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Dedicated to the memory
of Professor Milan Kolibiar

EFFECT ALGEBRA COUNTEREXAMPLES

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ABSTRACT. Two effect algebra counterexamples are presented. The first shows that the standard effect algebra of operators on a Hilbert space is not a lattice and the second shows that the tensor product of two effect algebras need not exist.

1. Introduction

Effect algebras (or D-posets) have been recently introduced as an axiomatic model for the foundations of quantum mechanics ([2], [9], [10], [17]). The most important effect algebra is the set $\mathcal{E}(H)$ of all self-adjoint operators $\mathbf{A}$ on a Hilbert space H satisfying $\mathbf{O} \leq \mathbf{A} \leq \mathbf{I}$. The partial order on $\mathcal{E}(H)$ is defined by setting $\mathbf{A} \leq \mathbf{B}$ if $\langle \mathbf{A}x, x\rangle \leq \langle \mathbf{B}x, x\rangle$ for all $x \in H$. This effect algebra is the basis for a widely employed approach to quantum mechanics called the operational approach ([3], [4], [12], [16], [20]). In this note, it is shown that if $\dim H \geq 2$, then $\mathcal{E}(H)$ is not a lattice. This substantiates a long held opinion in the folklore of the subject ([2], [3], [4], [10], [16]).

Tensor products of effect algebras are important because they are used to describe coupled physical systems [1], [7], [8], [15] and various results concerning the existence of these tensor products have been obtained ([5], [6], [8], [19], [21], [22]). However, whether the tensor product of two arbitrary effect algebras exists has remained an open question ([5], [9]). This note also presents a counterexample that answers this question negatively.

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An *effect algebra* is a system $(L, 0, 1, \oplus)$, where 0, 1 are distinct elements in L and $\oplus$ is a partial binary operation on L satisfying the following conditions:

(1) If $a \oplus b$ is defined, then $b \oplus a$ is defined and $b \oplus a = a \oplus b$.

(2) If $a \oplus b$ and $(a \oplus b) \oplus c$ are defined, then $b \oplus c$ and $a \oplus (b \oplus c)$ are defined and $a \oplus (b \oplus c) = (a \oplus b) \oplus c$.

(3) For any $a \in L$, there is a unique $b \in L$ such that $a \oplus b$ is defined and $a \oplus b = 1$.

(4) If $a \oplus 1$ is defined, then $a = 0$.

It is easy to check that $\mathscr{E}(H)$ is an effect algebra, where $\mathbf{A} \oplus \mathbf{B}$ is defined for $\mathbf{A}, \mathbf{B} \in \mathscr{E}(H)$ if $\mathbf{A} + \mathbf{B} \le \mathbf{I}$ and, in this case, $\mathbf{A} \oplus \mathbf{B} = \mathbf{A} + \mathbf{B}$.

Let P, Q and R be effect algebras. A mapping $\beta\colon\ P \times Q \to R$ is called a *bimorphism* if the following conditions are satisfied:

(1) $\beta(1,1) = 1$.

(2) If $a \oplus b$ is defined, then $\beta(a,c) \oplus \beta(b,c)$ is defined for all $c \in Q$ and $\beta(a,c) \oplus \beta(b,c) = \beta(a \oplus b, c)$.

(3) If $c \oplus d$ is defined, then $\beta(a,c) \oplus \beta(a,d)$ is defined for all $a \in P$ and $\beta(a,c) \oplus \beta(a,d) = \beta(a, c \oplus d)$.

The precise definition of the tensor product is not needed in this note (cf. [5], [6], [9]). Roughly speaking, the tensor product of P and Q is a pair (T, τ), where T is an effect algebra and $\tau\colon\ P \times Q \to T$ is a bimorphism satisfying a universality condition. We shall show that P and Q need not admit a bimorphism in which case their tensor product does not exist.

