Mathematica Slovaca

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Mathematica Slovaca, Vol. 44 (1994), No. 1, 1--19

Persistent URL: http://dml.cz/dmlcz/129035

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BOOLEAN POWERS AND STOCHASTIC SPACES

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(Communicated by Lubomír Kubáček)

ABSTRACT. In [6], [7], D. Scott made the first attempt to connect Non-Standard Analysis and Boolean-valued models and at the same time he introduced Boolean Analysis, which had been developed subsequently mainly by G. Takeuti [10].

In this paper we investigate the relationship between the Boolean power $\mathbb{R}[\mathbb{B}]$ of $\mathbb{R}$ and the elementary stochastic space E in the sense of Kappos [3]. We obtain here that these two spaces are isomorphic. In this way, we obtain a stochastic interpretation of the Boolean power structure. The development is similar to Takeuti's Boolean analysis. The main difference lies in the fact that we use a full Boolean-valued model, known as Boolean power, and a two-step procedure: First we develop a restrictive model (a discrete or a kind of first order model), the Boolean power, in which all the axioms of the reals can be transferred immediately, and then we complete it using Cauchy sequences or Dedekind cuts in order to get a model isomorphic to the stochastic space V. In this way, we avoid the general Scott-Solovay model and we get instead a model which is more appropriate for generalizing the Robinsonian Infinitesimal Analysis to Boolean Analysis.

1. Boolean powers

In the following we give the main concepts and results from the theory of Boolean powers. Let $\mathcal{U} = (A, R)$ be a relational structure, i.e. A is a non-empty set called the *Universe*, and R is a binary relation on A. Everything we say for binary relations can be easily extended to more general ones.

AMS Subject Classification (1991): Primary 03C90. Secondary 60B99.
Key words: Boolean powers, Elementary stochastic spaces, Boolean non-standard reals, Completions.

Let $\mathcal{L}$ be a first order language with equality, the usual logical symbols $\wedge$, $\vee$, $\neg$, $\exists$, $\forall$, and variables x, y, v_1, v_2, ... In $\mathcal{L}$ we add also a predicate symbol $\mathcal{R}$ which in A is interpreted as R. We extend $\mathcal{L}$ to $\mathcal{L}(A)$ by adding a constant symbol $\mathfrak{a}$ for each element a in A. In the following we shall not make any difference between $\mathcal{R}$ and R or between $\mathfrak{a}$ and a and use the same symbol for both.

Let also $(\mathbb{B}, \wedge, \vee, \neg, 1_{\mathbb{B}}, 0_{\mathbb{B}})$ be a complete Boolean algebra. A $\mathbb{B}$-*extension*

$$\mathcal{U}[\mathbb{B}] = \big(A[\mathbb{B}], R[\mathbb{B}]\big)$$

of the structure $\mathcal{U}$ is defined as follows:

$$A[\mathbb{B}] = \Big\{f \in \mathbb{B}^A \, ; \;\; a \neq b \to f(a) \wedge f(b) = 0_{\mathbb{B}} \text{ and } \bigvee_{a \in A} f(a) = 1_{\mathbb{B}}\Big\},$$

and the Boolean interpretations of the equality and the binary relation are functions from $A \times A$ into $\mathbb{B}$, defined by:

$$E[\mathbb{B}](f,g) = \bigvee_{x \in A} f(x) \wedge g(x)\,,$$

$$R[\mathbb{B}](f,g) = \bigvee_{x,y \in A : R(x,y)} f(x) \wedge g(y)\,.$$

The truth value function $\|\cdot\|$ is defined for atomic formulas:

$$\|f = g\| = E[\mathbb{B}](f,g)\,,$$

$$\|R(f,g)\| = R[\mathbb{B}](f,g)\,,$$

and for any other formulas ϕ, ψ of $\mathcal{L}\big(A[\mathbb{B}]\big)$ inductively:

$$\|\phi \wedge \psi\| = \|\phi\| \wedge \|\psi\|\,,$$

$$\|\neg\phi\| = \neg\|\phi\|\,,$$

$$\|\exists\, x, \phi(x)\| = \bigvee_{f \in A[\mathbb{B}]} \|\phi(f)\|\,.$$

This Boolean-valued structure is called the *Boolean power* or the *Boolean extension* of the structure $\mathcal{U}$. We say that a sentence ϕ is $\mathbb{B}$-valid if and only if $\|\phi\| = 1_{\mathbb{B}}$.

In order to verify that $\mathcal{U}[\mathbb{B}]$ is really a Boolean-valued structure, one can easily check the following:

1.1. THEOREM.

(i) $\|f = g\| = 1_{\mathbb{B}}$ *if and only if* $f = g$ *as functions,*

(ii) $\|f = g\| = \|g = f\|$,

(iii) $\|f = g\| \wedge \|g = h\| \leq \|f = h\|$,

(iv) $\|f = g\| \wedge \|\phi(f)\| \leq \|\phi(g)\|$

for every f, g, h *in* $A[\mathbb{B}]$ *and every formula* ϕ *with a free variable* x . *Note that* $\|\phi\| \leq \|\psi\|$ *is equivalent to* $\|\phi\| \to \|\psi\| = 1_{\mathbb{B}}$ *in every Boolean algebra* $\mathbb{B}$.

NOTATION. From now on, we use the symbol $A^{\#}$ to denote $A[\mathbb{B}]$ and the same symbol is used for all the Boolean extensions of relations or functions that we are going to use in the following.

