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A THREE-POINT BOUNDARY VALUE PROBLEM FOR THIRD ORDER DIFFERENTIAL EQUATIONS

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Introduction

In the present paper we investigate a nonlinear boundary value problem at 3 points with linear boundary conditions of the following type

$$x''' = f(t, x, x', x''), \quad (t, x, x', x'') \in [a_1, a_3] \times R^3, \tag{1}$$

$$\begin{gathered} \alpha_1 x(a_1) + \alpha_2 x'(a_1) + \alpha_3 x''(a_1) = A_1 \\ \beta_1 x(a_2) + \beta_2 x'(a_2) + \beta_3 x''(a_2) = A_2 \\ \gamma_1 x(a_3) + \gamma_2 x'(a_3) + \gamma_3 x''(a_3) = A_3, \\ a_i,\ \alpha_i,\ \beta_i,\ \gamma_i,\ A_i \in R,\ i = 1, 2, 3,\ a_1 < a_2 < a_3, \\ \sum_{i=1}^{3} |\alpha_i| > 0, \quad \sum_{i=1}^{3} |\beta_i| > 0, \quad \sum_{i=1}^{3} |\gamma_i| > 0. \end{gathered} \tag{2}$$

We use similar methods as K. Schmitt in [3], where he has proved existence theorems for the boundary value problem

$$\begin{gathered} x'' = f(t, x, x'), \\ \alpha_1 x(a_1) - \alpha_2 x'(a_1) = A_1, \quad \beta_1 x(a_2) + \beta_2 x'(a_2) = A_2, \\ \alpha_i,\ \beta_i \geqq 0, \quad i = 1, 2,\ \alpha_1 + \alpha_2 > 0, \quad \beta_1 + \beta_2 > 0, \quad \alpha_1 + \beta_1 > 0. \end{gathered}$$

We obtain a certain generalization of Schmitt's results.

1. The linear boundary value problem

Consider differential equations

$$x''' = 0, \tag{3}$$

$$x''' = r(t), \quad r(t) \in C_0(I = [a_1, a_3]). \tag{4}$$

Lemma 1. *A homogeneous boundary value problem* (3) *and* (2) *for* $A_1 = A_2 = A_3 = 0$ *has only the trivial solution if and only if*

$$\Delta = \begin{vmatrix} \alpha_1, & \alpha_1 a_1 + \alpha_2, & \alpha_1 a_1^2 + 2\alpha_2 a_1 + 2\alpha_3 \\ \beta_1, & \beta_1 a_2 + \beta_2, & \beta_1 a_2^2 + 2\beta_2 a_2 + 2\beta_3 \\ \gamma_1, & \gamma_1 a_3 + \gamma_2, & \gamma_1 a_3^2 + 2\gamma_2 a_3 + 2\gamma_3 \end{vmatrix} \neq 0 \tag{5}$$

Further we always assume that condition (5) is satisfied.

M. Greguš in [1] has solved a linear nonhomogeneous boundary value problem of the n-th order at m points (n, $m \geqq 2$) using special Green's function. His results concerning the boundary value problem (4) and (2) are summarized in the next lemma.

Lemma 2. (a) *For each point* $s \in (a_k, a_{k+1})$, $k = 1, 2$, *there exist functions* $G_k = G_k(t, s)$ *(Green's functions) such that*

1. G_k, $\dfrac{\partial G_k}{\partial t} = G_{kt}$ *are continuous in* t *on* I.

2. $\dfrac{\partial^2 G_k}{\partial t^2} = G_{ktt}$ *is continuous in* t *on* I *except the point* s *which is a discontinuity point of the* 1*st kind, and* $G_{ktt}(s+0, s) - G_{ktt}(s-0, s) = 1$.

3. G_k, *as a function of* t, *is a solution of* (3) *on the intervals* $[a_1, s)$, $(s, a_3]$, *and satisfies the homogeneous boundary conditions* (2) *for* $A_1 = A_2 = A_3 = 0$.

4. *The functions* G_k *are uniquely determined by properties* 1., 2., *and* 3.

(b) *The solution* x *of the boundary value problem* (4) *and* (2) *for* $A_1 = A_2 = A_3 = 0$ *is of the form*

$$x(t) = \sum_{k=1}^{2} \int_{a_k}^{a_{k+1}} G_k(t, s) r(s)\, \mathrm{d}s, \quad t \in I.$$

(c) *The solution* x *of the boundary value problem* (4) *and* (2) *is of the form*

$$x(t) = \varphi(t) + \sum_{k=1}^{2} \int_{a_k}^{a_{k+1}} G_k(t, s) r(s)\, \mathrm{d}s, \quad t \in I,$$

where φ *is a solution of the boundary value problem* (3) *and* (2).

