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THE MOORE-PENROSE INVERSE OF A PARTITIONED MORPHISM IN AN ADDITIVE CATEGORY

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ABSTRACT. R. E. Cline (1969) published a formula for a Moore-Penrose inverse of a partitioned matrix. R. Börger (1987) and R. Puystjens and D. W. Robinson (1984) generalized the Moore-Penrose inverse of a matrix to the Moore-Penrose inverse of a morphism. In this paper the notion of partitioned morphism is introduced and a formula for its Moore-Penrose inverse is derived.

Further, the problems of Moore-Penrose inverses of partitioned matrices in the set of all matrices with complex entries with an arbitrary involution are solved here. Using the results of this paper, we get an algorithm for the computation of the Moore-Penrose inverse of a matrix without using the full-rank factorization in the set of matrices with an arbitrary involution.

1. Introduction

The notion of the Moore-Penrose inverse is well-known in matrix theory. It is a generalization for all matrices of the inverse of nonsingular matrices. Let us mention that the Moore-Penrose inverse of a complex matrix A is a complex matrix X satisfying the following axioms:

$$\begin{aligned} AXA &= A\,, &\qquad (AX)^* &= AX\,,\\ XAX &= X\,, &\qquad (XA)^* &= XA\,, \end{aligned}$$

where * is the conjugate transpose.

It has been proved that the Moore-Penrose inverse exists for an arbitrary matrix and that it is unique. When proving its existence full-rank factorization

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is usually used. In 1960, Greville [7] published a recurrent algorithm that proves the existence of the Moore-Penrose inverse without using the full-rank factorization. In 1969, Cline [6] published a formula for the Moore-Penrose inverse of a partitioned matrix in an article extending Greville's work. In the article [3] Börger occupies himself with this notion in a general category with an involution and generalizes the notion of the Moore-Penrose inverse of a matrix to the notion of the Moore-Penrose inverse of a morphism. Some others (e.g.. Puystjens and Robinson [10]) investigate the Moore-Penrose inverse of morphisms in an additive category with an involution. which seems to be more appropriate. In this paper, which explores further the problems considered in the articles mentioned above, the notion of partitioned morphism is introduced and a formula for its Moore-Penrose inverse is derived. L. Skula [11] described all the involutions on the set of all matrices with complex entries. The problems of Moore-Penrose inverses of partitioned matrix in this set of matrices with an arbitrary involution is solved here. Finally, it can be seen that an algorithm for computation of the Moore-Penrose inverse of a matrix without using the full-rank factorization on this set of all matrices with an arbitrary involution follows immediately from the results presented in this paper.

2. Some basic notions of category theory

In this section, we would like to remind the reader of some basic notions of category theory. These and other relevant material can be found in [8], [1] or [4].

Category:
A category $\mathcal{K}$ is a class $\operatorname{ob}\mathcal{K}$ whose elements are called objects together with a class $\mathcal{M}$ which is a disjoint union of the form

$$\mathcal{M} = \bigcup_{(A,B)\in \operatorname{ob}\mathcal{K}\times\operatorname{ob}\mathcal{K}} \mathcal{K}(A,B)\,,$$

where $\mathcal{K}(A,B)$ is a set. The elements of $\mathcal{K}(A,B)$ are called morphisms from A to B and for $f \in \mathcal{K}(A,B)$ we write $f\colon A \to B$.

Moreover, the morphisms have the following properties:

1. $(\forall A \in \operatorname{ob}\mathcal{K})(\exists\,\text{morphism } \operatorname{id}_A\colon A \to A)$,
2. $(\forall f\colon A \to B,\ g\colon B \to C)(\exists\, fg\colon A \to C)$,
3. $(\forall f\colon A \to B)(\operatorname{id}_A f = f,\ f\operatorname{id}_B = f)$,
4. $(\forall f\colon A \to B,\ g\colon B \to C,\ h\colon C \to D)((fg)h = f(gh))$.

Note. Let $A \in \operatorname{ob}\mathcal{K}$. If f is a morphism to A and g is a morphism from A, then we say that the morphism g is a *conformable morphism* to f.

Isomorphism:
A morphism $f\colon A \to B$ is called an isomorphism, if there exists a morphism $g\colon B \to A$ such that $fg = \mathrm{id}_A$ and $gf = \mathrm{id}_B$. The morphism g is denoted as f^{-1}.

Idempotent morphism:
If $f\colon A \to A$ and $ff = f$, then the morphism f is called idempotent.

Sum of objects:
Let A, B be objects in a category $\mathcal{K}$. A sum of A and B is ordered triad $(A+B, e_A, e_B)$, where $A+B \in \operatorname{ob}\mathcal{K}$, $e_A\colon A \to A+B$ and $e_B\colon B \to A+B$ satisfy: for any object X and arbitrary morphisms $f\colon A \to X$ and $g\colon B \to X$ there exists a unique morphism $h\colon A+B \to X$ such that $f = e_A h$ and $g = e_B h$.

Note. The sum of objects need not exist, but if it exists, then it is unique up to an isomorphism.