The development of Boolean powers is actually a generalization of the "power" part of the "ultrapower" construction. So a generalization of the Łoś theorem is desirable:

1.2. THEOREM. *Let* ϕ *be a formula of the language* $\mathcal{L}(A)$ *with free variables* $x_1, \ldots, x_n$, *and* $f_1, \ldots, f_n$ *are in* $A^{\#}$. *Then*

$$\|\phi(f_1, \ldots, f_n)\| = \bigvee_{\substack{a_1, \ldots, a_n \in V: \\ V \models \phi(a_1, \ldots, a_n)}} \left[\bigwedge_{i=1}^{n} f_i(a_i) \right].$$

We recall also from [4] the maximum principle:

1.3. THEOREM. *For every formula* ϕ *with one free variable* x , *there always exists a* g *in* $A^{\#}$ *such that*

$$\|\exists\, x, \phi(x)\| = \|\phi(g)\| .$$

1.4. DEFINITION. *For every* $a \in A$ *we define* $\widehat{a}$ *in* $A^{\#}$ *to be the function*

$$\widehat{a}(x) = \begin{cases} 1_{\mathbb{B}} & \textit{if } x = a\,, \\ 0_{\mathbb{B}} & \textit{otherwise}\,. \end{cases}$$

It is clear that for every a in A and for every f in $A^{\#}$

$$\|f = \widehat{a}\| = f(a)\,.$$

1.5. THEOREM. *The mapping* $\widehat{\ }: A \to A^{\#}$ *is a* $\mathbb{B}$*-elementary embedding, so for every* $\mathcal{L}(A)$*-formula* ϕ *whose free variables are among* $x_1, x_2, \ldots, x_n$ *and for any* $a_1, a_2, \ldots, a_n$ *we have:*

$$\mathcal{U} \models \phi(a_1, \ldots, a_n) \iff \|\phi(\widehat{a}_1, \ldots, \widehat{a}_n)\| = 1_{\mathbb{B}} .$$

If, in addition, we denote $\widehat{\phi}$ the formula of $\mathcal{L}(A^{\#})$ derived from an $\mathcal{L}(A)$-formula ϕ by putting a "hat"on each constant c included in ϕ, we have the following:

1.6. THEOREM. (*Transfer*)

$$\mathcal{U} \models \phi \iff \|\widehat{\phi}\| = 1_{\mathbb{B}} .$$

The following *Mixing Properties* give us a characterization of the elements of $A^{\#}$:

1.7. THEOREM. (*Mixing Properties*)

(1) *Let* $T = \{t_i : i \in I\}$ *be a pairwise disjoint collection from* $\mathbb{B}$ *and* $\{f_i : i \in I\}$ *is any collection from* $A^{\#}$. *Then there is an* $f \in A^{\#}$ *such that*

$$\|f = f_i\| \geq t_i \qquad \text{for all} \quad i .$$

If, in addition, $\bigvee_{i \in I} t_i = 1_{\mathbb{B}}$ (*then* T *is called a resolution of identity*), *this* f *is unique and can be written in the form*

$$f(x) = \bigvee_{i \in I} \big(f_i(x) \wedge t_i\big) \qquad \text{or} \qquad f = \bigvee_{i \in I} f_i \wedge t_i ,$$

or, using the "sum" notation,

$$f = \sum_{i \in I} f_i \cdot t_i .$$

(2) *If* $\{a_i : i \in I\} \subseteq A$ *and* $T = \{t_i : i \in I\}$ *is a resolution of identity, then there is a unique function* $f \in A^{\#}$ *such that, for all* i, $\|f = \widehat{a}_i\| = t_i$. *This* f *will then be denoted by*

$$\sum_{i \in I} \widehat{a}_i \cdot t_i .$$

1.8. Remark. By (1), (2) and the definition of $A^{\#}$, it follows that every function $f \in A^{\#}$ can be written in the above form for appropriate pairwise different a_i's in A and T a resolution of identity in $\mathbb{B}$. In addition, we may suppose that the resolution is strictly positive ($t_i \neq 0_{\mathbb{B}}$, for all i). This form is called the *reduced representation* of f by its values. Such an f is actually the function

$$f(x) = \begin{cases} t_i & \text{if } x = a_i\,, \\ 0_{\mathbb{B}} & \text{otherwise}\,. \end{cases}$$

The next theorem gives us an interpretation of subsets of $A^{\#}$:

1.9. THEOREM. *Let $S = \{x \in A : \phi(x)\}$ be a subset of A. Then $S^{\#}$ is isomorphic to the set $\{f \in A^{\#} : \|\phi(f)\| = 1_{\mathbb{B}}\}$.*

Proof. By definition we have

$$S^{\#} = \{f \in \mathbb{B}^S : f \text{ partitions unity in } \mathbb{B}\}\,.$$

Let

$$C := \{f \in A^{\#} : \|\phi(f)\| = 1_{\mathbb{B}}\}\,.$$

We need to prove that $C = S^{\#}$. Indeed, we have (by 1.2):

$$\|\phi(f)\| = \bigvee_{\substack{x \in A: \\ \phi(x)}} f(x) = \bigvee_{x \in S} f(x)\,. \tag{1}$$

Now let $f \in C$, then $\|\phi(f)\| = 1_{\mathbb{B}}$ so $\bigvee_{x \in S} f(x) = 1_{\mathbb{B}}$ and $f(x) = 0_{\mathbb{B}}$ for every $x \in A - S$. So f takes positive values only on elements of S, and therefore it can be thought as an element of $S^{\#}$.

Conversely, if $f \in S^{\#}$, then $\|\phi(f)\| = 1_{\mathbb{B}}$ (by (1)) and f can be extended from S to A by putting $f(x) = 0_{\mathbb{B}}$ for each $x \in A - S$. So f is in $A^{\#}$ and $f \in C$.