Further, denote $g(s) = \frac{1}{2}\alpha_1(s - a_1)^2 - \alpha_2(s - a_1) + \alpha_3$, $h(s) = \frac{1}{2}\beta_1(s - a_2)^2 - \beta_2(s - a_2) + \beta_3$, and by Δ_{ij} the corresponding signed minor of Δ. Using the above listed properties of Green's functions G_k and function φ, we can express them explicitly as follows:

If $s \in (a_1, a_2)$, then

$$G_1(t, s) = \begin{cases} \dfrac{g(s)}{\Delta}(\Delta_{11} + \Delta_{12} t + \Delta_{13} t^2) - \dfrac{1}{2}(s - t)^2, & a_1 \leqq t \leqq s \\ \dfrac{g(s)}{\Delta}(\Delta_{11} + \Delta_{12} t + \Delta_{13} t^2), & s < t \leqq a_3. \end{cases} \tag{6}$$

If $s \in (a_2, a_3)$, then $G_2(t, s) =$

$$= \begin{cases} \frac{g(s)}{\Delta}(\Delta_{11} + \Delta_{12}t + \Delta_{13}t^2) + \frac{h(s)}{\Delta}(\Delta_{21} + \Delta_{22}t + \Delta_{23}t^2) - \frac{1}{2}(s-t)^2, \ a_1 \leqq t < s \\ \frac{g(s)}{\Delta}(\Delta_{11} + \Delta_{12}t + \Delta_{13}t^2) + \frac{h(s)}{\Delta}(\Delta_{21} + \Delta_{22}t + \Delta_{23}t^2), \\ \qquad s \leqq t \leqq a_3. \end{cases} \tag{7}$$

$$\varphi(t) = \frac{1}{\Delta} \sum_{i=1}^{3} A_i(\Delta_{i1} + \Delta_{i2}t + \Delta_{i3}t^2), \quad t \in I. \tag{8}$$

2. The existence theorem for the nonlinear boundary value problem

Throughout the rest of this paper we assume that the function $f(t, x, x', x'')$ is continuous on $I \times R^3$.

According to Lemma 2, the solution $x(t)$ of the boundary value problem (1) and (2) is a solution of the integro-differential equation

$$x(t) = \varphi(t) + \sum_{k=1}^{2} \int_{a_k}^{a_{k+1}} G_k(t, s) f(s, x(s), x'(s), x''(s))\, ds, \tag{9}$$

and vice versa.

Theorem 1. *Let $M > 0$ be a constant such that*

$$|f(t, x, x', x'')| \leqq M, \quad \forall (t, x, x', x'') \in I \times R^3.$$

Then the boundary value problem (1) *and* (2) *has at least one solution.*

Proof. Let $C_2(I)$ be the Banach space endowed with the norm

$$\|x\| = \sum_{i=0}^{2} \max_I |x^{(i)}(t)|.$$

Define an operator T: $C_2(I) \to C_2(I)$ as follows

$$Tx(t) = \varphi(t) + \sum_{k=1}^{2} \int_{a_k}^{a_{k+1}} G_k(t, s) f(s, x(s), x'(s) x''(s))\, ds.$$

Further, define constans

$$K = \max_I |\varphi(t)|, \quad K' = \max_I |\varphi'(t)|, \quad K'' = \max_I |\varphi''(t)|,$$

$$N = (a_3 - a_1) \max \{ \sup_{I \times (a_1, a_2)} |G_1(t, s)|, \sup_{I \times (a_2, a_3)} |G_2(t, s)| \},$$

$$N' = (a_3 - a_1) \max \{ \sup_{I \times (a_1, a_2)} |G_{1t}(t, s)|, \sup_{I \times (a_2, a_3)} |G_{2t}(t, s)| \},$$

$$N'' = (a_3 - a_1)\max\{\sup_{I\times(a_1,a_2)} |G_{1tt}(t,s)|, \sup_{I\times(a_2,a_3)} |G_{2tt}(t,s)|\}.$$

Then

$$|Tx(t)| \leqq K + MN, \quad |(Tx)'(t)| \leqq K' + MN',$$
$$|(Tx)''(t)| \leqq K'' + MN'', \quad \forall t \in I.$$

Define a set E by

$$E = \{x \in C_2(I): |x| \leqq K + MN, \quad |x'| \leqq K' + MN', \quad |x''| \leqq K'' + MN''.$$

Then E is a closed convex subset of $C_2(I)$, $TE \subset E$, TE is relatively compact, and T is a continuous operator. Thus T satisfies all the assumptions of the Schauder fixed point theorem (cf. [2], p. 476), and hence there exists at least one fixed point $x(t) \in E$ of T such that (9) holds true. The fixed point is a solution of the boundary value problem (1) and (2).