Additive category:
Let each set $\big(\mathcal{K}(A,B), +\big)$ be an abelian group, subject to the following conditions:

1. If $f, g \in \mathcal{K}(A,B)$, $x \in \mathcal{K}(X,A)$ and $y \in \mathcal{K}(B,Y)$, then

$$x(f+g) = xf + xg\,, \tag{1}$$
$$(f+g)y = fy + gy\,. \tag{2}$$

2. The zero element 0_{AB} of $\mathcal{K}(A,B)$ satisfies

$$x0_{AB} = 0_{XB}\,, \tag{3}$$
$$0_{AB}y = 0_{AY} \tag{4}$$

and it is called *zero morphism.*

Then we call $\mathcal{K}$ an additive category.

Involution:
An involution in a category $\mathcal{K}$ is a unary operation * which with each $f\colon A \to B$ associates $f^*\colon B \to A$ such that

$$(f^*)^* = f\,, \tag{5}$$
$$(fg)^* = g^* f^* \tag{6}$$

for any conformable morphism g.

If the category $\mathcal{K}$ is additive, then in addition

$$(f+h)^* = f^* + h^* \tag{7}$$

holds for all $h\colon A \to B$.

Note. Clearly, for all $A, B \in \operatorname{ob}\mathcal{K}$ the identities $\mathrm{id}_A^* = \mathrm{id}_A$ and $0_{AB}^* = 0_{BA}$ hold.

Symmetric morphism:
A morphism $f\colon A \to A$ is called symmetric with respect to the involution * provided that $f = f^*$.

EXAMPLE. Category $\mathcal{MAT}$.
The objects of this category are positive integers and morphisms from m to n are complex matrices of size $m \times n$. The composition of morphisms is ordinary multiplication of matrices. In $\mathcal{MAT}$ the morphism id_m is identity matrix I_m of order m.

Isomorphisms are nonsingular matrices and idempotent morphisms are idempotent matrices.

The sum of objects m and n can be the ordered triad $\big(m+n, [\,R_m \quad 0_{m,n}\,], [\,0_{n,m} \quad S_n\,]\big)$, where $0_{m,n}$ is the zero matrix of size $m \times n$ and R_m and S_n are any nonsingular matrices of orders m and n, respectively.

The category $\mathcal{MAT}$ can be considered as an additive category with ordinary addition of matrices. Zero morphisms are the zero matrices.

Conjugate transpose forms an involution in this category.

3. Moore-Penrose inverse of a morphism

The following definition, Lemma 3.1 and Lemma 3.2 were published in [3].

Moore-Penrose inverse of a morphism:
Let $\mathcal{K}$ be a category and let $f\colon A \to B$ be a morphism. A morphism $g\colon B \to A$ is called the Moore-Penrose inverse (abbreviated to MP-inverse) of the morphism f with respect to the involution * if the following conditions hold:

$$fgf = f\,, \tag{8}$$
$$gfg = g\,, \tag{9}$$
$$(fg)^* = fg\,, \tag{10}$$
$$(gf)^* = gf\,. \tag{11}$$

LEMMA 3.1. *If an MP-inverse of morphism f exists, then it is unique.*

The MP-inverse of morphism f is denoted by f^+.

If the MP-inverse of morphism f exists, then f is called *MP-invertible.*

Lemma 3.2.

1. *If f is an isomorphism, then f is MP-invertible and $f^+ = f^{-1}$.*
2. *If f is MP-invertible, then f^*, f^+ are also MP-invertible, and $(f^+)^+ = f$, $(f^+)^* = (f^*)^+$.*

Lemma 3.3. *Let $\mathcal{K}$ be an additive category.*

1. *For each $A, B \in \operatorname{ob}\mathcal{K}$ the morphism 0_{AB} is MP-invertible and $\left(0_{AB}\right)^+ = 0_{BA}$.*
2. *If $f\colon A \to B$, $g\colon B \to C$ are MP-invertible and $fg = 0_{AC}$, then $(fg)^+ = g^+f^+ = 0_{CA}$.*
3. *If $f\colon A \to B$ is an arbitrary MP-invertible morphism, then $\mathrm{id}_A - ff^+$ and $\mathrm{id}_B - f^+f$ are symmetric and idempotent morphisms.*

Proof.

1. We get this statement from (8), ... ,(11).
2. Using (6),(8), ... ,(11) and Lemma 3.2 we get

$$\begin{aligned} g^+f^+ &= g^+gg^+f^+ff^+ \\ &= g^+(gg^+)^*(f^+f)^*f^+ = g^+(g^+)^*g^*f^*(f^+)^*f^+ = g^+(g^+)^*(fg)^*(f^+)^*f^+ \\ &= 0_{CA} = (fg)^+ . \end{aligned}$$

3. This statement follows easily from the definitions of symmetric and idempotent morphism. □

4. Partitioned morphism and its Moore-Penrose inverse

Definition 4.1. Partitioned morphism. Let A, B be objects in the category $\mathcal{K}$, let $(A+B, e_A, e_B)$ be their sum and let the morphisms e_A, e_B have the following properties

$$e_Ae_A^* = \mathrm{id}_A , \qquad e_Be_B^* = \mathrm{id}_B , \tag{12}$$

$$e_A^*e_A + e_B^*e_B = \mathrm{id}_{A+B} . \tag{13}$$

Suppose that $f\colon A \to X$ and $g\colon B \to X$ are morphisms in the category $\mathcal{K}$. Then we call the morphism $h\colon A+B \to X$ with the property $f = e_Ah$ and $g = e_Bh$ the partitioned morphism of f and g with respect to the sum $(A+B, e_A, e_B)$.