The following theorem, used by Grätzer [1; p. 147] in the definition of Boolean powers, gives us an interpretation of functions or operators from the main structure to its Boolean extension:

1.10. THEOREM. *The Boolean extension of any function $F\colon A^n \to A$ takes the form:*

$$F^{\#}\colon (A^{\#})^n \to A^{\#}\,,$$

and $F^{\#}(g_1, \ldots, g_n) = g$, where g is defined by

$$g(a) := \bigvee_{\substack{a_1,\ldots,a_n \in A:\\ F(a_1,\ldots,a_n)=a}} \Big(\bigwedge_{i=1}^{n} g_i(a_i) \Big)$$

If $n = 1$ and $f = \sum_{i\in I} \widehat{x}_i \cdot t_i$, then $F^{\#}(f) = \sum_{i\in I} \widehat{F(x_i)} \cdot t_i$.

For example, if $\mathcal{U}$ is the structure of the real numbers and $F = \sin$, then $\sin^{\#}(f) = \sum_{i\in I} \widehat{\sin(x_i)} \cdot t_i$, and the usual properties of the sine function remain valid in $\mathcal{U}^{\#}$ (i.e. $\left\| \big(\sin^{\#}(f)\big)^2 + \big(\cos^{\#}(f)\big)^2 = \widehat{1} \right\| = 1_{\mathfrak{B}}$).

Note that $\sum_{i\in I} \widehat{F(x_i)} \cdot t_i$ may not be the reduced representation of $F^{\#}(f)$, for some of the $F(x_i)$ may be equal. Then there is an easy way to find such a representation by "adding" all the corresponding b_i's. This is the reason that we are allowed to use the "sum" notation without loss of generality. If F is a function between subsets of A, thus $F\colon S \to R$, then $F^{\#}$ is a function from $S^{\#}$ to $R^{\#}$.

By now we are allowed to use the transfer principle only for first-order formulas of $\mathcal{U}$ and $\mathcal{U}^{\#}$. So, formulas like

$$\psi = \big(\forall x\big) \big[x \in S \leftrightarrow \phi(x)\big]$$

which involve variables ranging over subsets of A cannot be transferred from the main structure to its Boolean extension and conversely. One then has to use the superstructures over A and $A^{\#}$. This project is under preparation for future publication. In this paper, we are only interested in the study of the Boolean power of $\mathbb{R}$ in connection with the study of the elementary stochastic space.

2. The Boolean power of $\mathbb{R}$

From now on it is supposed that the structure we are interested in is

$$\mathcal{R} := (\mathbb{R}, \leq, +, \cdot)\,.$$

Then the binary relation is just $\leq$, and there are also functions $+$ and $\cdot$ that can be thought as ternary relations so that for every $f, g, h \in \mathbb{R}^{\#}$ we have:

$$\|f \leq g\| = \bigvee_{\substack{x, y \in \mathbb{R}: \\ x \leq y}} f(x) \wedge g(y)\,,$$

$$\|f + g = h\| = \bigvee_{\substack{x, y, z \in \mathbb{R}: \\ x + y = z}} f(x) \wedge g(y) \wedge h(z)\,,$$

$$\|f \cdot g = h\| = \bigvee_{\substack{x, y, z \in \mathbb{R}: \\ x \cdot y = z}} f(x) \wedge g(y) \wedge h(z)\,.$$

If $f = \sum\limits_{i \in I} \widehat{a}_i \cdot t_i$ and $g = \sum\limits_{j \in J} \widehat{b}_j \cdot s_j$, then using Theorem 1.10 we can define

$$f +^{\#} g = \sum_{(i,j) \in I \times J} \widehat{a_i + b_j} \cdot (t_i \cdot s_j)\,,$$

$$f \cdot^{\#} g = \sum_{(i,j) \in I \times J} \widehat{a_i \cdot b_j} \cdot (t_i \cdot s_j)\,,$$

where $t_i \cdot s_j$ is the sum-product notation of $t_i \wedge s_j$.

One can see that

$$\|f + g = h\| = 1_{\mathbb{B}} \iff h = f +^{\#} g\,.$$

In the following, we shall use instead of $f +^{\#} g$ the simpler form $f + g$.

By the transfer principle, it follows that $\mathcal{R}^{\#}$ is a Boolean-valued model of all the first order axioms of the reals, thus the axioms of the totally ordered field remain true (take truth value $1_{\mathbb{B}}$). For instance

$$\|(\forall\, x)(\forall\, y)\,[x + y = y + x]\| = 1_{\mathbb{B}}\,,$$

or

$$\bigwedge_{x \in \mathbb{R}^{\#}} \bigwedge_{y \in \mathbb{R}^{\#}} \|x + y = y + x\| = 1_{\mathbb{B}}\,,$$

which means that for all $x, y \in \mathbb{R}^{\#}$

$$\|x + y = y + x\| = 1_{\mathbb{B}}\,.$$

Note that "trichotomy" law also holds for

$$\bigwedge_{x \in \mathbb{R}^{\#}} \bigwedge_{y \in \mathbb{R}^{\#}} \|x < y \vee y < x \vee x = y\| = 1_{\mathbb{B}}\,,$$

or for every $x, y \in \mathbb{R}^{\#}$

$$\|x < y\| \vee \|y < x\| \vee \|x = y\| = 1_{\mathbb{B}} .$$

This, in the usual sense, may seem to be a contradiction, because two functions may not be comparable. This will be again discussed in 3.8 and the resolution of this will be apparent.

By Transfer, $\mathbb{R}^{\#}$ is a $\mathbb{B}$-Archimedean, where the set of naturals is just $\mathbb{N}^{\#}$. Alternatively, one can see that $\mathbb{R}^{\#}$ is also Archimedean since the following is true:

$$\bigwedge_{n \in \mathbb{N}} \|x \leq \widehat{n}\| = 1_{\mathbb{B}}$$

for every $x \in \mathbb{R}^{\#}$.

