Math. Slovaca 41, 1991, No. 1, 3—9

LATIN PARALLELEPIPEDS NOT COMPLETING TO A CUBE

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ABSTRACT. In this paper we construct a latin $(n \times n \times (n-d))$-parallelepiped that cannot be extended to a latin cube of order n, for every $d \geq 3$ and $n \geq 6d$ or $n = 3d, 4d, 5d$. For $d = 2$, it is similar to the construction already known.

1. Introduction

A latin square of the elements $z_1, \dots, z_n$ is an $n \times n$ array such that the entries are members of $\{z_1, \dots, z_n\}$ and no member occurs in any row or column more than once. Moreover, if some cells may be empty we have an incomplete latin square of the elements $z_1, \dots, z_n$.

Let $A_1 = [a_{i,j,1}]$, $A_2 = [a_{i,j,2}]$, ..., $A_k = [a_{i,j,k}]$ be latin squares of the elements $z_1, \dots, z_n$. The ordered k-tuple $A = (A_1, A_2, \dots, A_k)$ is called a latin $(n \times n \times k)$-parallelepiped of elements $z_1, \dots, z_n$ if the elements $a_{i,j,1}, \dots, a_{i,j,k}$ are mutually distinct, for every $1 \leq i, j \leq n$. In the case $k = n$, A is called a latin cube of the elements $z_1, \dots, z_n$.

Usually $z_i = i$, $1 \leq i \leq n$. In this case we speak in abbreviation about latin squares or cubes of order n and about $(n \times n \times k)$-parallelepipeds (and do not use the words "of elements 1, 2, ..., n").

A latin cube A' of order n is an extension of a latin $(n \times n \times k)$-parallelepiped $A = (A_1, \dots, A_k)$ if there exist latin squares $A_{k+1}, \dots, A_n$ such that $A' = (A_1, \dots, A_k, A_{k+1}, \dots, A_n)$.

The following problem (see [4]) was mentioned during the Sixth Hungarian Colloquium on Cmbinatorics, Eger 1981. Given a latin $(n \times n \times k)$-parallelepiped A, does there exist a latin cube of order n, which is an extension of A? An analogous problem for latin rectangles was answered in the affirmative by Hall in [3]. On the contrary there are known constructions of the latin $(n \times n \times (n-2))$-parallelepipeds that cannot be extended to a latin cube of order n: these constructions are done for $n = 2^k$, $k \geq 3$, in [4], for $n = 6$ and

AMS Subject Classification (1985): Primary 05 B 15
Key words: Latin square, Latin parallelepiped

$n \geq 12$ in [1] and for $n \geq 5$ in [5]. This is the best possible result, because it is impossible to construct such parallelepipeds for $n \leq 4$ (see [5]). In this paper we construct for every $d \geq 3$ and $n = 3d, 4d, 5d$ or $n \geq 6d$ a latin $(n \times n \times (n-d))$-parallelepiped that cannot be extended to a latin cube of order n.

2. Construction

In this part we prove the following theorem.

Theorem: *Let $d \geq 3$, $n = 3d$, $4d$, $5d$ or $n \geq 6d$. Then there exists a latin $(n \times n \times (n-d))$-parallelepiped that cannot be extended to a latin cube of order n.*

Proof: Let $d \geq 3$. Take a latin cube $B = (B_1, \ldots, B_d)$ of order d such that $b_{i,j,k}$, the entry in the i-th row and the j-th column of B_k satisfies $b_{i,j,k} \equiv \equiv i + j + k - 2 \pmod{d}$, and $b_{i,j,k} = d$ if $i + j + k - 2 \equiv 0 \pmod{d}$.

Replace, in the latin cube B, each number $t \in \{1, \ldots, d\}$ by an arbitrary latin $(3 \times 3 \times 2)$-parallelepiped $C^{(t)}$ of the elements t, $d+t$, $2d+t$. We get a latin $(3d \times 3d \times 2d)$-parallelepiped. The same idea will be used in the following construction.

Let $d \geq 3$. Let φ be a map of $\{\langle i, j \rangle;\ 1 \leq i, j \leq d\}$ onto the five element set $\{p, r, s, t, u\}$ satisfying:

$\varphi\langle 1, 1\rangle = p$,
$\varphi\langle i, 1\rangle = r$, for $2 \leq i \leq d$,
$\varphi\langle 1, j\rangle = s$, for $2 \leq j \leq d$,
$\varphi\langle 2, j\rangle = t$, for $2 \leq j \leq d$,
$\varphi\langle i, j\rangle = u$, for $3 \leq i \leq d$, $2 \leq j \leq d$.