Let us denote $h = \begin{pmatrix} f \\ g \end{pmatrix}$.

For $f\colon X \to A$ and $g\colon X \to B$ we define $(f \quad g) = \begin{pmatrix} f^* \\ g^* \end{pmatrix}^*$.

Note. $\begin{pmatrix} f \\ g \end{pmatrix} : A+B \to X$, $\quad (f \quad g) : X \to A+B$.

EXAMPLE 4.1. The partitioned morphism in category $\mathcal{MAT}$.
Let $(m+n, [R_m \quad 0_{m,n}], [0_{n,m} \quad S_n])$ be a sum of objects in category $\mathcal{MAT}$. It can easily be seen that the relations (12) and (13) hold if and only if R_m and S_n are unitary matrices. Let $x \in \operatorname{ob}\mathcal{MAT}$ (x is a positive integer) and let $f\colon m \to x$ and $g\colon n \to x$ be morphisms in $\mathcal{MAT}$ (f is a matrix A of size $m \times x$ and g is a matrix B of size $n \times x$). In this case, the partitioned morphism $\begin{pmatrix} f \\ g \end{pmatrix}$ is the partitioned matrix $\begin{bmatrix} R_m^* A \\ S_n^* B \end{bmatrix}$.

LEMMA 4.1. *Let the sums $(A+B, e_A, e_B)$ and $(C+D, e_C, e_D)$ exist and let all partitioned morphisms be given with respect to either $(A+B, e_A, e_B)$ or $(C+D, e_C, e_D)$.*

1. *If $f\colon A \to X$, $g\colon B \to X$, $k\colon X \to Y$, then*

$$\begin{pmatrix} f \\ g \end{pmatrix} k = \begin{pmatrix} fk \\ gk \end{pmatrix} : A+B \to Y. \tag{14}$$

2. *If $x\colon Y \to A$, $y\colon Y \to B$, $k\colon X \to Y$, then*

$$k(x \quad y) = (kx \quad ky) : X \to A+B. \tag{15}$$

3. *If $f\colon A \to X$, $g\colon B \to X$, $x\colon Y \to A$, $y\colon Y \to B$, then*

$$(x \quad y)\begin{pmatrix} f \\ g \end{pmatrix} = xf + yg\colon Y \to X. \tag{16}$$

4. *If $f\colon A \to X$, $g\colon B \to X$, $x\colon X \to C$, $y\colon X \to D$, then*

$$\begin{pmatrix} f \\ g \end{pmatrix}(x \quad y) = \begin{pmatrix} (fx & fy) \\ (gx & gy) \end{pmatrix} = \left(\begin{pmatrix} fx \\ gx \end{pmatrix} \begin{pmatrix} fy \\ gy \end{pmatrix}\right).$$

Proof.

1. From the definition of partitioned morphism it follows that $fk = e_A \begin{pmatrix} f \\ g \end{pmatrix} k$ and $gk = e_B \begin{pmatrix} f \\ g \end{pmatrix} k$.

From the same definition, there exists one and only one morphism $\begin{pmatrix} fk \\ gk \end{pmatrix}$: $A+B \to Y$ such that $e_A \begin{pmatrix} fk \\ gk \end{pmatrix} = fk$, $e_B \begin{pmatrix} fk \\ gk \end{pmatrix} = gk$.

2. Similarly to 1.

3. From the definition of the partitioned morphism we have $f = e_A \begin{pmatrix} f \\ g \end{pmatrix}$, $g = e_B \begin{pmatrix} f \\ g \end{pmatrix}$ and $(x \quad y)e_A^* = x$, $(x \quad y)e_B^* = y$.

So

$$xf + yg = (x \;\vdots\; y)e_A^* e_A \begin{pmatrix} f \\ g \end{pmatrix} + (x \quad y)\, e_B^* e_B \begin{pmatrix} f \\ g \end{pmatrix}$$
$$= (x \quad y)(e_A^* e_A + e_B^* e_B) \begin{pmatrix} f \\ g \end{pmatrix} = (x \quad y) \begin{pmatrix} f \\ g \end{pmatrix} \qquad \text{according (13).}$$

4. This assertion follows from part 1. and 2. □

Let us denote

$$\begin{pmatrix} f \\ g \end{pmatrix} (x \quad y) = \begin{pmatrix} fx & fy \\ gx & gy \end{pmatrix} : A + B \to C + D\,. \tag{17}$$

From the definition of partitioned morphism it is easy to see that

$$\begin{pmatrix} fx & fy \\ gx & gy \end{pmatrix}^* = \begin{pmatrix} (fx)^* & (gx)^* \\ (fy)^* & (gy)^* \end{pmatrix}. \tag{18}$$

ASSUMPTION. Let $\mathcal{K}$ be an additive category with involution * and $A \in \operatorname{ob} \mathcal{K}$.

For the following theorem let us suppose that if $f\colon A \to A$ is an arbitrary morphism, then $\mathrm{id}_A + ff^*$ is an isomorphism.