2. $\mathscr{E}(H)$ is not a lattice

It has often been stated in the literature that the standard effect algebra $\mathscr{E}(H)$ is not a lattice for $\dim H \ge 2$ ([2], [3], [4], [10], [16]). However, until very recently [18], no explicit counterexample seems to have been given. Instead, the authors have referred to previous sources such as [13], [14]. For example, let $\mathscr{S}(H)$ denote the set of self-adjoint operators on H. It is shown in [13] that for $\mathbf{E}, \mathbf{F} \in \mathscr{E}(H)$, $\mathbf{E} \wedge_{\mathscr{S}} \mathbf{F}$ exists if and only if $\mathbf{E}$ and $\mathbf{F}$ are comparable. However, it cannot be concluded from this that $\mathbf{E} \wedge_{\mathscr{E}} \mathbf{F}$ does not exist when $\mathbf{E}$ and $\mathbf{F}$ are not comparable. For instance, it is also shown in [13] that $\mathbf{E} \wedge_{\mathscr{E}} \mathbf{F}$ exists for any two projection operators $\mathbf{E}$ and $\mathbf{F}$. We now show that $\mathscr{E}(H)$ is not a lattice by characterizing those pairs $\mathbf{A}, \mathbf{B} \in \mathscr{E}(\mathbb{C}^2)$ such that $\mathbf{A} \wedge \mathbf{B}$ exists, where $\mathbf{A} \wedge \mathbf{B}$ means $\mathbf{A} \wedge_{\mathscr{E}} \mathbf{B}$ in the sequel.

Let $\mathscr{E} = \mathscr{E}(\mathbb{C}^2)$, and let $\mathbf{A} \in \mathscr{E}(\mathbb{C}^2)$ with

$$\mathbf{A} = \begin{bmatrix} a & b \\ \bar{b} & c \end{bmatrix}, \qquad a, c \in \mathbb{R}, \quad b \in \mathbb{C}.$$

It is easy to show that $\mathbf{A} \geq \mathbf{O}$ if and only if $a, c \geq 0$ and $ac \geq |b|^2$ ([13]). It follows that $\mathbf{A} \in \mathscr{E}$ if and only if $0 \leq a \leq 1$, $0 \leq c \leq 1$, $ac \geq |b|^2$ and $(1-a)(1-c) \geq |b|^2$. If $\mathbf{B} \in \mathscr{E}$ has the form

$$\mathbf{B} = \begin{bmatrix} d & e \\ \bar{e} & f \end{bmatrix}, \qquad d, f \in \mathbb{R}, \quad e \in \mathbb{C},$$

we conclude that $\mathbf{B} \leq \mathbf{A}$ if and only if $d \leq a$, $f \leq c$ and $(a-d)(c-f) \geq |b-e|^2$. For $a, b \in \mathbb{R}$ we use the notation $a \wedge b = \min(a, b)$.

Lemma 1. *If $\mathbf{A} \in \mathscr{E}$ is a multiple of a 1-dimensional projection, then $\mathbf{A} \wedge \mathbf{B}$ exists for every $\mathbf{B} \in \mathscr{E}$.*

Proof. We can assume that $\mathbf{A}$ is diagonal, so $\mathbf{A}$ and $\mathbf{B}$ have the form

$$\mathbf{A} = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} b & c \\ \bar{c} & d \end{bmatrix}.$$

Suppose that $\mathbf{C} \leq \mathbf{A}, \mathbf{B}$, where $\mathbf{C} \in \mathscr{E}$. Then $\mathbf{C}$ has the form

$$\mathbf{C} = \begin{bmatrix} e & 0 \\ 0 & 0 \end{bmatrix}, \qquad 0 \leq e \leq a, b; \;\; (b-e)d \geq |c|^2.$$

Now define

$$f = \begin{cases} b & \text{if } d = 0, \\ b - |c|^2/d & \text{if } d \neq 0. \end{cases}$$

It follows that

$$\begin{bmatrix} a \wedge f & 0 \\ 0 & 0 \end{bmatrix} = \mathbf{A} \wedge \mathbf{B}.$$

□

Theorem 2. *For $\mathbf{A}, \mathbf{B} \in \mathscr{E}$, $\mathbf{A} \wedge \mathbf{B}$ exists if and only if $\mathbf{A}$ and $\mathbf{B}$ are comparable, or either $\mathbf{A}$ or $\mathbf{B}$ is a multiple of a 1-dimensional projection.*

Proof. Sufficiency follows from Lemma 1. For necessity, suppose that $\mathbf{A}, \mathbf{B} \in \mathscr{E}$ are incomparable and neither is a multiple of a 1-dimensional projection. First assume that $\mathbf{A}$ and $\mathbf{B}$ commute, and hence, they can be simultaneously diagonalized

$$\mathbf{A} = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} c & 0 \\ 0 & d \end{bmatrix},$$