3. Measure algebras and elementary stochastic spaces

We recall some definitions and ideas from K a p p o s [3]. Let $(\Omega, \mathcal{A}, P)$ be a P-complete probability space, and

$$\mathcal{N} := \{N \in \mathcal{A} : \ P(N) = 0\}$$

be the σ-ideal of sets of measure zero. We define on $\mathcal{A}$ an equivalence relation $\approx$ by

$$A \approx B \overset{\text{def}}{\Longleftrightarrow} A \triangle B \in \mathcal{N} ,$$

and let $\mathbb{B} := \mathcal{A}/\approx$ be the quotient algebra. If γ is the canonical σ-homomorphism from $\mathcal{A}$ to $\mathbb{B}$ (where $\gamma(A) := A/\approx$), then:

$$\gamma\left(\bigcup_{i=1}^{\infty} A_i\right) = \bigvee_{i=1}^{\infty} \gamma(A_i) ,$$

$$\gamma\left(\bigcap_{i=1}^{\infty} A_i\right) = \bigwedge_{i=1}^{\infty} \gamma(A_i) ,$$

$$\gamma(\neg A) = \neg\gamma(A) .$$

$\mathbb{B}$ is clearly a Boolean σ-algebra, elements of $\mathbb{B}$ are called *events*, $1_{\mathbb{B}} \equiv \emptyset/\approx$ is the *sure*, and $0_{\mathbb{B}} \equiv \Omega/\approx$ is the *impossible* event.

On $\mathbb{B}$, we can define the reduction p of the probability measure P by:

$$p(a) := P(A) , \qquad \text{where} \quad \gamma(A) = a .$$

It is clear that this definition is independent from the choice of the representative A. The pair $(\mathbb{B}, p)$ is called a *measure algebra.*

3.1. Definition. *A random experiment or trial $T \subseteq \mathbb{B}$ is a positive partition of unity in $\mathbb{B}$ (or resolution of identity, as in 1.7), i.e. if $T = \{t_i : i \in I\}$, then t_i's satisfy the following conditions:*

(i) $(\forall i \in I)\,[t_i \neq 0_{\mathbb{B}}]$,
(ii) $t_i \wedge t_j = 0_{\mathbb{B}}$ *for* $i \neq j$,
(iii) $\bigvee_{i \in I} t_i = 1_{\mathbb{B}}$.

The set of all trials is denoted by $\mathcal{T}$. This set can be ordered by the following relation:

$$T \preceq S \overset{\text{def}}{\iff} (\forall t_i \in T)(\exists s_j \in S)\,[t_i \leq s_j].$$

Note that for any two trials T, S the *common refinement*

$$T \wedge S := \{t_i \wedge s_j \neq 0_{\mathbb{B}} : t_i \in T,\ s_j \in S\}$$

always exist. Thus $(\mathcal{T}, \preceq)$ becomes a directed set.

3.2. Proposition. *The Boolean σ-algebra $\mathbb{B}$ satisfies the countable chain condition, and hence it is complete.*

P r o o f. See [3].

3.3. Proposition. *Let $T = \{t_i : i \in I\}$ be a trial. Then I is at most countable.*

P r o o f. It follows from countable chain condition.

3.4. Definition. *Let $T = \{t_i : i \in I\}$ be a trial. Then any function $X\colon T \to \mathbb{R}$ is called an elementary random variable (e.r.v.) on $(\mathbb{B}, p)$.*

Some special cases of e.r.v.'s are the following:

(1) If $T = \{1_{\mathbb{B}}\}$, then any X defined on T is called a *constant r.v.*
(2) If $T = \{a, \neg a\}$, $X(a) = 1$ and $X(\neg a) = 0$, then X is called an *indicator r.v.* and is denoted by I_a.
(3) If T is finite, then any e.r.v. defined on T is called a *simple r.v.*

We denote $\mathcal{E}$ the set of all e.r.v.'s defined on all elements of $\mathcal{T}$ and call it the *elementary stochastic space* on $(\mathbb{B}, p)$.

3.5. R e m a r k. For every $X \in \mathcal{E}$ there exists a trial $T = \{t_i : i \in I\}$ and a collection $\{x_i : i \in I\}$ of pairwise different reals such that $X(t_i) = x_i$ for all $i \in I$, where I is at most countable. We denote

$$X = \sum_{i \in I} x_i \cdot I_{t_i}. \tag{2}$$

This representation is called the *canonical* (or *reduced*) representation of X by indicators.

3.6. Proposition. *Let $\mathbb{R}^{\#}$ be the Boolean power of $\mathbb{R}$, where $\mathbb{B}$ is the measure algebra defined above. Then for every e.r.v. X there corresponds a unique element of $\mathbb{R}^{\#}$ and conversely.*

Proof. Let X be an e.r.v. with reduced representation as in (2). Then the corresponding f_X is

$$f_X(x) = \begin{cases} t_i & \text{if and only if } x = x_i\,, \\ 0_{\mathbb{B}} & \text{otherwise}\,, \end{cases}$$

which is unique by 1.8.

Conversely, if $f = \sum_{i \in I} \widehat{x}_i \cdot t_i$, then the corresponding X_f is defined on the trial $T = \{t_i : i \in I\}$, and its reduced representation is just (2).