THEOREM 4.1. GENERALIZED CLINE'S THEOREM. *Let the above assumption be satisfied. Further, let $u\colon X \to A$, $v\colon X \to B$ and let u be MP-invertible. Let us denote*

$$c = (\mathrm{id}_X - uu^+)v\,, \qquad c\colon X \to B\,, \tag{19}$$

and assume c is MP-invertible. Furthermore, let us denote

$$k = \left(\mathrm{id}_B + (\mathrm{id}_B - c^+c)v^*(u^+)^*u^+v(\mathrm{id}_B - c^+c)\right)^{-1}, \qquad k\colon B \to B\,. \tag{20}$$

If the sum $(A + B, e_A, e_B)$ exists and if the morphism

$$f = (u \quad v)\,, \qquad f\colon X \to A + B\,,$$

is a partitioned morphism of u and v with respect to $(A + B, e_A, e_B)$, then it is MP-invertible and

$$f^+ = \begin{pmatrix} u^+ - u^+vc^+ - u^+v(\mathrm{id}_B - c^+c)kv^*(u^+)^*u^+(\mathrm{id}_X - vc^+) \\ c^+ + (\mathrm{id}_B - c^+c)kv^*(u^+)^*u^+(\mathrm{id}_X - vc^+) \end{pmatrix}$$

with respect to $(A + B, e_A, e_B)$.

Note. Since the morphism $\mathrm{id}_B - c^+c$ is symmetric (see Lemma 3.3), the morphism k exists (see Assumption).

Proof. Let us denote the following morphism as f_0:

$$f_0 = \begin{pmatrix} u^+ - u^+vc^+ - u^+v(\mathrm{id}_B - c^+c)q \\ c^+ + (\mathrm{id}_B - c^+c)q \end{pmatrix}, \tag{21}$$

where $q\colon B \to X$ is a morphism. We would like to choose the morphism q in such a way that the morphism f_0 be the MP-inverse of f. Let us verify the conditions (8), ... ,(11).

1. The condition (10): $(ff_0)^* = ff_0$.
Using (16) we get

$$\begin{aligned} ff_0 &= uu^+ - uu^+vc^+ - uu^+v(\mathrm{id}_B - c^+c)q + vc^+ + v(\mathrm{id}_B - c^+c)q \\ &= uu^+ - uu^+v\big(c^+ + (\mathrm{id}_B - c^+c)q\big) + v\big(c^+ + (\mathrm{id}_B - c^+c)q\big) \\ &= uu^+ + (v - uu^+v)\big(c^+ + (\mathrm{id}_B - c^+c)q\big) \\ &= uu^+ + c\big(c^+ + (\mathrm{id}_B - c^+c)q\big) \\ &= uu^+ + cc^+ . \end{aligned}$$

Clearly, the morphism $uu^+ + cc^+$ is a symmetric morphism according to (7) and (10).

The condition (10) is satisfied for arbitrary $q\colon B \to X$.

2. The condition (8): $ff_0f = f$.
By the result above and (15) we have $ff_0f = \big((uu^+ + cc^+)u \;\; (uu^+ + cc^+)v\big)$.

Moreover,

$$u^+c = u^+(\mathrm{id}_X - uu^+)v = 0_{AB}. \tag{22}$$

According to the Lemma 3.3, part 2, we have that

$$c^+u = 0_{BA}. \tag{23}$$

Further, by (19), $cc^+v = (cc^+)^*v = c^{+*}c^*v = c^{+*}v^*(\mathrm{id}_X - uu^+)^*v$.

Since $(\mathrm{id}_X - uu^+)$ is symmetric and idempotent (see Lemma 3.3, part 3), we get

$$\begin{aligned} c^{+*}v^*(\mathrm{id}_X - uu^+)^*v &= c^{+*}v^*(\mathrm{id}_X - uu^+)^*(\mathrm{id}_X - uu^+)v \\ &= c^{+*}c^*c = cc^+c = c. \end{aligned}$$

So $cc^+v = c$ and by (9)

$$c^+v = c^+c. \tag{24}$$

Using (19) we get $uu^+v + cc^+v = uu^+v + c = uu^+v + (\mathrm{id}_X - uu^+)v = v$.

So, the condition (8) holds for each $q\colon B \to X$.

3. The condition (9): $f_0ff_0 = f_0$.
We use the formula $f_0ff_0 = f_0(uu^+ + cc^+)$. Then

$$f_0ff_0 = \begin{pmatrix} u^+ - u^+vc^+ - u^+v(\mathrm{id}_B - c^+c)q(uu^+ + cc^+) \\ c^+ + (\mathrm{id}_B - c^+c)q(uu^+ + cc^+) \end{pmatrix}$$

according to (9), (22) and (23). Therefore, for (9) to hold we must have

$$q(uu^+ + cc^+) = q\,, \tag{25}$$

which is true only in some cases (see below).

4. The condition (11): $(f_0 f)^* = f_0 f$.

Let us make the composition of morphisms f_0 and f. Using formulas (23) and (24) we get

$$f_0 f = \begin{pmatrix} u^+u - u^+v(\mathrm{id}_B - c^+c)qu & u^+v(\mathrm{id}_B - c^+c)(\mathrm{id}_B - qv) \\ (\mathrm{id}_B - c^+c)qu & c^+c + (\mathrm{id}_B - c^+c)qv \end{pmatrix}.$$

For the formula $(f_0 f)^* = f_0 f$ to hold, the relation

$$\big(u^+v(\mathrm{id}_B - c^+c)(\mathrm{id}_B - qv)\big)^* = (\mathrm{id}_B - c^+c)qu \tag{26}$$

must be satisfied according to (18) and the morphisms

$$u^+v(\mathrm{id}_B - c^+c)qu \qquad \text{and} \qquad (\mathrm{id}_B - c^+c)qv$$

must be symmetric.