3. Special boundary value problem
Definition of lower and upper solutions

In the remaining sections we investigate a boundary value problem of the type (1) and (2) satisfying the following special boundary conditions

$$\begin{aligned} \alpha_1 x(a_1) - \alpha_2 x'(a_1) + \alpha_3 x''(a_1) &= A_1 \\ \beta_2 x'(a_2) - \beta_3 x''(a_2) &= A_2 \\ \gamma_2 x'(a_3) + \gamma_3 x''(a_3) &= A_3, \end{aligned} \tag{10}$$

$$\alpha_i, \beta_i, \gamma_i \geqq 0, \quad i = 2, 3, \quad \alpha_1 > 0,$$
$$\beta_2 + \beta_3 > 0, \quad \gamma_2 + \gamma_3 > 0, \quad \beta_2 + \gamma_2 > 0.$$

Denote

$$h = a_3 - a_1, \quad h_1 = a_2 - a_1, \quad h_2 = a_3 - a_2.$$

The determinant Δ from Lemma 1, corresponding to conditions (10), satisfies condition (5) and we have

$$\Delta = 2\alpha_1(\beta_2\gamma_2 h_2 + \beta_2\gamma_3 + \beta_3\gamma_2) > 0. \tag{11}$$

We say that a function $\alpha \in C_3(I)$ is a *lower solution* of the boundary value problem (1) and (10) if

$$\alpha''' \geqq f(t, \alpha, \alpha', \alpha''), \tag{12}$$

and

$$\begin{aligned} \alpha_1 \alpha(a_1) - \alpha_2 \alpha'(a_1) + \alpha_3 \alpha''(a_1) &\leqq A_1 \\ \beta_2 \alpha'(a_2) - \beta_3 \alpha''(a_2) &\leqq A_2 \\ \gamma_2 \alpha'(a_3) + \gamma_3 \alpha''(a_3) &\leqq A_3 \end{aligned} \tag{13}$$

holds true.

Similarly, we say that a function $\beta \in C_3(I)$ is an *upper solution* of the boundary value problem (1) and (10) if

$$\beta''' \leqq f(t, \beta, \beta', \beta''), \tag{14}$$

and

$$\begin{aligned} \alpha_1\beta(a_1) - \alpha_2\beta'(a_1) + \alpha_3\beta''(a_1) &\geqq A_1 \\ \beta_2\beta'(a_2) - \beta_3\beta''(a_2) &\geqq A_2 \\ \gamma_2\beta'(a_3) + \gamma_3\beta''(a_3) &\geqq A_3 \end{aligned} \tag{15}$$

holds true.

Using the assumption of the existence of lower and upper solutions we shall prove existence theorems for the boundary value problem (1) and (10).

4. A modification of the differential equation $x''' = f(t, x, x', x'')$

Let f be a function satisfying assumptions of Theorem 1, let α, $\beta \in C_3(I)$ be functions satisfying (12), (14), and

$$\alpha(a_1) \leqq \beta(a_1), \quad \alpha'(t) \leqq \beta'(t), \quad \forall t \in I. \tag{16}$$

Consider the following modification of the differential equation (1)

$$x''' = F(t, x, x', x''), \tag{17}$$

where F is defined on $I \times R^3$ by

$$F(t, x, x', x'') =$$

$$= \begin{cases} f(t, \beta(t), \beta'(t), x'') + \dfrac{x' - \beta'(t)}{1 + x'^2}, & x > \beta(t), \quad x' > \beta'(t) \ldots (\mathrm{I}) \\ f(t, x, \beta'(t), x'') + \dfrac{x' - \beta'(t)}{1 + x'^2}, & \alpha(t) \leqq x \leqq \beta(t), \quad x' > \beta'(t) \ldots (\mathrm{II}) \\ f(t, \alpha(t), \beta'(t), x'') + \dfrac{x' - \beta'(t)}{1 + x'^2}, & x < \alpha(t), \quad x' > \beta'(t) \ldots \vdots \\ f(t, \beta(t), x', x''), & x > \beta(t), \quad \alpha'(t) \leqq x' \leqq \beta'(t) \ldots \\ f(t, x, x', x''), & \alpha(t) \leqq x \leqq \beta(t), \quad \alpha'(t) \leqq x' \leqq \beta'(t) \ldots \\ f(t, \alpha(t), \alpha'(t), x'') + \dfrac{x' - \alpha'(t)}{1 + x'^2}, & x < \alpha(t), \quad x' < \alpha'(t) \ldots \\ f(t, x, \alpha'(t), x'') + \dfrac{x' - \alpha'(t)}{1 + x'^2}, & \alpha(t) \leqq x \leqq \beta(t), \quad x' < \alpha'(t) \ldots \\ f(t, \beta(t), \alpha'(t), x'') + \dfrac{x' - \alpha'(t)}{1 + x'^2}, & x > \beta(t), \quad x' < \alpha'(t) \ldots \vdots \\ f(t, \alpha(t), x', x''), & x < \alpha(t), \quad \alpha'(t) \leqq x' \leqq \beta'(t) \ldots (\mathrm{IX}) \end{cases} \tag{18}$$

Function F is continuous and bounded on $I \times R^3$ and we have

$$M_1 + M_3 \leqq F(t, x, x', x'') \leqq M_2 + M_4, \quad \forall (t, x, x', x'') \in I \times R^3, \tag{19}$$

where

$$\begin{aligned} M_1 &= \inf_{I \times R^3} f(t, x, x', x''), \quad M_2 = \sup_{I \times R^3} f(t, x, x', x''), \\ 0 > M_3 &= \min_{\substack{t \in I \\ x' < \alpha'(t)}} \frac{x' - \alpha'(t)}{1 + x'^2}, \quad 0 < M_4 = \max_{\substack{t \in I \\ x' > \beta'(t)}} \frac{x' - \beta'(t)}{1 + x'^2}. \end{aligned} \tag{20}$$