The function f_X corresponding to an e.r.v. X is actually the qualitative (or Boolean) density function of this e.r.v. The quantitative density function would then be:

$$\varphi_X(x) = \begin{cases} p(t_i) & \text{if and only if } x = x_i\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

In $\mathcal{E}$ we can define:

$$\|X = Y\| := \bigvee_{x \in \mathbb{R}} \left[X^{-1}(x) \wedge Y^{-1}(x)\right],$$

$$\|X \leq Y\| := \bigvee_{\substack{x, y \in \mathbb{R}: \\ x \leq y}} \left[X^{-1}(x) \wedge Y^{-1}(y)\right].$$

Also, we may extend all the relations of $\mathbb{R}$ to $\mathcal{E}$ in an analogous way, and define

$$X + Y := \sum_{(i,j) \in I \times J} (x_i + y_j) \cdot I_{t_i s_j}\,,$$

$$X \cdot Y := \sum_{(i,j) \in I \times J} (x_i \cdot y_j) \cdot I_{t_i s_j}\,.$$

By these definitions, $\mathcal{E} \equiv (\mathcal{E}\,, =\,, \leq\,, +\,, \cdot)$ becomes a Boolean-valued structure which can be seen as an extension of the reals. It is clear that

$$\|X = Y\| = \|f_X = f_Y\|\,,$$

and

$$\|X \leq Y\| = \|f_X \leq f_Y\|\,.$$

Also $X + Y$ (or $X \cdot Y$) corresponds to $f_X + f_Y$ (or $f_x \cdot f_Y$) and conversely.

From 3.6 and the above discussion the following theorem has been established:

3.7. THEOREM. *The structures $\mathcal{E}$ and $\mathcal{R}^{\#}$ are isomorphic and they both extend the structure of the real numbers.*

3.8. Remarks. We have already seen that, by the theory of Boolean powers, $\mathbb{R}^{\#}$ (or E) becomes a Boolean-valued model of all the first order axioms of the reals. If we consider again the trichotomy law, in this case, it means that the union of the sets that the e.r.v. X (or its $\mathbb{B}$-density f_X) is greater than, less than or equal to the e.r.v. Y (or its $\mathbb{B}$-density f_Y) is the whole probability space.

We have already seen that the elementary stochastic space E and the corresponding Boolean power $\mathbb{R}^{\#}$ are made up by discrete r.v.'s. We would like to extend E to the stochastic space V of all r.v.'s (see [3]) and accordingly to get a completion of the Boolean power of $\mathbb{R}$. In this work, we propose two methods for doing that: The first one is based on the notion of the $\mathbb{B}$-distribution function and it is equivalent to the Dedekind cuts procedure, while the second is based on the Cauchy sequences.

4. Completions of $\mathbb{R}^{\#}$

4.1. $\mathbb{B}$-Distributions functions.

We recall some definitions from the theory of Boolean algebras:

Let $\mathbb{B}$ be a Boolean algebra. Then, if $\{b_n\}_{n\in\mathbb{N}}$ is a sequence in $\mathbb{B}$, then:

$$\underline{\lim}\, b_n := \bigvee_{n=1}^{\infty} \bigwedge_{k\geq n} b_k \qquad \text{(lower limit)}\,,$$

$$\overline{\lim}\, b_n := \bigwedge_{n=1}^{\infty} \bigvee_{k\geq n} b_k \qquad \text{(upper limit)}\,,$$

and, if $\underline{\lim}\, b_n = \overline{\lim}\, b_n = b$, then we say that the limit of $\{b_n\}_{n\in\mathbb{N}}$ is b and write:

$$\lim_{n\to\infty} b_n = b\,.$$

Let $b\colon \mathbb{R} \to \mathbb{B}$ be a function. We say that

$$\lim_{x\to x_0} b(x) = a$$

if and only if for every sequence x_n of reals which converges to x_0 we have $\lim\limits_{n\to\infty} b(x_n) = a$. One can also define the limit of the function as x goes to x_0 from above or below and write $b(x+0)$ and $b(x-0)$ respectively. Note that for monotone functions $b(x+0)$ and $b(x-0)$ always exist.

Continuity, continuity from the left or from the right are defined analogously.

4.1.1. DEFINITION. *Let $f \in \mathbb{R}^{\#}$. Then the $\mathbb{B}$-valued function F_f defined for all reals by:*

$$F_f(x) := \|f \leq \widehat{x}\| = \bigvee_{t \leq x} f(t)$$

is called the $\mathbb{B}$-distribution function (BDF) of f.

It is clear that F_f is increasing, thus $x \leq y$ implies that $F_f(x) \leq F_f(y)$ and has also the following properties:

4.1.2. PROPOSITION. *If F_f is the BDF of f, then:*

$$\begin{aligned} &\lim_{x \to -\infty} F_f(x) = 0_{\mathbb{B}}\,, \qquad &&\lim_{x \to \infty} F_f(x) = 1_{\mathbb{B}}\,, \\ &F_f(x) = \lim_{t \downarrow x} F_f(t) \qquad &&\textit{for all} \quad x \in \mathbb{R}\,. \end{aligned} \tag{3}$$

P r o o f . Obvious if we use the definition of the limit in Boolean algebras and the fact that F_f is increasing.

Next we give a well-known theorem for Boolean algebras:

4.1.3. THEOREM. *Let $\mathbb{B}$ be a complete Boolean algebra that satisfies the countable chain condition. Then for any set $A \subseteq \mathbb{B}$ there is a set $B \subseteq A$ which is at most countable and*

$$\bigvee B = \bigvee A\,, \qquad \bigwedge B = \bigwedge A\,.$$

By Proposition 4.1.2, it follows that

$$\begin{aligned} &\bigwedge_{x \in \mathbb{R}} F_f(x) = 0_{\mathbb{B}}\,, \qquad &&\bigvee_{x \in \mathbb{R}} F_f(x) = 1_{\mathbb{B}}\,, \\ &F_f(x) = \bigwedge_{t > x} F_f(t) \qquad &&\text{for all} \quad x \in \mathbb{R}\,, \end{aligned} \tag{4}$$

and by Theorem 4.1.3, these infimum's and supremum's are the same if the index sets restrict to countable ones. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, and F_f is increasing, one can show that the required countable sets can be subsets of $\mathbb{Q}$, so properties (4) can be written equivalently:

$$\begin{aligned} &\bigwedge_{r \in \mathbb{Q}} F_f(r) = 0_{\mathbb{B}}\,, \qquad &&\bigvee_{r \in \mathbb{Q}} F_f(r) = 1_{\mathbb{B}}\,, \\ &F_f(x) = \bigwedge_{\substack{t \in \mathbb{Q}: \\ t > x}} F_f(t) \qquad &&\text{for all} \quad x \in \mathbb{R}\,, \end{aligned} \tag{5}$$