Let us denote

$$q = kv^*(u^+)^*u^+(\mathrm{id}_X - vc^+)\,, \tag{27}$$

where k is given by (20).

Since $u^+(\mathrm{id}_X - vc^+)uu^+ = u^+$ according to (9) and (23) and $u^+(\mathrm{id}_X - vc^+)cc^+ = -u^+vc^+$ according to (22) and (9) the formula

$$\begin{aligned} q(uu^+ + cc^+) &= kv^*(u^+)^*u^+(\mathrm{id}_X - vc^+)(uu^+ + cc^+) \\ &= kv^*(u^+)^*u^+ + kv^*(u^+)^*(-u^+vc^+) \\ &= kv^*(u^+)^*u^+(\mathrm{id}_X - vc^+) = q \end{aligned}$$

holds for q given by (27) and the formula (25) is satisfied.

Further we know by the Lemma 3.3, that the morphism $\mathrm{id}_B - c^+c$ is symmetric and idempotent. Hence it easy to see that the morphism $\mathrm{id}_B - c^+c$ commutes with the morphism $\mathrm{id}_B + (\mathrm{id}_B - c^+c)v^*(u^+)^*u^+v(\mathrm{id}_B - c^+c)$. This shows that the morphism $\mathrm{id}_B - c^+c$ also commutes with morphism k given by (20). Clearly

$$\big((\mathrm{id}_B - c^+c)k\big)^* = (\mathrm{id}_B - c^+c)k\,, \tag{28}$$

because the morphism k is symmetric.

Let us show now that the morphism $u^+v(\mathrm{id}_B - c^+c)qu$ is symmetric for q given by (27). We get

$$\begin{aligned} u^+v(\mathrm{id}_B - c^+c)qu &= u^+v(\mathrm{id}_B - c^+c)kv^*(u^+)^*u^+(\mathrm{id}_X - vc^+)u \\ &= u^+v(\mathrm{id}_B - c^+c)kv^*(u^+)^*u^+u \\ &= u^+v(\mathrm{id}_B - c^+c)kv^*(u^+)^* \end{aligned}$$

and this morphism is clearly symmetric according to (28).

Furthermore we will show that the morphism $(\mathrm{id}_B - c^+c)qv$ is also symmetric for q given by (27). Clearly $(\mathrm{id}_B - c^+c)qv = (\mathrm{id}_B - c^+c)kv^*(u^+)^*u^+(\mathrm{id}_X - vc^+)v$. Since the morphism $\mathrm{id}_B - c^+c$ is symmetric and idempotent we get

$$\begin{aligned}(\mathrm{id}_B - c^+c)qv &= (\mathrm{id}_B - c^+c)k(\mathrm{id}_B - c^+c)v^*(u^+)^*u^+(\mathrm{id}_X - vc^+)v\\ &= (\mathrm{id}_B - c^+c)k(k^{-1} - \mathrm{id}_B)\\ &= (\mathrm{id}_B - c^+c)(\mathrm{id}_B - k)\end{aligned}$$

using (28). Together,

$$(\mathrm{id}_B - c^+c)qv = (\mathrm{id}_B - c^+c)(\mathrm{id}_B - k)\,. \tag{29}$$

Since $\mathrm{id}_B - k$ is clearly symmetric, the morphism $(\mathrm{id}_B - c^+c)qv$ is also symmetric.

Finally we will show that the relation (26) also holds for q given by (27). We have

$$\begin{aligned}u^+v(\mathrm{id}_B - c^+c)(\mathrm{id}_B - qv) &= u^+v(\mathrm{id}_B - c^+c)k\\ &= \left((\mathrm{id}_B - c^+c)kv^*(u^+)^*\right)^*.\end{aligned}$$

On the other hand,

$$\begin{aligned}(\mathrm{id}_B - c^+c)qu &= (\mathrm{id}_B - c^+c)kv^*(u^+)^*u^+(\mathrm{id}_X - vc^+)u\\ &= (\mathrm{id}_B - c^+c)kv^*(u^+)^*\,.\end{aligned}$$

Now it is easy to see that the formula (26) holds for q given by (27).

Altogether, we have shown that the formula $f^+ = f_0$ holds for q given by (27). □

Note. To prove the theorem, we need not suppose that for every $A \in \mathrm{ob}\,\mathcal{K}$ and for every morphism $f\colon A \to A$ the morphism $\mathrm{id}_A + ff^*$ is an isomorphism; it is sufficient to suppose that the morphism k is an isomorphism.

5. Involutions on the set of all matrices

L. Skula described in [11] all involutions on the set of all matrices with complex entries. In this section we would like to recall some important results of the paper.

From now on the following notation will be used:

$\mathbf{C}$ the field of complex numbers,
$\mathbf{C}^n$ the vector space of all n dimensional vectors.

$\mathcal{M}$ the set of all matrices,
$\mathcal{M}_{mn}$ the set of all $m \times n$ matrices,
$\mathcal{M}_n$ the set of all square matrices of order n,
$i_{\mathbf{C}}$ the identity mapping of $\mathbf{C}$,

Remark. We have $\mathbf{C}^n = \mathcal{M}_{n1}$.

