the border (the sphere) ∂B of the ball B. For $|\zeta_1| + |\zeta_2| + |z| = A$, this follows from Proposition 3. For $|d_1 - \bar{c}_1| = r_1$ and $|\zeta_1| + |\zeta_2| + |z| \le A$, we have $A \ge \|\zeta_1\|_\infty + \|\zeta_2\|_\infty + \|z\|_\infty$ and

$$\begin{aligned}\left|\int_0^T f\big(d_1 + \lambda\zeta_1(t) - \lambda\zeta_2(t) + \lambda z(0,t)\big)\,\mathrm{d}t - \theta_1\right| &\ge T|f(d_1) - f(\bar{c}_1)| - K_1 T A \\ &\ge (k_1 r_1 - K_1 A)T > 0\,.\end{aligned}$$

Similarly for $|d_2 - \bar{c}_2| = r_2$ and $|\zeta_1| + |\zeta_2| + |z| \le A$. Consequently, by using Propositions 1 and 2, we can apply the Leray-Schauder degree theory to the system (24)–(27). Indeed, let us denote by $\tilde{\Psi}_1(d_1, d_2, \zeta_1, \zeta_2, z, \lambda)$ the left-hand side of (27) and by $\Psi_2(d_1, d_2, \zeta_1, \zeta_2, z, \lambda)$ the right-hand side of (24)–(26), respectively. We put

$$\Psi_1 := \tilde{\Psi}_1(d_1, d_2, \zeta_1, \zeta_2, z, \lambda) + (d_1 - \theta_1, d_2 - \theta_2)\,.$$

Then by using Propositions 1 and 2, the operators

$$\Psi_1 \colon X \times [0,1] \to \mathbb{R}^2\,, \qquad \Psi_2 \colon X \times [0,1] \to Y^2_{\omega,0} \times X_{\omega,0}$$

are compact and continuous. Moreover, by putting

$$\psi := (d_1, d_2, \zeta_1, \zeta_2, z)\,, \qquad \Psi(\psi,\lambda) := \big(\Psi_1(\psi,\lambda), \Psi_2(\psi,\lambda)\big)\,,$$

system (24)–(27) has the form

$$\psi - \Psi(\psi,\lambda) = 0\,.$$

We already know that $\psi - \Psi(\psi,\lambda) \ne 0$ on $\psi \in \partial B$ for any $\lambda \in [0,1]$. Hence we can define the Leray-Schauder degree $\deg\big(I - \Psi(\cdot,\lambda), B, 0\big)$. Now from system

(24) (27) for $\lambda = 1$, we get (2), while for $\lambda = 0$, we get

$$\zeta_1(t) = -\tfrac{4}{\pi} L_2 P \left\{ \int_0^{\pi/4} p(x,t)\, \mathrm{d}x + f(d_1) + g(d_2) \right\},$$

$$\zeta_2(t) = -\tfrac{12}{\pi} L_2 P \left\{ \tfrac{8}{\pi} \int_0^{\pi/4} p(x,t)\left(x - \tfrac{\pi}{8}\right) \mathrm{d}x - f(d_1) + g(d_2) \right\}, \tag{28}$$

$$z(x,t) = 0,$$

$$f(d_1) = \theta_1/T,$$

$$g(d_2) = \theta_2/T.$$

Since $\bar{c}_1$, $\bar{c}_2$ are simple roots of the equations $f(\bar{c}_1) = \theta_1/T$ and $g(\bar{c}_2) = \theta_2/T$, we see that the system (28) is solvable for $d_i = \bar{c}_i$, $i = 1,2$, and also the corresponding Leray-Schauder degree or the coincidence topological degree $\deg\bigl(I - \Psi(\cdot,0), B, 0\bigr)$ is nonzero (see [5]). Since

$$\deg\bigl(I - \Psi(\cdot,1), B, 0\bigr) = \deg\bigl(I - \Psi(\cdot,0), B, 0\bigr) \neq 0,$$

system (24)–(27) is solvable in the ball B. The proof is finished. □

For instance, if $f(x) = g(x) = Kx + \varepsilon x^3$ for constants $K > 0$ and ε, then Theorem 1 is applicable when

$$\eta := 4K\left(\frac{16}{\pi\omega\sqrt{\omega^2 + \delta^2}} + M_2 \right) < 1$$