where $0 < a, b, c, d \leq 1$. Since $\mathbf{A}$ and $\mathbf{B}$ are incomparable, we can assume without loss of generality that $a < c$, $b > d$. Let $\varepsilon = ad/(a+b)$ and let $\beta = (c-a) \wedge (b-d)$. Then $\varepsilon > 0$, $\beta > 0$, and we let $\delta = (\beta\varepsilon + \varepsilon^2)^{1/2}$ so $\delta > \varepsilon$. Suppose that $\mathbf{A} \wedge \mathbf{B}$ exists, and let

$$\mathbf{C} = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}.$$

Since $\mathbf{C} \in \mathscr{E}$ and $\mathbf{C} \leq \mathbf{A}, \mathbf{B}$, we have $\mathbf{C} \leq \mathbf{A} \wedge \mathbf{B} \leq \mathbf{A}, \mathbf{B}$. It follows that $\mathbf{A} \wedge \mathbf{B} = \mathbf{C}$. Now let

$$\mathbf{D} = \begin{bmatrix} a - \varepsilon & \delta \\ \delta & d - \varepsilon \end{bmatrix} .$$

To show that $\mathbf{D} \in \mathscr{E}$, we have $0 \leq a - \varepsilon \leq 1$, $0 \leq d - \varepsilon \leq 1$. Also, $ad = (a + b)\varepsilon$, so

$$ad - (a + d)\varepsilon + \varepsilon^2 = (b - d)\varepsilon + \varepsilon^2 \geq \beta\varepsilon + \varepsilon^2 = \delta^2$$

and $(1 - a)(1 - d) \geq (a + b - 2)\varepsilon$, so

$$(1 - a)(1 - d) + (2 - a - d)\varepsilon + \varepsilon^2 \geq (b - d)\varepsilon + \varepsilon^2 \geq \beta\varepsilon + \varepsilon^2 = \delta^2 .$$

Hence, $(1 - a + \varepsilon)(1 - d + \varepsilon) \geq \delta^2$, so $\mathbf{D} \in \mathscr{E}$. Now

$$\mathbf{A} - \mathbf{D} = \begin{bmatrix} \varepsilon & -\delta \\ -\delta & b - d + \varepsilon \end{bmatrix} .$$

Since $\varepsilon \geq 0$, $b - d + \varepsilon \geq 0$ and

$$\varepsilon(b - d + \varepsilon) = (b - d)\varepsilon + \varepsilon^2 \geq \beta\varepsilon + \varepsilon^2 = \delta^2 ,$$

we have $\mathbf{A} - \mathbf{D} \geq \mathbf{O}$, so $\mathbf{D} \leq \mathbf{A}$. Moreover,

$$\mathbf{B} - \mathbf{D} = \begin{bmatrix} c - a + \varepsilon & -\delta \\ -\delta & \varepsilon \end{bmatrix} .$$

Since $c - a + \varepsilon$, $\varepsilon \geq 0$ and

$$(c - a + \varepsilon)\varepsilon = (c - a)\varepsilon + \varepsilon^2 \geq \beta\varepsilon + \varepsilon^2 = \delta^2 ,$$

we have $\mathbf{B} - \mathbf{D} \geq 0$, so $\mathbf{D} \leq \mathbf{B}$. We conclude that $\mathbf{D} \leq \mathbf{C}$. But

$$\mathbf{C} - \mathbf{D} = \begin{bmatrix} \varepsilon & -\delta \\ -\delta & \varepsilon \end{bmatrix}$$

and $\varepsilon^2 < \delta^2$, which is a contradiction.

Now consider the case in which $\mathbf{A}$ and $\mathbf{B}$ do not necessarily commute. Assume that $\mathbf{C} = \mathbf{A} \wedge \mathbf{B}$ exists and, without loss of generality, that $\mathbf{C}$ is diagonal. Then $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ have the forms

$$\mathbf{A} = \begin{bmatrix} a & b \\ \overline{b} & c \end{bmatrix} , \qquad \mathbf{B} = \begin{bmatrix} d & e \\ \overline{e} & f \end{bmatrix} , \qquad \mathbf{C} = \begin{bmatrix} g & 0 \\ 0 & h \end{bmatrix} .$$