Matrix A^φ:
Let φ be an automorphism on $\mathbf{C}$ such that $\varphi^2 = i_{\mathbf{C}}$. For $A = [A_{ij}] \in \mathcal{M}_{mn}$ we define

$$A^\varphi = [b_{kl}]_{1\le k\le n,\, 1\le l\le m},$$

where $b_{kl} = \varphi(a_{lk})$ $(1 \le k \le n,\ 1 \le l \le m)$.

φ-Hermitian matrix:
The matrix A is called φ-Hermitian if $A^\varphi = A$.

Clearly, if $A \in \mathcal{M}_{mn}$, then $A^\varphi \in \mathcal{M}_{nm}$. Hence a φ-Hermitian matrix must be a square matrix.

THEOREM 5.1. *Let $^*\colon \mathcal{M} \to \mathcal{M}$ be a unary operation. The operation * is an involution on $\mathcal{M}$ if and only if there exists an automorphism φ on $\mathbf{C}$ such that $\varphi^2 = i_{\mathbf{C}}$ and for each positive integer m there exists a nonsingular $A_m \in \mathcal{M}_m$, which is φ-Hermitian, such that for each $X \in \mathcal{M}_{pq}$ we have*

$$X^* = A_q X^\varphi A_p^{-1}.$$

Note. Let φ be an automorphism on $\mathbf{C}$ such that $\varphi^2 = i_{\mathbf{C}}$ and let $\mathcal{A} = \{A_n\}_{n=1}^\infty$ be a sequence of nonsingular φ-Hermitian matrices $A_n \in \mathcal{M}_n$. We say that the pair $[\mathcal{A}, \varphi]$ defines the involution $^*\colon \mathcal{M} \to \mathcal{M}$ if for each $X \in \mathcal{M}_{pq}$ we have $X^* = A_q X^\varphi A_p^{-1}$.

THEOREM 5.2. *Let φ and ψ be automorphisms on $\mathbf{C}$ such that $\varphi^2 = \psi^2 = i_{\mathbf{C}}$. Let $\mathcal{A} = \{A_n\}_{n=1}^\infty$ and $\mathcal{B} = \{B_n\}_{n=1}^\infty$ be sequences of nonsingular matrices A_n and B_n of order n which are φ-Hermitian and ψ-Hermitian, respectively. The pairs $[\mathcal{A}, \varphi]$ and $[\mathcal{B}, \psi]$ define the same involution $^*\colon \mathcal{M} \to \mathcal{M}$ if and only if the following conditions are satisfied:*

1. *$\varphi = \psi$,*
2. *there exists a complex number c, $c \ne 0$, with the following properties: $\varphi(c) = c = \psi(c)$, and for each positive integer n we have $B_n = cA_n$.*

In the remainder of this section, let us suppose that φ is an automorphism on $\mathbf{C}$ such that $\varphi^2 = i_{\mathbf{C}}$ and $\varphi \ne i_{\mathbf{C}}$.

Field of φ-real numbers:
The field $F(\varphi) = \{\alpha \in \mathbf{C} :\ \varphi(\alpha) = \alpha\}$. The complex number $\alpha \in F(\varphi)$ will be called φ-real.

THEOREM 5.3. *The field $F(\varphi)$ is a linearly ordered field with the positive cone*

$$P = \{\rho^2 : \ \rho \in F(\varphi)\}.$$

Note. The above formula says that if $a, b \in F(\varphi)$, then $a \geq b$ if and only if $a - b \in P$.

φ-unitary matrix:
A nonsingular matrix is called φ-unitary if $U^{-1} = U^{\varphi}$.

THEOREM 5.4. *A square matrix $H \in \mathcal{M}_n$ is φ-Hermitian if and only if there exists a φ-unitary matrix $U \in \mathcal{M}_n$ and a diagonal matrix $D = \operatorname{diag}\{\lambda_1, \dots, \lambda_n\}$ such that for each $1 \leq k \leq n$, λ_k is φ-real, and*

$$H = U^{\varphi} D U.$$

The numbers $\lambda_1, \dots, \lambda_n$ are called the eigenvalues of H.

Positive [negative] φ-definite matrix:
Let H be a φ-hermitian matrix of order n. It is called positive [negative] φ-definite if for all $x \in \mathbf{C}^n$, $x \neq 0$,

$$x^{\varphi} H x > 0 \quad [x^{\varphi} H x < 0] \qquad \text{in} \quad F(\varphi)$$

is satisfied.

Note. If for all $x \in \mathbf{C}^n$, $x \neq 0$, $x^{\varphi} H x \neq 0$ in $F(\varphi)$, then the matrix H is called φ-definite.

THEOREM 5.5. *Let $H \in \mathcal{M}_n$ be φ-Hermitian with eigenvalues $\lambda_1, \dots, \lambda_n$. Then H is positive [negative] φ-definite if and only if $\lambda_1 > 0, \dots, \lambda_n > 0$ [$\lambda_1 < 0, \dots, \lambda_n < 0$] in $F(\varphi)$.*

COROLLARY. *If a matrix H is positive [negative] φ-definite, then the matrix H^{-1} is also positive [negative] φ-definite.*

Note. The matrix H is φ-definite if and only if $\lambda_1 \neq 0, \dots, \lambda_n \neq 0$ in $F(\varphi)$.

