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SUBADDITIVE MAXIMAL ERGODIC THEOREM

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The paper is aimed to generalize the so called maximal ergodic theorem, which can be formulated (see, e.g., [1]) as follows:

Let (X, S, μ, T) be a dynamical system, that is $X \neq \emptyset$, S is a σ — algebra on X, μ — measure on S and $T\colon X \to X$ a measure preserving transformation. Let f be an μ — integrable function defined on a set X. Let us denote $E = \{x \in X;\ \exists k \in N\colon f(x) + \ldots + f(T^{k-1}x) \geqq 0\}$. Then

$$\int f\chi_E d\mu \geqq 0. \tag{1}$$

The proof of this maximal ergodic theorem is based on the assertion, whose formal modifications (see [1], [2], [3]) can be formulated in common form as:

Let us denote $E_n = \{x \in X;\ \exists k \leqq n\colon f(x) + \ldots + f(T^{k-1}x) \geqq 0\}$. Then

$$\int f\chi_{E_n} d\mu \geqq 0. \tag{2}$$

The proof of the assertion (1) is based only on the suitable limitation by using (2). Therefore (2) can be regarded as a kernel of a maximal ergodic theorem. If f is a nonnegative integrable function, then (2) is trivial. In this case a nontrivial consequence of (2) is the following assertion:

Let $a \geqq 0$, $f \geqq 0$. Let us denote

$$F_n = \{x \in X;\ \exists k \leqq n\colon f(x) + \ldots + f(T^{k-1}x) \geqq ka\},$$

or

$$H_n = \{x \in X;\ \exists k \leqq n\colon f(x) + \ldots + f(T^{k-1}x) \leqq ka\} \quad \text{resp.}$$

Then

$$\int f\chi_{F_n} d\mu \geqq a\mu(F_n), \tag{3}$$

resp.

$$\int f\chi_{H_n} d\mu \leqq a\mu(H_n). \tag{4}$$

The generalization of the mentioned classical theorem is based on the changing of the integral with respect to a measure (that is of some linear functional defined on a set of integrable functions) by a sublinear functional I defined on a set F which is a subset of a set R^X, where X is a nonempty set. A measure preserving transformation is replaced by a transformation to which the functional I is invariant. The exact formulation of the demands on the objects X, F, I, T is formulated in § 1.

In § 2 there are formulated some lemmas.

§ 3 is devoted to the generalized variant of a maximal ergodic theorem.

§ 4 contains an example using an integral with respect to a premeasure which is one of the representatives of the nonlinear functional I. The main importance of this example lies in showing the impossibility of the generalization of (4).

§ 1. Fundamental properties of a system (X, F, I, T)

1. Let us assume X to be a nonempty set. Let us denote a subset of a set of all real functions defined on X by the symbol F. The following conditions must be fulfilled:
   a) If f, g are elements of F, then $f+g \in F$, $\max(f, g) = f \vee g \in F$, $\min(f, g) = f \wedge g \in F$.
   b) If $c \in R$, $f \in F$, then $cf \in F$.
   c) If e: $X \to R$ is the map defined by the formula $e(x) = 1$, then $e \in F$.
2. Let us denote by the symbol I a functional I: $F \to R$ with the following properties:
   a) If f, $g \in F$, $f \leqq g$, then $I(f) \leqq I(g)$.
   b) If $f \in F$, $c \in R$, then $I(cf) = cI(f)$.
   c) If $f \in F$, $g \in F$, $f \geqq 0$, $g \geqq 0$, then $I(f+g) \leqq I(f) + I(g)$ (subadditivity).
   d) If $a \in R$, $a \geqq 0$, $f \in F$, then $I(f) = I(f \wedge a) + I(f - f \wedge a)$ (additivity in a horizontal sense).
   e) Let $f_n \in F$ for $n \in N$, $f_n \nearrow f \in R^X$. Let a sequence $\{I(f_n)\}$ be upper bounded by a constant K. Then $f \in F$ and $I(f) \leqq K$.
3. Let us denote by the symbol T a transformation T: $X \to X$. The following conditions on T must be fulfilled:
   a) If $f \in F$, Tf: $X \to R$, $x \mapsto f(Tx)$, then $Tf \in F$.
   b) If $f \in F$, then $I(Tf) = I(f)$.

§ 2. Some elementar properties

This paragraph is devoted to simple lemmas, which aim at securing the correction of the following considerations.

Lemma 1. *Let $f \in F$, $f \geqq 0$, $a \geqq 0$. Let us denote*

$$M = \{x \in X; f(x) > 0\}.$$

Then $I(f + a\chi_M) = I(f) + aI(\chi_M)$.

Proof: The assertion of lemma is the straightforward consequence of a property 2d from § 1 (additivity in a horizontal sense) of the functional I.

$$I(f + a\chi_M) = I((f + a\chi_M) \wedge a) + I((f + a\chi_M) - (f + a\chi_M) \wedge a) = \\ = I(a\chi_M) + I(f) = I(f) + aI(\chi_M). \qquad \text{Q.E.D.}$$

Lemma 2. *Let $f \in F$. Let us denote*

$$M = \{x \in X;\, f(x) > 0\}.$$

Then $\chi_M \in F$.