By assumption, we have $\mathbf{C} \neq \mathbf{A}, \mathbf{B}$, $ac > |b|^2$ and $df > |e|^2$. Since $\mathbf{C} \leq \mathbf{A}, \mathbf{B}$, we have $g \leq a$, $h \leq c$, $(a - g)(c - h) \geq |b|^2$, $g \leq d$, $h \leq f$, $(d - g)(f - h) \geq |e|^2$. Now at most one of the following equalities holds: $g = a$, $h = c$, $g = d$, $h = f$. Indeed, assume that $g = a$, so $b = 0$. Then, if $g = d$, we have $e = 0$, so $\mathbf{A}$ and $\mathbf{B}$ are comparable. If $h = c$, then $\mathbf{C} = \mathbf{A}$. If $h = f$, then $e = 0$, so $\mathbf{A}$ and $\mathbf{B}$ are diagonal, and this case was treated earlier. The other cases are similar. Hence, we can assume without loss of generality that $h < c$ and $h < f$. If $g \neq 0$, let

$$0 < \delta < \min\left(1 - h, \frac{g(c-h)}{a}, \frac{g(f-h)}{d}\right),$$

and let

$$\mathbf{D} = \begin{bmatrix} 0 & 0 \\ 0 & h+\delta \end{bmatrix}.$$

Then $\mathbf{D} \not\leq \mathbf{C}$. Moreover, $\mathbf{D} \leq \mathbf{A}$ because

$$h + \delta \leq h + g(c-h)/a \leq h + c - h = c$$

and

$$a(c - h - \delta) = (a-g)(c-h) + g(c-h) - a\delta \geq |b|^2 + g(c-h) - a\delta \geq |b|^2 .$$

Also, $\mathbf{D} \leq \mathbf{B}$ because

$$h + \delta \leq h + g(f-h)/d \leq h + f - h = f$$

and

$$d(f - h - \delta) = (d-g)(f-h) + g(f-h) - d\delta \geq |e|^2 + g(f-h) - d\delta \geq |e|^2 .$$

If $g = 0$, let

$$0 < \varepsilon < \min\left(a - \frac{|b|^2}{c}, d - \frac{|e|^2}{f}\right),$$

and let

$$\mathbf{D} = \begin{bmatrix} \varepsilon & 0 \\ 0 & 0 \end{bmatrix}.$$

Then $\mathbf{D} \not\leq \mathbf{C}$. Moreover, $\mathbf{D} \leq \mathbf{A}$ because

$$\varepsilon \leq a - |b|^2/c \leq a$$

and

$$(a - \varepsilon)c = ac - \varepsilon c \geq |b|^2 ,$$

also, $\mathbf{D} \leq \mathbf{B}$ because

$$\varepsilon \leq d - |e|^2/f \leq f$$

and

$$(d - \varepsilon)f = df - \varepsilon f \geq |e|^2 .$$

$\square$

Theorem 2 also holds for $\mathscr{E}(\mathbb{R}^2)$ with essentially the same proof. We conclude that there are many pairs $\mathbf{A}, \mathbf{B}$ in $\mathscr{E}(\mathbb{C}^2)$ and $\mathscr{E}(\mathbb{R}^2)$ such that $\mathbf{A} \wedge \mathbf{B}$ does not exist. A simple concrete example is

$$\mathbf{A} = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 3/4 & 0 \\ 0 & 1/4 \end{bmatrix}.$$

Related results are given in [18]. For example, letting $\mathbf{A}' = \mathbf{I} - \mathbf{A}$, it is shown in [18] that there exists an $\mathbf{A} \in \mathscr{E}(\mathbb{C}^2)$ such that $\mathbf{A} \wedge \mathbf{A}'$ does not exist. But the verification of such counterexamples follow directly from Theorem 2. For instance, letting $\mathbf{B}$ be defined as above, we see that $\mathbf{B}$ and $\mathbf{B}'$ are incomparable and neither is a multiple of a 1-dimensional projection. Applying Theorem 2, we conclude that $\mathbf{B} \wedge \mathbf{B}'$ does not exist. Now let H be a real or complex Hilbert space with $\dim H \geq 2$, and let ϕ and ψ be orthogonal unit vectors in H. Define $\mathbf{A}, \mathbf{B} \in \mathscr{E}(H)$ by $\mathbf{A}\phi = \frac{1}{2}\phi$, $\mathbf{A}\psi = \frac{1}{2}\psi$, $\mathbf{B}\phi = \frac{3}{4}\phi$, $\mathbf{B}\psi = \frac{1}{4}\psi$, and $\mathbf{A}\gamma = \mathbf{B}\gamma = \boldsymbol{O}$ for all γ in the orthogonal complement of the span of $\{\phi, \psi\}$. It follows from Theorem 2 that $\mathbf{A} \wedge \mathbf{B}$ does not exist, so $\mathscr{E}(H)$ is not a lattice. Theorem 2 might be useful in solving the following.