THEOREM 5.6. *Let φ and ψ be automorphisms on $\mathbf{C}$ such that $\varphi^2 = \psi^2 = i_{\mathbf{C}}$ and let involutions * and $^\sharp$ be defined by the pairs $[\mathcal{A} = \{A_n\}_{n=1}^{\infty}, \varphi]$ and $[\mathcal{B} = \{B_n\}_{n=1}^{\infty}, \psi]$, respectively. Further, let us denote the MP-inverses of matrix X with respect involutions * and $^\sharp$ as $\mathrm{MP}_*(X)$ and $\mathrm{MP}_\sharp(X)$, respectively. Then following statements are equivalent:*

1. $\mathrm{MP}_*(X) = \mathrm{MP}_\sharp(X)$ *for each* $X \in \mathcal{M}$.
2. $\varphi = \psi$, *and for each positive integer m there exists a complex number $c(m)$ such that $A_m = c(m) B_m$.*

6. The Moore-Penrose inverse of a partitioned morphism in the category $\mathcal{MAT}$

In this section, let us consider the category $\mathcal{MAT}$ with an involution * defined by a pair $[\mathcal{A} = \{A_n\}_{n=1}^{\infty}, \varphi]$, where φ is an automorphism on $\mathbf{C}$ such that $\varphi^2 = i_{\mathbf{C}}$ and $\varphi \neq i_{\mathbf{C}}$.

Lemma 6.1. *Let m be a fixed object in category $\mathcal{MAT}$ with the involution * where for each positive integer n the matrix A_n is φ-definite.*

Then just one of the following conditions hold:

1. *For any $f\colon m \to 1$, $f \neq 0_{m1}$ is $f^*f > 0$ in $F(\varphi)$.*
2. *For any $f\colon m \to 1$, $f \neq 0_{m1}$ is $f^*f < 0$ in $F(\varphi)$.*

Proof. Let m be an object in the category $\mathcal{MAT}$. Suppose that the matrices A_1 and A_m are both positive φ-definite. If $f\colon m \to 1$, $f \neq 0_{m1}$ is a morphism, then $f^*f = A_1 f^{\varphi} A_m^{-1} f$ by 5.1 and by definition of positive φ-definite matrix $f^*f > 0$ for every f.

If the matrices A_1 and A_m are not both positive φ-definite, then we prove the statement similarly. □

Note. If $f\colon m \to 1$, then let us denote $|f|_m = f^*f$. Observe that either $|f|_m > 0$ or $|f|_m < 0$ in $F(\varphi)$ and these inequalities behave for given m in the same way for arbitrary f.

Lemma 6.2. *Let $f\colon m \to 1$ be a morphism in the category $\mathcal{MAT}$ with the involution * where for all positive integers n the matrices A_n are φ-definite. The morphism f is MP-invertible and if $f \neq 0_{m1}$, then*

$$f^+ = \frac{f^*}{|f|_m}.$$

Proof. The result follows by verification of the conditions (8). … ,(11). □

We would like to find the answer to this question: Which involutions in the category $\mathcal{MAT}$ satisfy the assumption of Theorem 1? The following theorem gives the answer.

Theorem 6.1. *The following statements are equivalent:*

1. *The morphism $\mathrm{id}_m + FF^*$ is an isomorphism for each $m \in \mathrm{ob}\,\mathcal{MAT}$ and for each morphism $F\colon m \to m$.*
2. *For all positive integers n the matrices A_n are φ-definite.*

Note. Let us remind that m is a positive integer, $F \in \mathcal{M}_m$, the morphism id_m is the identity matrix of order m and isomorphisms are just nonsingular matrices in $\mathcal{MAT}$.

Proof.

I. Let us suppose that condition 2 holds. Consider any $m \in \mathrm{ob}\,\mathcal{MAT}$ and any morphism $F\colon m \to m$. For each $x\colon m \to 1$ ($x \in \mathbf{C}^m$), we have $x^*(\mathrm{id}_m + FF^*)x = x^*x + x^*FF^*x = |x|_m + |F^*x|_m$. It follows by 6.1 that $x^*(\mathrm{id}_m + FF^*)x = 0$ if and only if $x = 0$. Hence $\mathrm{id}_m + FF^*$ is an isomorphism.

II. Conversely, let us suppose that there exists positive integer m such that the matrix A_m is φ-Hermitian, but it is neither positive φ-definite nor negative φ-definite. Hence, there exists a φ-unitary matrix $U_m \in \mathcal{M}_m$ and a diagonal matrix $D_m = \mathrm{diag}\{\lambda_1, \dots, \lambda_m\}$ such that for each $1 \le k \le m$, λ_k is φ-real, and $A_m = U_m^\varphi D_m U_m$ according to Theorem 5.4 and using Theorem 5.5 we get that there exist $1 \le k, l \le m$ such that $\lambda_k > 0$ and $\lambda_l < 0$ in $F(\varphi)$.

Let $y \in \mathbf{C}^m$ be the vector with kth entry equal to 1, lth entry equal to $\sqrt{-(\lambda_k/\lambda_l)}$ and other entries zero and let us put $x = U_m^\varphi y$ and $F = 0_{mm}$.