4.1.4. Theorem. *Let f, g be in $\mathbb{R}^{\#}$. If $F_f(r) = F_g(r)$ for every r in $\mathbb{Q}$, then for every $x \in \mathbb{R}$ we have $F_f(x) = F_g(x)$.*

P r o o f. Obvious, using (5).

4.2. Homomorphisms of e.r.v.'s.

For every $f \in \mathbb{R}^{\#}$ there is a function $H_f : \mathcal{D} \to \mathbb{B}$, where $\mathcal{D}$ is the set of all intervals of the real line of the form $(-\infty, x]$, defined by:

$$H_f\big((-\infty, x]\big) := F_f(x).$$

This is an order-preserving function that can be extended to a σ-homomorphism between $\mathcal{B}$ (the Borel σ-algebra on the real line) and $\mathbb{B}$. The following holds:

$$H_f(\neg a) = \neg\big(H_f(a)\big) \quad \text{and} \quad H_f\left(\bigvee_{i=1}^{\infty} a_i\right) = \bigvee_{i=1}^{\infty} H_f(a_i) \tag{6}$$

for all $a, a_1, \dots, a_n \in \mathcal{B}$.

4.3. Completion of $\mathbb{R}^{\#}$.

Until now we have seen that every e.r.v. defines a BDF and a σ-homomorphism between $\mathcal{B}$ and $\mathbb{B}$. However, the opposite is not always true. This will lead us to the desired completion of $\mathbb{R}^{\#}$. First we need some definitions:

4.3.1. Definition. *Let $F : \mathbb{R} \to \mathbb{B}$ be an increasing function with the properties* (5). *Then F is called a $\mathbb{B}$-distribution function (BDF).*

4.3.2. Definition. *Let $H : \mathcal{B} \to \mathbb{B}$ be an order-preserving function for which* (6) *holds. Then H is called a σ-homomorphism from $\mathcal{B}$ to $\mathbb{B}$.*

Suppose now that F is a BDF taking only at most countable pairwise different values $\{b_i : i \in Z\}$. Then, since F is increasing and continuous from the right, all numbers in intervals of the form $[x_i, x_{i+1})$ must take a constant value b_i. Define:

$$f(x) = \begin{cases} b_i - b_{i-1} & \text{if and only if } x = x_i\,, \\ 0_{\mathbb{B}} & \text{otherwise}. \end{cases}$$

Then f is clearly in $\mathbb{R}^{\#}$. Thus, a BDF of this form defines a unique e.r.v. by:

$$f(x) = F(x) - \bigvee_{\substack{t \in \mathbb{Q}: \\ t < x}} F(t).$$

The same thing is true if H is a σ-homomorphism taking only at most countable pairwise different values $\{b_i : i \in Z\}$ for all elements of $\mathcal{D}$. Then all intervals $(-\infty, x]$, where $x \in [x_i, x_{i+1})$ take value b_i, and the corresponding f is defined analogously.

Let $\mathbb{D}$ be the class of all BDF that take at most countable pairwise different values. Then, if we define:

$$\|F = G\| := \bigvee_{x \in \mathbb{R}} \left[\Big(F(x) - \bigvee_{\substack{q \in \mathbb{Q}: \\ q < x}} F(q) \Big) \wedge \Big(G(x) - \bigvee_{\substack{q \in \mathbb{Q}: \\ q < x}} G(q) \Big) \right].$$

$$\|F \le G\| := \bigvee_{\substack{x, y \in \mathbb{R}: \\ x \le y}} \left[\Big(F(x) - \bigvee_{\substack{q \in \mathbb{Q}: \\ q < x}} F(q) \Big) \wedge \Big(G(y) - \bigvee_{\substack{q \in \mathbb{Q}: \\ q < y}} G(q) \Big) \right].$$

and the operations $+$, $\cdot$ analogously, then the structures $(\mathbb{D}, =, \le, +, \cdot)$ and $\mathcal{R}^{\#}$ are isomorphic.

If we extend $\mathbb{D}$ to $\mathbb{D}_*$, which is the set of all BDF's, then $\mathbb{R}^{\#}$ is extended to $\mathbb{R}^{\#}_*$ by adding all the "abstract" r.v.'s that correspond to elements of $\mathbb{D}_* - \mathbb{D}$. Then we have to extend the truth value function to all sentences of $\mathcal{L}(\mathbb{R}^{\#}_*)$. The following theorem is very helpful:

4.3.3. THEOREM. *Every BDF defines a class of almost everywhere (a.e.) equal random variables on* $(\Omega, \mathcal{A}, P)$.

P r o o f. See [3].

So every element of $\mathbb{D}_*$ defines a class of a.e. equal r.v.'s on Ω, and especially elements of $\mathbb{D}$ define classes of discrete r.v.'s. Then, if $f, g \in \mathbb{R}^{\#}$ correspond to $F, G \in \mathbb{D}$, which by 4.3.3 define X, Y, r.v.'s on Ω, we have

$$\|f = g\| = \gamma\big(\{\omega \in \Omega : X(\omega) = Y(\omega)\}\big),$$
$$\|f \le g\| = \gamma\big(\{\omega \in \Omega : X(\omega) \le Y(\omega)\}\big).$$

This can also be a definition of $\|\cdot\|$ for atomic formulas of the language of $\mathbb{R}^{\#}$, and this definition can be easily extended to the language of the complete model $\mathbb{R}^{\#}_*$. The following question now arises naturally: *What kind of formulas remain true* ($\mathbb{B}$*-valid*) *during this extension*? We are not going to answer this question in detail because the reader may find them in D. S c o t t [6], where, using the notion of Borel functions, it is proved that all the axioms of the real line (totally ordered complete field) remain true in $\mathbb{R}^{\#}_*$.