Proof: Let us denote $g_n = \min(n \max(f, 0), e)$ for $n \in N$. It is easy to see $g_n \in F$, $g_n \leqq e$. Therefore the sequence $\{I(g_n)\}$ is upper bounded by the constant $I(e)$. Moreover $g_n \nearrow \chi_M$. By using the property 2e we obtain $\chi_M \in F$.

Q.E.D.

Lemma 3. *Let $f \in F$, $f \geqq 0$, $A \subset X$, $\chi_A \in F$. Then $f\chi_A \in F$.*

Proof: Let us denote $h_n = \min(f, n\chi_A)$, $n \in N$.

Evidently $h_n \in F$; $I(h_n) \leqq I(f)$. Therefore $\{I(h_n)\}$ is an upper bounded sequence. Moreover $h_n \nearrow f\chi_A$. By using 2e we obtain $f\chi_A \in F$.

Q.E.D.

§ 3. Maximal ergodic theorem

An assertion analogical to the classical maximal ergodic theorem will be proved in the next paragraph. For its formulation we need some notations. Let $f \in F$, $f \geqq 0$, $a \geqq 0$, $k \in N$. Let us denote

$$S_k = f + Tf + \ldots + T^{k-1}f - ka$$
$$M_n = \max(0, S_1, \ldots, S_n)$$
$$A_n = \{x \in X;\, M_n(x) > 0\}.$$

By lemmas 2, 3 the functions S_k, M_n, χ_{A_n} are elements of F.

Theorem 1. *Let $f \in F$, $f \geqq 0$. Let f be a bounded function.*
Let $a \geqq 0$. Then $I(f\chi_{A_n}) \geqq aI(\chi_{A_n})$.

Proof. For $a = 0$ an assertion of the theorem is trivial.

Let us assume $a > 0$. It is evident that $M_n \geqq S_k$. However, $TS_k = Tf + \ldots + T^k f - ka = S_{k+1} - f + a$ thus it follows that $f + TM_n \geqq S_{k+1} + a$. All these relationships are valid for $k = 1, 2, \ldots, n$.

It is easy to see that $f + TM_n \geqq f - a + a = S_1 + a$. Hence

$$(f + TM_n)\chi_{A_n} \geqq \max(S_1, S_2, \ldots, S_{n+1})\chi_{A_n} + a\chi_{A_n} = \\ = \max(0, S_1, \ldots, S_{n+1})\chi_{A_n} + a\chi_{A_n} = \\ = M_{n+1}\chi_{A_n} + a\chi_{A_n}$$

and then

$$\begin{aligned} f\chi_{A_n} &\geqq \chi_{A_n} M_{n+1} - (TM_n)\chi_{A_n} + a\chi_{A_n} \geqq \\ &\geqq \chi_{A_n} M_n - (TM_n)\chi_{A_n} + a\chi_{A_n} = \\ &= M_n - (TM_n)\chi_{A_n} + a\chi_{A_n}. \end{aligned}$$

For obtaining the last inequality, the relationship $M_{n+1} \geqq M_n$ and $M_n(x) > 0$ iff $x \in A_n$ has been employed. Let $k > 1$ such that $M_n - (TM_n)\chi_{A_n} + ka\chi_{A_n} \geqq 0$. This is possible because of the assumption $a > 0$ and the fact that the functions f, M_n, TM_n are bounded and all functions occurring in this inequality are equal to zero for $x \notin A_n$. Moreover there is valid

$$\begin{aligned} I(M_n + ka\chi_{A_n}) &= I(M_n - (TM_n)\chi_{A_n} + ka\chi_{A_n} + (TM_n)\chi_{A_n}) \leqq \\ &\leqq I(M_n - (TM_n)\chi_{A_n} + ka\chi_{A_n}) + I((TM_n)\chi_{A_n}) \leqq \\ &\leqq I(M_n - (TM_n)\chi_{A_n} + ka\chi_{A_n}) + I(TM_n) = \\ &= I(M_n - (TM_n)\chi_{A_n} + a\chi_{A_n} + (k-1)a\chi_{A_n}) + I(M_n) \leqq \\ &\leqq I(f\chi_{A_n} + (k-1)a\chi_{A_n}) + I(M_n) \leqq I(f\chi_{A_n}) + a(k-1)I(\chi_{A_n}) + I(M_n). \end{aligned}$$

From lemma 1 it follows that

$$I(M_n + ka\chi_{A_n}) = I(M_n) + kaI(\chi_{A_n}).$$

By applying the last inequality we obtain after a short arrangement the assertion of the theorem.

Q.E.D.