Then

$$\begin{aligned} x^*(\mathrm{id}_m + FF^*)x &= x^*x \\ &= A_1 x^\varphi A_n^{-1} x \\ &= A_1 x^\varphi (U_m^\varphi D_m U_m)^{-1} x \\ &= A_1 (U_m^\varphi y)^\varphi (U_m^\varphi D_m U_m)^{-1} U_m^\varphi y \\ &= A_1 y^\varphi D_n^{-1} y \\ &= A_1 \left(\lambda_k + \lambda_l \left(\sqrt{-(\lambda_k/\lambda_l)}\right)^2\right) = 0 . \end{aligned}$$

Hence $\mathrm{id}_m + FF^*$ is not an isomorphism for any $F\colon m \to m$ and hence the matrices A_n must be φ-definite for all positive integers n. □

The following theorem gives instructions for getting a partition morphism of given morphisms in the category $\mathcal{MAT}$ with an arbitrary involution.

THEOREM 6.2. *Let the matrices A_n be positive φ-definite for all positive integers n or negative φ-definite for all positive integers n. Moreover, let us put $A_n = U_n^\varphi D_n U_n$ for each positive integer n, where matrix $U_n \in \mathcal{M}_n$ is a φ-unitary and the matrix $D_n = \mathrm{diag}\{\lambda_1, \dots, \lambda_n\}$ is a diagonal matrix.*

Let $a, b, x \in \mathrm{ob}\,\mathcal{MAT}$, $f\colon a \to x$ and $g\colon b \to x$ be arbitrary morphisms in $\mathcal{MAT}$.

Let the matrix

$$e_a = U_a^\varphi \left[\sqrt{D_a E_a} \quad 0_{a,b}\right] U_{a+b} \qquad \text{and} \qquad e_b = U_b^\varphi \left[0_{b,a} \quad \sqrt{D_b F_b}\,\right] U_{a+b},$$

where if the matrix $D_{a+b} = \operatorname{diag}\{\lambda_1, \dots, \lambda_{a+b}\}$, *then the matrix*

$$E_a = \left(\operatorname{diag}\{\lambda_1, \dots, \lambda_a\}\right)^{-1} \qquad \textit{and} \qquad F_b = \left(\operatorname{diag}\{\lambda_{a+1}, \dots, \lambda_{a+b}\}\right)^{-1}.$$

Then the matrix

$$h = U^{\varphi}_{a+b} \begin{bmatrix} \sqrt{D_a E_a}^{-1} & 0_{a,b} \\ 0_{b,a} & \sqrt{D_b F_b}^{-1} \end{bmatrix} \begin{bmatrix} U_a & 0_{a,b} \\ 0_{b,a} & U_b \end{bmatrix} \begin{bmatrix} f \\ g \end{bmatrix}$$

is a partitioned morphism of f *and* g *with respect to the sum* $(a+b, e_a, e_b)$.

P r o o f. First we must prove that the morphisms e_a and e_b satisfy the relations (12) and (13) and then we can easily show that $f = e_a h$ and $g = e_b h$. As the proof is not difficult we do not present it here. □

Note. Using the relations $f = e_a h$ and $g = e_b h$ we can obtain simply the morphisms f and g from given partitioned morphism h.

Final remark. Generalized Greville's algorithm. Lemma 6.1 together with Theorem 4.1 and Theorem 6.2 give an algorithm for computation of the MP-inverse of an arbitrary morphism in the category $\mathcal{MAT}$ with an involution $*$ defined by the pair $[\mathcal{A} = \{A_n\}_{n=1}^{\infty}, \varphi]$ where the matrices A_n are either positive φ-definite for all positive integers n or negative φ-definite for all positive integers n without using the full-rank factorization.

If the matrices A_n are φ-definite for all positive integers n we obtain this algorithm by using Theorem 5.6 in addition.

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REFERENCES

[1] ADÁMEK, J.: *Mathematical Structures and Categories*, SNTL, Praha, 1982. (Czech)

[2] ALBERT, A.: *Regression and the Moore-Penrose Pseudoinverse*, Academic Press, New York-London, 1972 [Russian translation: Moskva, 1977].

[3] BÖRGER, R.: *On the existence of Moore-Penrose-inverses in categories with involution.* In: Workshop on Category Theory. Math.-Arbeitspapiere Nr. 28, Uni Bremen, 1987.

[4] CALENKO, M. S.—SULGEJFER, E. G.: *Osnovy teorii kategorij*, Nauka, Moskva, 1974. (Russian)

[5] CHIPMAN, J. S.: *Specification problems in regression analysis.* In: Proceedings of the Symposium on Theory of Generalized Inverses of Matrices, Lubbock, Texas, 1968 (T. L. Boullion, P. L. Odel, eds.), 1969, pp. 114 176.

[6] CLINE, R. E.: *Representations for the generalized inverse of partitioned matrix*. J. Soc. Ind. Appl. Math. **12** (1964), 588-600.

[7] GREVILLE, T. N. E.: *Some applications of the pseudoinverse of a matrix*, SIAM Rev. **2** (1960), 15–22.

[8] MITCHELL, B.: *Theory of Categories*, Academic Press, New York-London, 1965.

[9] PEŠKA, P.: *Pseudoinverse Matrices and Algorithms of their Computation in MATLAB*. Thesis, Masaryk University, Brno, 1995. (Czech)

[10] PUYSTJENS, R.—ROBINSON, D. W.: *The Moore-Penrose inverse of a morphism in an additive category*. Comm. Algebra **12** (1984), 287–299.

[11] SKULA, L.: *Involution for matrices and generalized inverses*, Linear Algebra Appl. **271** (1998), 283-308.

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