Putting $m_1 = \max_I \alpha'(t)$, $m_2 = \min_I \beta'(t)$, we get

$$M_3 = -\frac{1}{2}(\sqrt{m_1^2 + 1} + m_1), \quad M_4 = \frac{1}{2}(\sqrt{m_2^2 + 1} - m_2)$$

Lemma 3. *Let the function $f(t, x, x', x'')$ be nonincreasing in x on R and let $x(t) \in C_3(I)$ be a solution of (17). Let $u(t) = x(t) - \alpha(t)$, $v(t) = x(t) - \beta(t)$, $\forall t \in I$. Then:*

(a) *Function $u'(t)$ ($v'(t)$) does not attain a negative local minimum (a positive local maximum) at any $t_0 \in (a_1, a_3)$.*

(b) *If $u'(t) < 0$ and $u''(t) \leqq 0$ ($v'(t) > 0$ and $v''(t) \geqq 0$) for some $t_1 \in [a_1, a_3)$, then $u'(t) < 0$ and $u''(t) < 0$ ($v'(t) > 0$ and $v''(t) > 0$) for all $t \in (t_1, a_3]$.*

(c) *If $u'(t_2) < 0$ and $u''(t_2) \geqq 0$ ($v'(t) > 0$ and $v''(t_2) \leqq 0$) for some $t_2 \in (a\ , a_3]$, then $u'(t) < 0$ and $u''(t) > 0$ ($v'(t) > 0$ and $v''(t) < 0$) for all $t \in [a_1, t_2)$.*

Proof. (a) Suppose that at some $t_0 \in (a_1, a_3)$ the function $u'(t)$ attains a negative local minimum (the function v' attains a positive local maximum). Then $u''(t_0) = 0$ ($v''(t_0) = 0$) and $u'''(t_0) \geqq 0$ ($v'''(t_0) \leqq 0$). But if $u''(t_0) = 0$ ($v''(t_0) = 0$), then it follows from the assumptions that $u'''(t_0) < 0$ ($v'''(t_0) > 0$), which is a contradiction. For instance, if $v''(t_0) = 0$, $\alpha(t_0) - \beta(t_0) \leqq v(t_0) \leqq 0$, then from (18)-(II) we get

$$\begin{aligned} v'''(t_0) = x'''(t_0) - \beta'''(t_0) \geqq f(t_0, x(t_0), \beta'(t_0), \beta''(t_0)) + \\ + \frac{x'(t_0) - \beta'(t_0)}{1 + x'^2(t_0)} - f(t_0, \beta(t_0), \beta'(t_0), \beta''(t_0)) > 0. \end{aligned}$$

Remark. From the above proof it follows that for $t_0 \in (a_1, a_3)$ we have: if $u'(t_0) < 0$, $u''(t_0) = 0$, then $u'''(t_0) < 0$ (if $v'(t_0) > 0$, $v''(t_0) = 0$, then $v'''(t_0) > 0$). The same result holds also for $t_0 = a_1, a_3$.

(b) We prove only the statement in parentheses. The proof of the other statement is analogous.

Let $v'(t_1) > 0$ and $v''(t_1) = 0$ for some $t_1 \in [a_1, a_3)$. It follows from Remark that $v'''(t_1) > 0$. Thus, there exists a δ, $t_1 < \delta \leqq a_3$ such that $\forall t \in (t_1, \delta)$ we have $v'(t) > 0$ and $v''(t) > 0$. Let $v''(\delta) = 0$ and $\delta < a_3$. Then again we have $v'''(\delta) > 0$. Consequently, the function v'' is negative in the left part of some deleted neighborhood of

δ, which is a contradiction. Hence $v''(\delta)>0$ and we may have $\delta=a_3$. However, it follows from Remark that $v''(a_3)>0$, and the assertion is proved.

If $v'(t_1)>0$ and $v''(t_1)>0$, then the proof is similar. Note that in this case it is not necessary to investigate the value $v'''(t_1)$.

(c) The proof of (c) is similar to that of (b) and is omitted.

The function F satisfies the assumptions of Theorem 1 and hence there exists a solution x of the boundary value problem (17) and (10). The next lemma gives an estimate for the value $x''(a_1)$ of the solution.

Lemma 4. *Let x be a solution of the boundary value problem* (17) *and* (10). *Then we have*

$$k_1 \leqq x''(a_1) \leqq k_2,$$

$$\begin{aligned} k_1 &= \frac{2\alpha_1}{\Delta}\left((\beta_2 A_3-\gamma_2 A_2+\gamma_2 h_2(M_1+M_3)\left(\frac{\beta_2}{2}h_2+\beta_3\right)\right)-(M_2+M_4)h, \\ k_2 &= \frac{2\alpha_1}{\Delta}\left(\beta_2 A_3-\gamma_2 A_2+\gamma_2 h_2(M_2+M_4)\left(\frac{\beta_2}{2}h_2+\beta_3\right)\right)-(M_1+M_3)h, \end{aligned} \tag{21}$$

where Δ satisfies (11) *and the constants M_1, M_2, M_3, and M_4 satisfy* (20).