4.4. Connection with Dedekind cuts.

Recall that a *Dedekind cut in* $\mathbb{R}$ is a subset a of $\mathbb{Q}$ such that:

(1) $\big(\exists s \in \mathbb{Q}\big)\,[s \in a]$,

(2) $\big(\exists s \in \mathbb{Q}\big)\,[s \notin a]$,

(3) for all $s \in \mathbb{Q}$, $(s \in a) \iff \big(\forall t \in \mathbb{Q}\big)\,[t > s \to t \in a]$.

If we transfer this in $\mathbb{R}^{\#}$, we get that a $\mathbb{B}$-*Dedekind cut* a has the properties (see also [9]):

(1) $\bigvee_{s \in \mathbb{Q}} \|s \in a\| = 1_{\mathbb{B}}$,

(2) $\bigwedge_{s \in \mathbb{Q}} \|s \in a\| = 0_{\mathbb{B}}$,

(3) for all $s \in \mathbb{Q}$, $\|s \in a\| = \bigvee_{\substack{t \in \mathbb{Q}:\\ t > s}} \|t \in a\|$,

and, if we denote $a(s) = \|s \in a\|$, then we have:

(1) $\bigvee_{s \in \mathbb{Q}} a(s) = 1_{\mathbb{B}}$,

(2) $\bigwedge_{s \in \mathbb{Q}} a(s) = 0_{\mathbb{B}}$,

(3) for all $s \in \mathbb{Q}$, $a(s) = \bigvee_{\substack{t \in \mathbb{Q}:\\ t > s}} a(t)$.

We may define that a function $a\colon \mathbb{Q} \to \mathbb{B}$ is a $\mathbb{B}$-*Dedekind cut* if and only if the above relations (1)–(3) hold. However, if we extend a from $\mathbb{Q}$ to $\mathbb{R}$ by putting

$$a(x) := \bigwedge_{\substack{t \in \mathbb{Q}:\\ t > x}} a(t)$$

(this extension is unique by 4.1.4), then (1)–(3) are the same with (5). So to every BDF there corresponds a $\mathbb{B}$-Dedekind cut and conversely, and the completion procedure of 4.3 is analogous to the Dedekind cuts one.

4.5. Completion using Cauchy sequences.

By Theorem 1.10, a sequence $s\colon \mathbb{N} \to \mathbb{R}$ can be extended to $s^{\#}\colon \mathbb{N}^{\#} \to \mathbb{R}^{\#}$, and any function $a\colon \mathbb{N}^{\#} \to \mathbb{R}^{\#}$ is a $\mathbb{B}$-sequence. However, in order to make the completion of $\mathbb{R}^{\#}$, we need usual sequences of the form:

$$s\colon \mathbb{N} \to \mathbb{R}^{\#}, \qquad \text{where} \quad s(n) = f_n .$$

We define that $\lim f_n = \widehat{0}$ if and only if $\|\lim f_n = \widehat{0}\,\| = 1_{\mathbb{B}}$, where:

$$\|\lim f_n = \widehat{0}\,\| := \bigwedge_{\substack{\varepsilon \in \mathbb{R}^{\#}:\\ \varepsilon > 0}} \ \bigvee_{n_0 \in \mathbb{N}} \ \bigwedge_{n \geq n_0} \|\,|f_n| < \varepsilon\,\| .$$

We also define that $\lim f_n = f$ if and only if $\|\lim(f_n - f) = \widehat{0}\,\| = 1_{\mathbb{B}}$.

4.5.1. PROPOSITION. *If the limit of a sequence f_n exists, then it is unique.*

Proof. Let $\| \lim(f_n - f) = \widehat{0} \| = \| \lim(f_n - g) = \widehat{0} \| = 1_{\mathbb{B}}$. Then

$$\| \lim(f_n - f) = \widehat{0} \| \wedge \| \lim(f_n - g) = \widehat{0} \| = 1_{\mathbb{B}} ,$$

and for a given positive ε in $\mathbb{R}^{\#}$ we have:

$$\bigvee_{n_0 \in \mathbb{N}} \bigwedge_{n \geq n_0} (\| |f_n - f| < \varepsilon \| \wedge \| |f_n - g| < \varepsilon \|) = 1_{\mathbb{B}} ,$$

and since

$$\| |f_n - f| < \varepsilon \| \wedge \| |f_n - g| < \varepsilon \| \leq \| |f_n - f| + |f_n - g| < 2\varepsilon \| ,$$

we have

$$\bigvee_{n_0 \in \mathbb{N}} \bigwedge_{n \geq n_0} \| |f - g| < \varepsilon \| = 1_{\mathbb{B}} .$$

But this equation holds for all positive ε, so

$$\| |f = g| = \widehat{0} \| = 1_{\mathbb{B}} \quad \text{or} \quad f = g$$

by Theorem 1.1. (i).

One can easily prove the following:

4.5.2. PROPOSITION.

(1) $\| \lim f_n = f \| \wedge \| \lim g_n = g \| \leq \| \lim(f_n + g_n) = f + g \|$.
(2) $\| \lim f_n = f \| \wedge \| \lim g_n = g \| \leq \| \lim(f_n \cdot g_n) = f \cdot g \|$.
(3) $\| \lim f_n = f \| \leq \| \lim(-f_n) = -f \|$.
(4) $\| \lim f_n = f \| \leq \| \lim |f_n| = |f| \|$.