We will use five distinct latin $(3 \times 3 \times 2)$-parallelepipeds $C^{(t,y)}$ (where $y \in \{p, r, s, t, u\}$) if $t = 1, 2$. Let us construct.

Construction A:

Take partial latin squares $D_x^{(t,y)}$ of the elements t, $d+t$, $2d+t$ (for $x \in \in \{2, 3\}$, $t \in \{1, 2\}$, $y \in \{p, r, s, t, u\}$) as it is illustrated in Fig. 1. We can check that there exist latin cubes $E^{(t,y)} = (E_1^{(t,y)}, E_2^{(t,y)}, E_3^{(t,y)})$ of the elements t, $d+t$, $2d+t$ for $t \in \{1, 2, \ldots, d\}$, $y \in \{p, r, s, t, u\}$ satisfying (1) and (2):

(1) *If $t = 1, 2$, then $E_x^{(t,y)}$ is an extension of $D_x^{(t,y)}$, where $x \in \{2, 3\}$, $y \in \in \{p, r, s, t, u\}$.*

(2) *If $t = 3, \ldots, d$, then the entry in the first row and the first column of $E_3^{(t,y)}$ is equal to t. Furthermore, all $E_3^{(t,y)}$ are the same for all $y \in \{p, r, s, t, u\}$.*

Then let us define $C^{(t,y)} = (E_1^{(t,y)}, E_2^{(t,y)})$, the $(3 \times 3 \times 2)$-parallelepiped of the elements t, $d+t$, $2d+t$ for any $t \in \{1, \ldots, d\}$, $y \in \{p, r, s, t, u\}$.

Construction B:

We have the latin cube $B = (B_1, \ldots, B_d)$, $B_k = [b_{i,j,k}]$, $1 \le k \le d$. Replace each $t = b_{i,j,k} \equiv i + j + k - 2 \pmod d$ by $C^{(t,\varphi\langle i,j\rangle)}$. We get a new latin $(3d \times 3d \times 2d)$-parallelepiped $F = (F_1, \ldots, F_{2d})$. The latin square F_{2k}, $k = 1, \ldots, d$, arises from B_k if we replace $t = b_{i,j,k}$ by $E_2^{(t,\varphi\langle i,j\rangle)}$. Similarly the latin square F_{2k-1}, $k = 1, \ldots, d$, arises from B_k if we replace $t = b_{i,j,k}$ by $E_1^{(t,\varphi(\langle i,j\rangle)}$.

Construction C:

Now we construct a new latin $(3d \times 3d \times 2d)$-parallelepiped G from F. Take the members 1, 2 from $F_2, F_4, \ldots, F_{2d}$ as shown in Fig. 2. for $d = 4$. More precisely, take the numbers 1, 2 which are in the intersections of the 1st, 5th, 7th, ..., $3(d-1) + 2$nd rows and the 2nd, ..., $3(l-1) + 2$nd, $3l + 1$st, $3(l+1) + 2$nd, ..., $3(d-1) + 2$nd columns of F_{2k}, where $l = d - k + 1$ if $k \ne 1$. In every $F_2, F_4, \ldots, F_{2d}$ we interchange this 1 and 2. We get new latin squares $G_2, G_4, \ldots, G_{2d}$.

Let $F_k = [f_{i,j,k}]$, $1 \le k \le 2d$. If $1 = f_{i,j,k}$ is interchanged in F_k by 2, then (3) or (4) holds:

(3) *There exists* $l \in \{2, 4, \ldots, 2d\}$ *such that* $f_{i,j,l} = 2$ *is interchanged in* F_l *by* 1.

(4) *No member* $f_{i,j,l}$ *is equal to* 2 *for any* $l \in \{1, \ldots, 2d\}$ (this follows from the condition (1) for $y \in \{s, t\}$).

Similarly, if 2 $= f_{i,j,k}$ is interchanged in F_k by 1, then (5) or (6) holds:

(5) *There exists* $l \in \{2, 4, \ldots, 2d\}$ *such that* $f_{i,j,l} = 1$ *is interchanged in* F_l *by* 2.

(6) *No member* $f_{i,j,l}$ *is equal to* 1 *for any* $l \in \{1, \ldots, 2d\}$.