Proof. According to (9), for the solution $x(t)$ we have

$$x''(t)=\varphi''(t)+\sum_{k=1}^{2}\int_{a_k}^{a_{k+1}} G_{ktt}(t,s)F(s,x(s),x'(s),x''(s))\,ds. \tag{22}$$

By (6), (7), and (8). with regard to the boundary conditions (10) we get:

for $s\in(a_1,a_2)$ we have

$$G_{1tt}(t,s)=\begin{cases}-1, & a_1\leqq t\leqq s\\ 0, & s<t\leqq a_3,\end{cases}$$

for $s\in(a_2,a_3)$ we have

$$G_{2tt}(t,s)=\begin{cases}\dfrac{2\alpha_1\gamma_2}{\Delta}(\beta_2(s-a_2)+\beta_3)-1, & a_1\leqq t<s\\ \dfrac{2\alpha_1\gamma_2}{\Delta}(\beta_2(s-a_2)+\beta_3), & s\leqq t\leqq a_3,\end{cases}$$

and for φ we have

$$\varphi''(t)=\frac{2\alpha_1}{\Delta}(\beta_2 A_3-\gamma_2 A_2).$$

First, we prove the right-hand side inequality in (21). From (22) we get

$$x''(a_1)=\varphi''(a_1)-\int_{a_1}^{a_2}F(s,x(s),x'(s),x''(s))\,ds+$$

$$+\int_{a_2}^{a_3}\left(\frac{2\alpha_1\gamma_2}{\Delta}(\beta_2(s-a_2)+\beta_3)-1\right)F(s,x(s),x'(s),x''(s))\,\mathrm{d}s=$$

$$=\varphi''(a_1)-\int_{a_1}^{a_3}F(s,x(s),x'(s),x''(s))\,\mathrm{d}s+$$

$$+\frac{2\alpha_1\gamma_2}{\Delta}\int_{a_2}^{a_3}(\beta_2(s-a_2)+\beta_3)F(s,x(s),x'(s),x''(s))\,\mathrm{d}s\leqq$$

$$\leqq\varphi''(a_1)-(M_1+M_3)h+\frac{2\alpha_1\gamma_2}{\Delta}(M_2+M_4)\left(\frac{\beta_2}{2}h_2^2+\beta_3h_2\right)=k_2.$$

The left-hand side inequality can be proved similarly.

5. The existence theorem via lower and upper solutions

Theorem 2. *Let the function $f(t,x,x',x'')$ satisfy the assumptions of Theorem 1 and let f be nonincreasing in x on R.*

Further, let there exist $\alpha,\beta\in C_3(I)$ which are lower and upper solutions, respectively, of the boundary value problem (1) *and* (10) *satisfying* (16) *and such that*

$$\beta''(a_1)\leqq k_1,\quad k_2\leqq\alpha''(a_1), \tag{23}$$

where k_1 and k_2 are constants defined by (21).

Then there exists at least one solution x of the boundary value problem (1) *and* (10) *such that*

$$\alpha(t)\leqq x(t)\leqq\beta(t),\quad \alpha'(t)\leqq x'(t)\leqq\beta'(t),\quad \forall t\in I. \tag{24}$$

Proof. Consider the modified differential equation (17). It follows from the previous section 4 that there exists at least one solution $x(t)$ of the boundary value problem (17) and (10). From (10), (13), and (15) we have that the functions $u=x-\alpha$ and $v=x-\beta$ satisfy the following conditions

$$\begin{aligned}
\alpha_1u(a_1)+\alpha_2u'(a_1)-\alpha_3u''(a_1)&\leqq0\ldots(\mathrm{I})\\
-\beta_2u'(a_2)+\beta_3u''(a_2)&\leqq0\ldots(\mathrm{II})\\
-\gamma_2u'(a_3)-\gamma_3u''(a_3)&\leqq0\ldots\vdots\\
-\alpha_1v(a_1)+\alpha_2v'(a_1)-\alpha_3v''(a_1)&\geqq0\ldots\\
-\beta_2v'(a_2)+\beta_3v''(a_2)&\geqq0\ldots\vdots\\
-\gamma_2v'(a_3)-\gamma_3v''(a_3)&\geqq0\ldots(\mathrm{VI})
\end{aligned} \tag{25}$$

We prove that $u(t)\geqq0$, $u'(t)\geqq0$, $v(t)\leqq0$, $v'(t)\leqq0$, $\forall t\in I$. This is equivalent to (24) and hence, according to (18)-(V), $x(t)$ is the needed solution of equation (1).

From (23) using Lemma 4 we get

$$u''(a_1) \leqq 0, \quad v''(a_1) \geqq 0. \tag{26}$$

Suppose that $v'(a_1) > 0$. It follows from Lemma 3(b) that $v'(a_3) > 0$ and $v''(a_3) > 0$, which contradict (25)-(VI). Thus $v'(a_1) \leqq 0$.