Next we give the definition of a Cauchy sequence and prove that every converging sequence is Cauchy, but the converse is not true.

4.5.3. DEFINITION. *Let f_n be a sequence of $\mathbb{R}^{\#}$. Then f_n is called a Cauchy sequence if and only if $\| f_n$ is Cauchy$\| = 1_{\mathbb{B}}$, where*

$$\| f_n \textit{ is Cauchy} \| := \bigwedge_{\substack{\varepsilon \in \mathbb{R}^{\#}: \\ \varepsilon > 0}} \bigvee_{n_0 \in \mathbb{N}} \bigwedge_{n,m \geq n_0} \| |f_n - f_m| < \varepsilon \| .$$

4.5.4. PROPOSITION. *If f_n and g_n are Cauchy sequences, then $f_n + g_n$, $f_n \cdot g_n$, etc. are also Cauchy.*

Proof. Easy and analogous to 4.5.2.

4.5.5. PROPOSITION. *If f_n is converging, then it is Cauchy, i.e.*

$$\|\lim f_n = f\| \leq \|f_n \text{ is Cauchy}\|.$$

Proof.

$$\begin{aligned}
&\|\lim f_n = f\| \\
&= \|\lim f_n = f\| \wedge \|\lim f_n = f\| \\
&= \Big(\bigwedge_{\substack{\varepsilon\in\mathbb{R}^{\#}:\\ \varepsilon>0}} \bigvee_{n_0\in\mathbb{N}} \bigwedge_{n\geq n_0} \Big\| |f_n - f| < \frac{\varepsilon}{2} \Big\|\Big) \wedge \Big(\bigwedge_{\substack{\varepsilon\in\mathbb{R}^{\#}:\\ \varepsilon>0}} \bigvee_{n_0\in\mathbb{N}} \bigwedge_{m\geq n_0} \Big\| |f_m - f| < \frac{\varepsilon}{2} \Big\|\Big) \\
&= \bigwedge_{\substack{\varepsilon\in\mathbb{R}^{\#}:\\ \varepsilon>0}} \bigvee_{n_0\in\mathbb{N}} \bigwedge_{n,m\geq n_0} \Big\| |f_n - f| < \frac{\varepsilon}{2} \wedge |f_m - f| < \frac{\varepsilon}{2} \Big\| \\
&\leq \bigwedge_{\substack{\varepsilon\in\mathbb{R}^{\#}:\\ \varepsilon>0}} \bigvee_{n_0\in\mathbb{N}} \bigwedge_{n,m\geq n_0} \| |f_n - f| + |f_m - f| < \varepsilon \| \\
&\leq \bigwedge_{\substack{\varepsilon\in\mathbb{R}^{\#}:\\ \varepsilon>0}} \bigvee_{n_0\in\mathbb{N}} \bigwedge_{n,m\geq n_0} \| |f_n - f_m| < \varepsilon \| \\
&= \|f_n \text{ is Cauchy}\|.
\end{aligned}$$

The converse is not true. It is not hard to construct a counter-example of a Cauchy sequence that is not convergent. Next we define:

$$f_n \approx g_n \overset{\text{def}}{\Longleftrightarrow} \|\lim(f_n - g_n) = \widehat{0}\,\| = 1_{\mathbb{B}}.$$

It is clear that $\approx$ is an equivalence relation on the set $\mathcal{C}$ of all the Cauchy sequences in $\mathbb{R}^{\#}$, and, if $\lim f_n = \lim g_n$, then $f_n \approx g_n$.

Consider the quotient $\mathcal{C}/\approx$ and let

$$\mathbb{R}^{\#}_{*} := \mathcal{C}/\approx.$$

Then $\mathbb{R}^{\#}_{*}$ is the desired completion of $\mathbb{R}^{\#}$ by means of the Cauchy sequences. This gives us a completion of E by the same time. From [3] it is well known that this completion leads to V, the stochastic space which is isomorphic to the set of all equivalence classes of a.e. equal ordinary r.v.'s defined on the probability space $(\Omega, \mathcal{A}, P)$. Similarly, $\mathbb{R}^{\#}_{*}$ is the set of all equivalence classes of $\mathbb{B}$-densities corresponding to r.v.'s in V. This remark shows us that in fact the two completions lead to the same result and this is the reason we use the same symbol for both. The completion by Dedekind cuts has the advantage that we can extend the truth value function in an obvious way, which may not be possible by means of Cauchy sequences.

5. Final remarks

(1) In this paper we have concentrated on the qualitative aspects of randomness without using the quantitative ones, i.e. the probability measure. In order to develop our theory, we only need a σ-ideal $\mathcal{N}$ in the σ-algebra $\mathcal{A}$. In connection with this, we quote a relevant remark by Halmos [2]: "... the mathematical theory of probability consists of the study of Boolean σ-algebras of sets. This is not to say that all σ-algebras are within the domain of probability theory. In general, statements concerning such algebras and the relations between their elements are merely quantitative. Probability theory differs from general theory in that it also studies the quantitative aspects of Boolean algebras." In future publications, the quantitative aspects such as stochastic sets, measures and integrals will be examined.

(2) The theory developed so far has a flavor of Infinitesimal Analysis. We can also define concepts like "internal" or "external" or use the Loeb measures to complete $\mathbb{R}^{\#}$, introducing a kind of blending of Infinitesimal and Boolean methods and deriving the general method of Boolean-valued models into two steps: The Boolean power followed by an infinitesimal step. These ideas will be developed in the near future.

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Received July 20, 1992

Revised October 25, 1992

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