Thus $G = (G_1, G_2, G_3, \ldots, G_{2d})$ is a latin $(3d \times 3d \times 2d)$-parallelepiped provided $G_{2k+1} = F_{2k+1}$ for $k = 0, \ldots, d-1$.

Now we prove that G cannot be extended to a latin cube of order $3d$. Let $G_k = [g_{i,j,k}]$, $1 \le k \le 2d$. Let us denote by $M_{i,j}(G)$ the subset of the members 1, 2, ..., $3d$ which do not occur in the set $\{g_{i,j,1}, g_{i,j,2}, \ldots, g_{i,j,2d}\}$, $1 \le i, j \le 3d$. $M_{i,j}(F)$ can be defined similarly.

From (1), (2) and the construction of $C^{(t,y)}$ it follows that:

$M_{3k+1,1}(F) = \{1, 2, \ldots, d\} (0 \le k \le d-1)$,

$M_{3k+1,3l+1}(F) = \{2, 3, \ldots, d, d+1\} (0 \le k \le d-1, 1 \le l \le d-1)$.

From the Construction C we can see that:

$M_{3k+1,1}(G) = M_{3k+1,1}(F) = \{1, 2, \ldots, d\} (0 \le k \le d-1\}$,

$M_{1,3l+1}(G) = \{1, 3, 4, \ldots, d, d+1\} (1 \le l \le d-1)$,

$M_{3k+1,3l+1}(G) = M_{3k+1,3l+1}(F) = \{2, 3, \ldots, d, d+1\} (1 \le k, l \le d-1)$.

Denote $I = \{\langle 3k+1, 3l+1\rangle; 0 \le k, l \le d-1\}$.

Let $H = [h_{i,j}]$ be a latin square of order $3d$ such that $h_{1,1} = 1$ and $h_{i,j} \in M_{i,j}(G)$ for all $1 \le i, j \le 3d$.

Since $h_{1,1} = 1$, there exists exactly one $\langle i, j\rangle \in I$ such that $h_{i,j} = 1$.

Clearly $h_{1,3l+1} \ne 2$ for any $l = 0, \ldots, d-1$. Thus there exist at most $d-1$ members $\langle i, j\rangle$ of I such that $h_{i,j} = 2$.

$D_3^{(1,p)} =$

1		

$D_2^{(1,p)} =$

	1	

$D_3^{(2,p)} =$

2		

$D_2^{(2,p)} =$

	2	

$D_3^{(1,r)} =$

1		

$D_2^{(1,r)} =$

	1	

$D_3^{(2,r)} =$

2		

$D_2^{(2,r)} =$

	2	

$D_3^{(1,s)} =$

v	1	

$D_2^{(1,s)} =$

1		

$D_3^{(2,s)} = D_2^{(2,p)}$ $\qquad$ $D_2^{(2,s)} = D_2^{(2,p)}$

Similarly there exist at most $d - 1$ members $\langle i, j \rangle$ of I such that $h_{i,j} = d + 1$. There exist at most $d(d-2)$ members $\langle i, j \rangle$ of I such that $h_{i,j} = 3, \dots, d$.

Thus there exist at most $d^2 - 1$ members $\langle i, j \rangle$ of I such that $h_{i,j} \in \in \{1, 2, \dots, d, d+1\}$. But if $\langle i, j \rangle \in I$, then $h_{i,j} \in \{1, 2, \dots, d, d+1\}$ — a contradiction with the fact that $|I| = d^2$. Thus G cannot be extended to a latin cube of order $3d$. Note that we do not know whether G can be extended to a latin

$D_3^{(1,t)} =$

v		
1		

$D_2^{(1,t)} =$

	1	

$D_3^{(2,t)} =$

2		
	2	

$D_2^{(2,t)} =$

2		

$D_3^{(1,u)} =$

v		

$D_2^{(1,u)} =$

	1	

$D_3^{(2,u)} = D_3^{(2,r)}$ $\qquad$ $D_2^{(2,u)} = D_2^{(2,r)}$

$$v = d + 1$$

Fig. 1

$(3d \times 3d \times (2d + 1))$-parallelepiped, but we know that G cannot be extended to a latin cube of order $3d$.

By H.—L. Fu [1], [2] every latin cube of order m can be embedded in a latin cube of order n for every $n \geq 2m$. Using this we can easily to see that G can be embedded in the latin $(n \times n \times (n - d))$-parallelepiped H, where $n \geq 6d$ and $M_{i,j}(G) = M_{i,j}(H)$ for $1 \leq i, j \leq 3d$. Therefore H cannot be extended to a latin cube of order n. Thus we have proved the theorem for $n = 3d$ and $n \geq 6d$.