Suppose that $v'(a_2) > 0$. and $\beta_3 > 0$ (if $\beta_3 = 0$, then it follows from (25)-(V) that $v'(a_2) \leqq 0$. From (25)-(V) we get $v''(a_2) \geqq 0$ and again, by Lemma 3(b), this would contradict (25)-(VI). Hence $v'(a_2) \leqq 0$. It will be shown later that it was not necessary to investigate the value $v'(a_2)$.

Suppose that $v'(a_3) > 0$ and $\gamma_3 > 0$ (if $\gamma_3 = 0$, then by (25)-(VI) we have $v'(a_3) \leqq 0$). Then by (25)-(VI) we have $v''(a_3) \leqq 0$. From Lemma 3(c) it follows that either $v'(a_2) > 0$, and $v''(a_2) < 0$, or $v''(a_1) < 0$, which contradicts (25)-(V) or (26). Thus $v'(a_3) \leqq 0$.

If $v'(t_1) > 0$ for some $t \in (a_1, a_3)$, then it follows from the above results that there exists a point $t_0 \in (a_1, a_3)$ at which $v'(t)$ attains a local maximum, which contradicts Lemma 3(a). Thus $v'(t) \leqq 0$ for all $t \in I$.

Since $v'(a_1) \leqq 0$ and $v''(a_1) \geqq 0$, it follows from condition (25)-(IV) that $v(a_1) \leqq 0$, and hence $v(t) \leqq 0$ for all $t \in I$.

Similarly it can be shown that $u(t) \geqq 0$ and $u'(t) \geqq 0$.

Example. Consider the following differential equation

$$x''' = -\sqrt[3]{\frac{x}{1+t+8|x|}}, \quad (= f(t, x, x', x''), (t, x, x', x'') \in [0, 1] \times R^3) \tag{27}$$

and choose the boundary conditions of the form

$$x(0) = 0, \quad x'(1/2) - x''(1/2) = 3/4, \quad x'(1) = 0. \tag{28}$$

The function f is continuous and bounded on $[0, 1] \times R^3$, $|f| < \frac{1}{2} = M_2 = -M_1$, and it is decreasing in x on R. It is easy to verify that functions $\alpha(t) = \frac{t}{12}(t^2 + 12t - 27)$ and $\beta(t) = -\frac{t}{12}(t^2 + 12t - 27)$ are a lower and an upper solution respectively, of the boundary value problem (27) and (28) for which $\Delta = 3$. The functions α and β satisfy (16) and (23), and $\beta''(0) = -2 < k_1 = -23/12$, $\alpha''(0) = 2 > k_2 = 11/12$ ($M_3 = -1/2$, $M_4 = 1/2$). Since in this case all the assumptions of Theorem 2 are satisfied, there exists at least one solution x of the boundary value problem (27) and (28), for which we obtain

$$\left.\begin{aligned} &\frac{t}{12}(t^2 + 12t - 27) \leqq x(t) \leqq -\frac{t}{12}(t^2 + 12t - 27), \\ &\frac{1}{4}(t^2 + 8t - 9) \leqq x'(t) \leqq -\frac{1}{4}(t^2 + 8t - 9), \end{aligned}\right\} \quad \forall t \in [0, 1].$$

6. The existence theorem without the assumption of the boundedness of f

In this section we prove a theorem analogous to Theorem 2, however, without the assumption of the boundedness of f.

Lemma 5. (Nagumo) *Let $\varphi(s)$, $0 \leqq s < \infty$ be a positive continuous function such that*

$$\int^{\infty} \frac{s \, ds}{\varphi(s)} = \infty.$$

Let $R_1 \geqq 0$, and let $x(t)$ be a function in $C_3([a, b])$ such that

$$|x'(t)| \leqq R_1, \quad |x'''(t)| \leqq \varphi(|x''(t)|), \quad \forall t \in [a, b].$$

Then there exists a constant R_2 (depending only on $\varphi(s)$, R_1, and $h = b - a$) such that

$$|x''(t)| \leqq R_2, \quad \forall t \in [a, b],$$

where R_2 satisfies equation

$$\int_{2R_1 h}^{R_2} \frac{s \, ds}{\varphi(s)} = 2R_1. \tag{29}$$

Since the statement and its proof can be reproduced substituting formally $x' \to x$, $x'' \to x'$, and $x''' \to x''$ in Lemma 5.1 in [2, p. 503], the proof of Lemma 5 is omitted.

Lemma 6. *Let α, $\beta \in C_3(I)$ be functions satisfying* (16) *and let L be a positive constant such that*

$$\begin{gathered} |f(t, x, x', x'') - f(t, x, x', y'')| \leqq L|x'' - y''|, \\ \forall (t, x, x') \in \omega = \{(t, x, x') : t \in I, \ \alpha(t) \leqq x \leqq \beta(t), \ \alpha'(t) \leqq x' \leqq \beta'(t)\} \end{gathered} \tag{30}$$

holds true.