and ε is sufficiently small. Indeed, we first take $\varepsilon = 0$. Hence $f(x) = g(x) = Kx$. Then $k_1 = k_2 = K_1 = K_2 = K$ in Theorem 1 for any s_1, s_2. We take $r_1 = r_2 = A/\eta$ to satisfy $k_i r_i > AK_i$, $i = 1,2$. We note $\bar{c}_i = \frac{\theta_i}{KT}$, $i = 1,2$. The condition of Theorem 1 for constants θ_1, θ_2 and function p now reads as follows

$$\left(\frac{5}{2\omega\sqrt{\omega^2 + \delta^2}} + M_2 \frac{\pi}{4} \right)|p| + \frac{|\theta_1| + |\theta_2|}{T}\left(\frac{16}{\pi\omega\sqrt{\omega^2 + \delta^2}} + M_2 \right) < A\frac{1-\eta}{2}.$$

Since we can always find $A > 0$ satisfying the above inequality, we see that when $\varepsilon = 0$, then (1) is solvable for any $h \in X_\omega$. Clearly the above inequalities remain also for ε small. This gives the solvability of (1) for any fixed $h \in X_\omega$ and ε small depending on h.

Now we present more constructive method than Theorem 1. We consider system (24)–(26) for $\lambda = 1$. Let $N(d_1, d_2, \zeta_1, \zeta_2, z)$ denote the right-hand side of (24)–(26) with $\lambda = 1$. Hence (24)–(26) with $\lambda = 1$ has the form

$$\tau = N(d_1, d_2, \tau) \tag{29}$$

for $\tau = (\zeta_1, \zeta_2, z)$ and d_1, d_2 are parameters. We intend to apply the Banach fixed point theorem to solve (29). For this reason, we consider on the Banach space $Y = Y^2_{\omega,0} \times X_{\omega,0}$ the ball

$$B_A = \left\{\tau = \left(\zeta_1(t), \zeta_2(t), z(x,t)\right) \in Y : \ |\zeta_1| + |\zeta_2| + |z| \le A\right\}.$$

The norm on Y is given by $|\tau| = |\zeta_1| + |\zeta_2| + |z|$. We suppose positive constants D_i, $i = 1, 2$, such that it holds that

$$\left(\frac{5}{2\omega\sqrt{\omega^2+\delta^2}} + M_2\frac{\pi}{4}\right)|p| + \left(F(A+D_1) + G(A+D_2)\right)\left(\frac{16}{\pi\omega\sqrt{\omega^2+\delta^2}} + M_2\right) \le A \tag{30}$$

and

$$\left(F'(A+D_1) + G'(A+D_2)\right)\left(\frac{16}{\pi\omega\sqrt{\omega^2+\delta^2}} + M_2\right) < 1. \tag{31}$$

The conditions (30) and (31) imply that for any $(d_1, d_2) \in B_D$ with

$$B_D := \left\{(d_1, d_2) \in \mathbb{R}^2 : \ |d_i| \le D_i, \ i = 1, 2\right\},$$

the mapping $N(d_1, d_2, \cdot)$ maps B_A to itself with the Lipschitz contraction constant

$$\Gamma := \left(F'(A+D_1) + G'(A+D_2)\right)\left(\frac{16}{\pi\omega\sqrt{\omega^2+\delta^2}} + M_2\right).$$

Hence (29) has a unique fixed point $\tau = \tau(d_1, d_2)$ in B_A for any $(d_1, d_2) \in B_D$. Moreover, mapping $\tau(d_1, d_2)$ is Lipschitz continuous with the constant $\Gamma/(1-\Gamma)$, i.e. it holds that

$$|\tau(d_1, d_2) - \tau(d_1', d_2')| \le \frac{\Gamma}{1-\Gamma}\left(|d_1 - d_1'| + |d_2 - d_2'|\right)$$

for any $(d_1, d_2), (d_1', d_2') \in B_D$. We consider in (30) and (31) the function p as a parameter for fixed A, D_1, D_2. Hence $\tau(d_1, d_2) = \tau(d_1, d_2, p)$. We plug this $\tau(d_1, d_2, p)$ into (27) with $\lambda = 1$ to get

$$\theta_1 = \int_0^T f\left(d_1 + \tau(d_1, d_2, p)(0, t)\right)\,\mathrm{d}t,$$

$$\theta_2 = \int_0^T g\left(d_2 + \tau(d_1, d_2, p)(\pi/4, t)\right)\,\mathrm{d}t,$$

where

$$\tau(d_1, d_2, p)(0, t) = \xi_1(t) - \xi_2(t) + z(0, t),$$
$$\tau(d_1, d_2, p)(\pi/4, t) = \xi_1(t) + \xi_2(t) + z(\pi/4, t)$$

for $\tau(d_1, d_2, p) = \big(\xi_1(t), \xi_2(t), z(x,t)\big)$.