OPEN PROBLEM. Characterize the pairs of elements $\mathbf{A}, \mathbf{B} \in \mathscr{E}(H)$ such that $\mathbf{A} \wedge \mathbf{B}$ exists.

Applying Theorem 2 and De Morgan's laws, we conclude that for $\mathbf{A}, \mathbf{B} \in \mathscr{E}(\mathbb{C}^2)$, $\mathbf{A} \vee \mathbf{B}$ exists if and only if $\mathbf{A}$ and $\mathbf{B}$ are comparable, or either $\mathbf{I} - \mathbf{A}$ or $\mathbf{I} - \mathbf{B}$ is a multiple of a 1-dimensional projection. Thus, $\mathbf{A} \wedge \mathbf{B}$ can exist while $\mathbf{A} \vee \mathbf{B}$ does not exist and vice versa. For example, letting

$$\mathbf{A} = \begin{bmatrix} 1/2 & 0 \\ 0 & 0 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 0 & 0 \\ 0 & 1/2 \end{bmatrix}$$

we conclude that $\mathbf{A} \wedge \mathbf{B}$ exists, but $\mathbf{A} \vee \mathbf{B}$ does not exist. Moreover, even if $\mathbf{A}$ and $\mathbf{B}$ are incomparable, it is possible that $\mathbf{A} \wedge \mathbf{B}$ and $\mathbf{A} \vee \mathbf{B}$ both exist. For example, this happens for

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}, \qquad a, b \in \mathbb{R}, \quad a \neq 1, \quad b \neq 0.$$

Finally, we have the converse of Lemma 1.

COROLLARY 3. *For $\mathbf{A} \in \mathscr{E}(\mathbb{C}^2)$, $\mathbf{A} \wedge \mathbf{B}$ exists for every $\mathbf{B} \in \mathscr{E}(\mathbb{C}^2)$ if and only if $\mathbf{A}$ equals $\mathbf{I}$ or a multiple of a 1-dimensional projection.*

3. Nonexistence of tensor products

Since $\oplus$ is an associative partial operation, we can write $(a \oplus b) \oplus c$ as $a \oplus b \oplus c$ when this expression is defined. It follows by induction that we do not need parentheses in expressions of the form $a_1 \oplus \cdots \ominus a_n$. If $a_1 \oplus \cdots \oplus a_n$ is defined, where $a_i = a$, $i = 1, \dots, n$, we denote this element by na. An *n-chain generated by* a is an effect algebra with elements $0, a, 2a, \dots, na$, where $na = 1$.

Let P_1 be the horizontal sum of three 4-chains generated by a, b and c, respectively, with the additional requirement that $a \oplus b \oplus c = 1$. The $\oplus$ table for P_1 is displayed below. In this table, a dash indicates that the sum is not defined and the trivial sums involving a 0 or 1 are not displayed. It is not hard to show that P_1 is indeed an effect algebra.

$\oplus$	a	$2a$	$3a$	b	$2b$	$3b$	c	$2c$	$3c$
a	$2a$	$3a$	1	$3c$	–	–	$3b$	–	–
$2a$	$3a$	1	–	–	–	–	–	–	–
$3a$	1	–	–	–	–	–	–	–	–
b	$3c$	–	–	$2b$	$3b$	1	$3a$	–	–
$2b$	–	–	–	$3b$	1	–	–	–	–
$3b$	–	–	–	1	–	–	–	–	–
c	$3b$	–	–	$3a$	–	–	$2c$	$3c$	1
$2c$	–	–	–	–	–	–	$3c$	1	–
$3c$	–	–	–	–	–	–	1	–	–

THEOREM 4. *If Q is the 4-chain generated by d, then P_1 and Q do not admit a bimorphism. Hence, the tensor product of P_1 and Q does not exist.*

Proof. Suppose a bimorphism $\beta\colon\ P_1 \times Q \to R$ exists. Then

$$\begin{aligned} 1 = \beta(1,1) = \beta(1,4d) &= 4\beta(1,d) = 4\beta(a \oplus b \oplus c, d) \\ &= 4\big[\beta(a,d) \oplus \beta(b,d) \oplus \beta(c,d)\big] \\ &= 4\beta(a,d) \oplus 4\beta(b,d) \oplus 4\beta(c,d) \\ &= \beta(4a,d) \oplus \beta(4b,d) \oplus \beta(4c,d) = 3\beta(1,d)\,. \end{aligned}$$