Then there exists a positive constant R_2 such that for each solution $x(t) \in C_3(I)$ of equation (1) *satisfying conditions $\alpha(t) \leqq x(t) \leqq \beta(t)$, $\alpha'(t) \leqq x'(t) \leqq \beta'(t)$, $\forall t \in I$ we have*

$$|x''(t)| \leqq R_2, \quad \forall t \in I. \tag{31}$$

Proof. Let $x(t)$ be a solution of (1) satisfying the assumptions of the lemma. From (30) we get

$$|f(t, x(t), x'(t), x''(t)) - f(t, x(t), x'(t), \beta''(t))| \leqq L|x''(t) - \beta''(t)|,$$

and consequently

$$\begin{aligned}|x'''(t)| &= |f(t, x(t), x'(t), x''(t))| \leqq \\ &\leqq L|x''(t) - \beta''(t)| + |f(t, x(t), x'(t), \beta''(t))| \leqq \\ &\leqq L|x''(t)| + L|\beta''(t)| + |f(t, x(t), x'(t), \beta''(t))|.\end{aligned}$$

Thus

$$|x'''(t)| \leqq L|x''(t)| + \max_{\omega} (L|\beta''(t)| + |f(t, x, x', \beta''(t))|).$$

Denote by m the maximum in the above expression and put $\varphi(s) = Ls + m$. Further, put

$$R_1 = \max (\max_I |\alpha'(t)|, \max_I |\beta'(t)|).$$

The existence of R_2 follows now from Lemma 5.

Rather than the calculation of R_2 using (29), an estimate of R_2 may appear to be more advantageous. For instance, if the function $s/(Ls + m)$ under the integral sign in (29) is replaced by a constant $\frac{2R_1/h}{2LR_1/h + m}$, then for s_0 satisfying equation

$$\int_{2R_1/h}^{s_0} \frac{2R_1/h \, \mathrm{d}s}{2LR_1/h + m} = 2R_1$$

we have $R_2 < s_0$. After calculating the value s_0, we get the following estimate

$$R_2 < 2LR_1 + hm + 2R_1/h. \tag{32}$$

Theorem 3. *Suppose that all the assumptions of Theorem 2 and Lemma 6 are satisfied except the one that f need not be bounded and condition* (23) *concerning the constants M_1 and M_2 is replaced by*

$$M_1 = \min_{\omega \times [-R_2, R_2]} f(t, x, x', x''), \quad M_2 = \max_{\omega \times [-R_2, R_2]} f(t, x, x', x''), \tag{33}$$

where R_2 is the constant from Lemma 6, and suppose that $R_2 \geqq |\alpha''(t)|$, $|\beta''(t)|$, $\forall t \in I$.

Then there exists at least one solution x of the boundary value problem (1) *and* (10) *which satisfies* (24).

Proof. Define a function $\Phi(t, x, x', x'')$ on $I \times R^3$ as follows:

for Φ on $\omega \times R$ put

$$\Phi(t, x, x', x'') = \begin{cases} f(t, x, x', R_2), & x'' > R_2 \\ f(t, x, x', x''), & |x''| \leqq R_2 \\ f(t, x, x', -R_2), & x'' < -R_2 \end{cases}$$

and extend Φ to its entire domain $I \times R^3$ by

$$\Phi(t, x, x', x'') =$$

$$= \begin{array}{lrr} \Phi(t, \beta(t), \beta'(t), x''), & x > \beta(t), & x' > \beta'(t) \\ \Phi(t, x, \beta'(t), x''), & \alpha(t) \leqq x \leqq \beta(t), & x' > \beta'(t) \\ \Phi(t, \alpha(t), \beta'(t), x''), & x < \alpha(t), & x' > \beta'(t) \\ \Phi(t, \beta(t), x', x''), & x > \beta(t), & \alpha'(t) \leqq x' \leqq \beta'(t) \\ \Phi(t, \alpha(t), \alpha'(t), x''), & x < \alpha(t), & x' < \alpha'(t) \\ \Phi(t, x, \alpha'(t), x''), & \alpha(t) \leqq x \leqq \beta(t), & x' < \alpha'(t) \\ \Phi(t, \beta(t), \alpha'(t), x''), & x > \beta(t), & x' < \alpha'(t) \\ \Phi(t, \alpha(t), x', x''), & x < \alpha(t), & \alpha'(t) \leqq x' \leqq \beta'(t). \end{array}$$

Further, consider the following modification of equation (1)

$$x''' = \Phi(t, x, x', x''). \tag{34}$$

The function Φ is on $I \times R^3$ continuous and bounded, $M_1 \leqq \Phi \leqq M_2$, and it is nonincreasing in x on R. The functions α and β are also a lower and an upper solution, respectively, of the boundary value problem (34) and (10), and they satisfy condition (23) with respect to Φ. Thus, for equation (34) all the assumptions of Theorem 2 are satisfied, and hence there exists at least one solution x of the boundary value problem (34) and (10) which satisfies (24). It follows from the definition of the function Φ that this solution is also a solution of equation (1).

From the above proof there follows an important fact concerning the solution x, namely that an estimate of the absolute value of its second derivative is given by (31).

Remark 1. In applications, as it can be seen from Example 1, condition (23) is important when investigating second derivatives of lower and upper solutions. Clearly, it is desirable to weaken the condition as much as possible. We show how it can be done.