By using the above formulas for θ_1, θ_2, from the splitting (22) for the function $h(x,t)$, we obtain

$$\begin{aligned} h(x,t) = {} & \frac{8}{T\pi}\Big(1 - \frac{12}{\pi}x\Big)\int_0^T g\big(d_2 + \tau(d_1,d_2,p)(\pi/4,t)\big)\,dt \\ & + \frac{16}{T\pi}\Big(\frac{6}{\pi}x - 1\Big)\int_0^T f\big(d_1 + \tau(d_1,d_2,p)(0,t)\big)\,dt + p(x,t) \end{aligned} \tag{32}$$

where $p(x,t) \in X_\omega$ satisfies (23). Summarizing, we get the following result.

THEOREM 2. *If there are positive constants A, B, D_1, D_2 such that*

$$\Big(\frac{5}{2\omega\sqrt{\omega^2+\delta^2}} + M_2\frac{\pi}{4}\Big)B + \big(F(A+D_1) + G(A+D_2)\big)\Big(\frac{16}{\pi\omega\sqrt{\omega^2+\delta^2}} + M_2\Big) \le A$$

and

$$\big(F'(A+D_1) + G'(A+D_2)\big)\Big(\frac{16}{\pi\omega\sqrt{\omega^2+\delta^2}} + M_2\Big) < 1,$$

then for any $(d_1, d_2) \in B_D$, $p(x,t) \in X_\omega$ satisfying (23) and $|p| \le B$, there is a unique $\tau(d_1,d_2,p) \in Y$ with $|\tau| \le A$ solving (29). Moreover, equation (1) has a T-periodic solution for the function $h(x,t)$ given by (32).

Since we use the Banach fixed point theorem to find $\tau(d_1,d_2,p) \in Y$ in B_A, we can construct $h(x,t)$ from (32). Moreover, by using the structure of functions from X_ω, $X_{\omega,0}$, Y_ω, $Y_{\omega,0}$, we can approximate this $h(x,t)$ by using the Fourier truncation method. We also note that the form (32) of the function $h(x,t)$ is not only sufficient but it is also, in some sense, necessary for the solvability of (1).

Finally we note that to verify the assumptions of Theorem 2, we can take the function

$$\Phi(x) := \big(F(x) + G(x)\big)\Big(\frac{16}{\pi\omega\sqrt{\omega^2+\delta^2}} + M_2\Big).$$

Now, if there is an $x_0 > 0$ such that $\Phi(x_0) < x_0$ and $\Phi'(x_0) < 1$, then for the validity of Theorem 2, we can take

$$\begin{aligned} & B = D_1 = D_2 = D = \frac{4\big(x_0 - \Phi(x_0)\big)\omega\sqrt{\omega^2+\delta^2}}{10 + \omega\sqrt{\omega^2+\delta^2}(M_2\pi + 4)}, \\ & A = x_0 - D. \end{aligned} \tag{33}$$

For instance, if $f(x) = g(x) = \Omega x^3$, $\Omega > 0$, then we get

$$\Phi(x) = 2\Omega x^3\Big(\frac{16}{\pi\omega\sqrt{\omega^2+\delta^2}} + M_2\Big)$$

and we can take

$$x_0 = \sqrt{\frac{\pi\omega\sqrt{\omega^2+\delta^2}}{12\Omega\left(16+\pi\omega\sqrt{\omega^2+\delta^2}M_2\right)}}\,.$$

Then we obtain

$$x_0 - \Phi(x_0) = \frac{5}{6}\sqrt{\frac{\pi\omega\sqrt{\omega^2+\delta^2}}{12\Omega\left(16+\pi\omega\sqrt{\omega^2+\delta^2}M_2\right)}}\,,$$

which according to (33) implies

$$B = \frac{5\omega\sqrt{\omega^2+\delta^2}}{10+\omega\sqrt{\omega^2+\delta^2}(M_2\pi+4)}\sqrt{\frac{\pi\omega\sqrt{\omega^2+\delta^2}}{27\Omega(16+\pi\omega\sqrt{\omega^2+\delta^2}M_2)}}\,.$$

This gives a relationship between the magnitude of the constant B and the parameter Ω.

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