Since $3\beta(1,d) \oplus \beta(1,d) = 1$, we conclude that $\beta(1,d) \oplus 1$ is defined. Hence, $\beta(1,d) = 0$. It follows that $0 = 3\beta(1,d) = 1$, which is a contradiction. □

It is clear that there are many counterexamples of this type. There are also counterexamples in which one of the effect algebras is an orthomodular poset or lattice. For example, let W be the 3×4 window ([11]), and let Q be a 4-chain. An argument similar to that given in Theorem 4 shows that W and Q do not admit a bimorphism. Moreover, if we replace d by $a \in P_1$ in the proof of Theorem 4, then we conclude that the tensor product of P_1 and P_1 does not exist. Nevertheless, we conjecture that the tensor product of W and W does exist. If this conjecture is true, it would give an example of a tensor product in which neither of its components possesses a state. A *state* on an effect algebra P is a map $\phi\colon P \to [0,1] \subseteq \mathbb{R}$ such that $\phi(1) = 1$, and if $a \oplus b$ is define, then $\phi(a)+\phi(b) \le 1$ and $\phi(a \oplus b) = \phi(a)+\phi(b)$. Such an example would be of interest because it is known that if two effect algebras P and Q each possess a state, then their tensor product exists ([6]).

REFERENCES

[1] AERTS, D.—DAUBECHIES, I.: *Physical justification of using tensor products to describe physical systems as one joint system*, Helv. Phys. Acta **51** (1978), 661–675.

[2] CATTANEO, G.—NISTICO, G.: *Brouwer-Zadeh posets and three-valued Lukasiewicz posets*, Internat. J. Fuzzy. Sets Sys. **33** (1989), 165–190.

[3] DAVIES, E. B.: *Quantum Theory of Open Systems*, Academic Press, London, 1976.

[4] DAVIES, E. B.—LEWIS, J. T.: *An operational approach to quantum probability*, Comm. Math. Phys. **17** (1970), 239–260.

[5] DVUREČENSKIJ, A.: *Tensor product of difference posets*, Trans. Amer. Math. Soc. **347** (1995), 1043–1057.

[6] DVUREČENSKIJ, A.—PULMANNOVÁ, S.: *Tensor products of D-posets and D-test spaces*, Rep. Math. Phys. **34** (1994), 4251–4275.

[7] FOULIS, D.: *Coupled physical systems*, Found. Phys. **19** (1989), 905–922.

[8] FOULIS, D.—BENNETT, M. K.: *Tensor products of orthoalgebras*, Order **10** (1993), 271–282.

[9] FOULIS, D.—BENNETT, M. K.: *Effect algebras and unsharp quantum logics*, Found. Phys. **24** (1994), 1325–1346.

[10] GIUNTINI, R.—GREULING, H.: *Toward a formal language for unsharp properties*, Found. Phys. **19** (1989), 931–945.

[11] GREECHIE, R.: *Orthomodular lattices admitting no states*, J. Combin. Theory **10** (1971), 119–132.

[12] HOLEVO, A. S.: *Probabilistic and Statistical Aspects of Quantum Theory*, North-Holland, Amsterdam, 1982.

[13] KADISON, R.: *Order properties of bounded self-adjoint operators*, Proc. Amer. Math. Soc. **34** (1951), 505–510.

[14] KAKUTANI, S.: *Concrete representation of abstract (M)-spaces*, Ann. of Math. **42** (1941), 994–1024.

[15] KLÄY, M.—RANDALL, C.—FOULIS, D.: *Tensor products and probability weights*, Internat. J. Theoret. Phys. **26** (1987), 199–216.

[16] KRAUS, K.: *States, Effects, and Operations*, Springer-Verlag, Berlin, 1983.

[17] KÔPKA, F.—CHOVANEC, F.: *D-posets*, Math. Slovaca **44** (1994), 21–34.

[18] LAHTI, P. J.—MACZYNSKI, M. J.: *Partial order of quantum effects*, J. Math. Phys. **36** (1995), 1673–1680.

[19] LOCK, R.: *The tensor product of generalized sample spaces which admit a unital set of dispersion-free weights*, Found. Phys. **20** (1990), 477–498.

[20] LUDWIG, G.: *Foundations of Quantum Mechanics I*, Springer-Verlag, Berlin, 1983.

[21] PULMANNOVÁ, S.: *Tensor products of quantum logics*, J. Math. Phys. **26** (1985), 1–5.

[22] WILCE, A.: *Tensor products of frame manuals*, Internat. J. Theoret. Phys. **29** (1990), 805–814.

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