Let us change the definition of the function F in section 4 as follows. replace the increments $(x' - \alpha'(t))/(1 + x'^2)$ in (18)-(VI), (VII), (VIII), and $(x' - \beta'(t))/(1 + x'^2)$ in (18)-(I), (II), (III) by $\varepsilon_1(x' - \alpha'(t))/(1 + x'^2)$ and $\varepsilon_2(x' - \beta'(t))/(1 + x'^2)$, respectively, where ε_1, $\varepsilon_2 > 0$. Denote the resulting function by $F_{\varepsilon_1, \varepsilon_2}$ and put

$$M_3 = M_3(\varepsilon_1) = \min_{\substack{I \in I \\ x' < \alpha'(t)}} \varepsilon_1 \frac{x' - \alpha'(t)}{1 + x'^2}, \quad M_4 = M_4(\varepsilon_2) = \max_{\substack{t \in I \\ x > \beta(t)}} \varepsilon_2 \frac{x' - \beta'(t)}{1 + x'^2}.$$

Then the function $F_{\varepsilon_1, \varepsilon_2}$ has the same properties as the function F, and hence Lemma 3, Lemma 4, Theorem 2, and Theorem 3 are valid for the corresponding modified equation $x''' = F_{\varepsilon_1, \varepsilon_2}(t, x, x', x'')$. Further, if we put in Lemma 4 $k_1 = k_1(\varepsilon_1, \varepsilon_2)$, $k_2 = k_2(\varepsilon_1, \varepsilon_2)$, we get

$$\sup_{0<\varepsilon_1,\,\varepsilon_2<\infty} k_1(\varepsilon_1, \varepsilon_2) = \frac{2\alpha_1}{\Delta}\left(\beta_2 A_3 - \gamma_2 A_2 + \gamma_2 h_2 M_1\left(\frac{\beta_2}{2} h_2 + \beta_3\right)\right) - M_2 h \stackrel{\text{def.}}{=} \bar{k}_1,$$
$$\inf_{0<\varepsilon_1,\,\varepsilon_2<\infty} k_2(\varepsilon_1, \varepsilon_2) = \frac{2\alpha_1}{\Delta}\left(\beta_2 A_3 - \gamma_2 A_2 + \gamma_2 h_2 M_2\left(\frac{\beta_2}{2} h_2 + \beta_3\right)\right) - M_1 h \stackrel{\text{def.}}{=} \bar{k}_2. \tag{35}$$

Condition (23) in Theorem 2 is then equivalent to condition

$$\beta''(a_1) < \bar{k}_1, \quad \bar{k}_2 < \alpha''(a_1). \tag{36}$$

Indeed, if (36) is satisfied, then there exist $\varepsilon_1, \varepsilon_2 > 0$ such that (23) holds true. Thus in Theorem 2 and Theorem 3 we can replace condition (23) by condition (36).

Remark 2. Finally, let us return to the estimates in Lemma 4. Using signs of the corresponding Green's functions we get new estimates. From the proof of Lemma 4 we get

$$G_{1tt}(a_1, s) = -1, \quad s \in (a_1, a_2),$$
$$G_{2tt}(a_1, s) = \frac{2\alpha_1\beta_2}{\Delta}(\gamma_2(s-a_3)-\gamma_3) \leqq 0, \quad s \in (a_2, a_3).$$

Consequently we get

$$x''(a_1) \leqq \varphi''(a_1) + (M_1 + M_3)\sum_{k=1}^{2}\int_{a_k}^{a_{k+1}} G_{ktt}(a_1, s)\,\mathrm{d}s =$$
$$= \varphi''(a_1) - (M_1 + M_3)\left(\frac{2\alpha_1\beta_2}{\Delta} h_2\left(\frac{\gamma_2}{2} h_2 + \gamma_3\right) + h_1\right) \stackrel{\text{def.}}{=} k_2.$$

Similarly we get $k_1 \leqq x''(a_1)$, where k_1 is obtained from k_2 when replacing $(M_1 + M_3)$ by $(M_2 + M_4)$.

Thus obtained estimates are sometimes better than those from Lemma 4. This can be seen in Example 1, where for the boundary value problem (27) and (28) we get $k_1 = -13/12$ and $k_2 = 1/12$.

Condition (23) corresponding to these estimates can be, according to Remark 1, weakened by replacing the nonstrict inequalities by the strict ones and leaving out the values M_3 and M_4.

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ТРЕХТОЧЕЧНАЯ КРАЕВАЯ ЗАДАЧА ДЛЯ ДИФФЕРЕНЦИАЛЬНОГО УРАВНЕНИЯ 3-ГО ПОРЯДКА

Ján Rusnák

В работе рассматривается трехточечная нелинейная краевая задача для дифференциального уравнения $x''' = f(t, x, x', x'')$ с линейными краевыми условиями. Методами теоремы о неподвижной точке, функции Грина и верхних и нижних решений этой задачи доказаны теоремы существования решения задачи для непрерывной и ограниченной или неограниченнои f.