The following theorem can be considered as a limit case of the preceding one. Firstly we must introduce the notations.

$$A = \{x \in X;\ \exists p \in N\colon f(x) + f(Tx) + \dots + f(T^{p-1}x) > pa\}.$$

Theorem 2. *Let $f \in F$, $f \geqq 0$, $a \geqq 0$. Let f be a bounded function. Then $I(f\chi_{A_n}) \geqq aI(\chi_A)$.*

Proof. For $a = 0$ the assertion of the theorem is trivial. Let us assume $a > 0$. Evidently

$$A_n \subset A_{n+1}$$

$$A = \bigcup_{n=1}^{\infty} A_n$$

and then

$$\chi_{A_n} \nearrow \chi_A$$

$$f\chi_{A_n} \nearrow f\chi_A.$$

Moreover $\chi_{A_n} \in F$, $I(\chi_{A_n}) \leqq I(e)$ and $\chi_A \in F$. By lemma 3 $f\chi_A \in F$. By using the above mentioned relationships and theorem 1 we obtain $I(f\chi_A) \geqq I(f\chi_{A_n})$. Therefore $\{I(\chi_{A_n})\}$ is a sequence bounded by $\frac{1}{a} I(f\chi_A)$. Hence from $I(\chi_A) \leqq \frac{1}{a} I(f\chi_A)$ the assertion of the theorem follows.

Q.E.D.

§ 4. Application of an integral with respect to a premeasure

In the preceding paragraph a positive result concerning the relation (3) was obtained. By applying an integral with respect to a premeasure a negative result concerning the relation (4) will be shown. Let us assume $f \in F$, $f \geqq 0$, $a \geqq 0$. Let us denote

$$N_n = \min(0, S_1, S_2, \ldots, S_n)$$
$$B_n = \{x \in X;\ N_n(x) < 0\}.$$

It will be shown that an arbitrary relation between the values $aI(\chi_{B_n})$ and $I(f\chi_{B_n})$ is allowed. The functional I will be replaced by an integral with respect to a premeasure which is introduced in [4]. Further properties of this integral have been worked out in [4], [5], [6]. The example given will be calculated by a method from [5], page 259. This method is applicable to the calculation of an integral of a real function of a real variable with respect to a premeasure μ. Let us denote this integral by a symbol $J_\mu(f)$. In order to get a better survey let us introduce a necessary formula

$$J_\mu(f) = \int_R g(t)\,d\lambda,$$

where λ is the Lebesgue measure and a function g is defined by the relation $g(t) = \mu(\{x \in X;\ f(x) > t\})$.

Let $X = \langle 0, 1)$, $\mu = \sqrt{\lambda}$. Let F be a set of all integrable functions with respect to the premeasure μ defined on the set $\langle 0, 1)$. The functional I will be defined as follows: $I(f) = J_\mu(f)$. As a transformation T define T: $\langle 0, 1) \to \langle 0, 1)$, $x \mapsto \left(x + \frac{1}{2}\right) \bmod 1$.

It is evident that this transformation is λ-preserving and, due to the definition of μ, premeasure μ preserving. It is easy to show that $J_\mu(f) = J_\mu(Tf)$. Moreover from the properties of the integral with respect to a premeasure it follows that all conditions required in a system (X, F, I, T) are fulfilled.

In the following example we shall work with the function $N = N_2 = \min(0, S_1, S_2)$ and with the set $B = \{x \in \langle 0, 1);\ N_2(x) < 0\}$.

Example. Let g: $\langle 0, 1) \to R$, $t \mapsto \sqrt{t}$, $f = g\chi\left\langle 0, \frac{1}{2}\right)$.

Let $a \in \left(0, \frac{1}{\sqrt{8}}\right\rangle$. After a short calculation we obtain

$$B = \langle 0, 4a^2) \cup \left\langle \frac{1}{2}, 1\right)$$

$$I(f\chi_B) = J_\mu(f\chi_B) = \pi a^2$$

$$aI(\chi_B) = aJ_\mu(f\chi_B) = a\sqrt{4a^2 + \frac{1}{2}}.$$

Let us denote

$$\xi = \frac{1}{\sqrt{2(\pi^2 - 4)}}.$$

It is easy to see that

$$I(f\chi_B) > aI(\chi_B) \quad \text{if} \quad a \in \left(\xi, \frac{1}{\sqrt{8}}\right\rangle$$

$$I(f\chi_B) = aI(\chi_B) \quad \text{if} \quad a = \xi$$

$$I(f\chi_B) < aI(\chi_B) \quad \text{if} \quad a \in (0, \xi).$$

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СУБАДДИТИВНАЯ МАКСИМАЛЬНАЯ ЭРГОДИЧЕСКАЯ ТЕОРЕМА

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Резюме

В работе приведено обобщение так называемой классической максимальной эргодической теоремы для случая сублинейного функционала. В примере использован нелинейный интеграл, основанный на понятии предмеры. На этом примере также показано, что с данной точки зрения не всегда возможно обобщение приведенной теоремы.