We prove the theorem if $n = 4d$. For this purpose let $V_x^{(t,y)}$ be the partial latin squares of the elements t, $d+t$, $2d+t$, $3d+t$ (for $x \in \{3, 4\}$, $t \in \{1, 2\}$, $y \in \{p, r, s, t, u\}$) satisfying (7) and (8):

(7) *The* 4*th row and the* 4*th column of* $V_x^{(t,y)}$ *are empty.*

(8) *Removing the* 4*th row and the* 4*th column of* $V_x^{(t,y)}$ *we get* $D_{x-1}^{(t,y)}$.

Analogously to the Construction A there exist latin cubes $Q^{(t,y)} = (Q_1^{(t,y)}, \ldots, Q_4^{(t,y)})$ of the elements t, $d+t$, $2d+t$, $3d+t$ for $t \in \{1, 2, \ldots, d\}$, $y \in \{p, r, s, t, u\}$ satisfying (9) and (10):

(9) *If* $t = 1, 2$, *then* $Q_x^{(t,y)}$ *is an extension of* $V_x^{(t,y)}$, where $x \in \{3, 4\}$, $y \in \{p, r, s, t, u\}$.

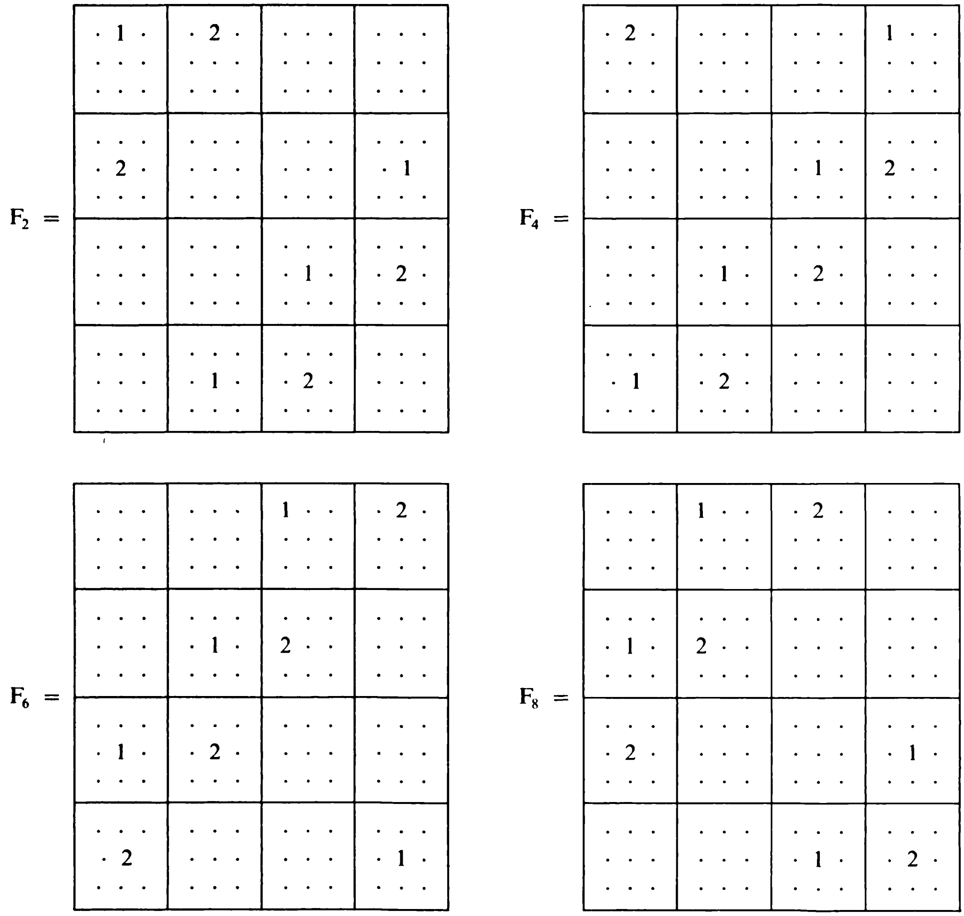

Fig. 2

(10) *If* $t = 3, \dots, d$, *then the entry in the first row and the first column of* $Q_4^{(t,y)}$ *is equal to* t. *Furthermore, all* $Q_4^{(t,y)}$ *are the same for all* $y \in \{p, r, s, t, u\}$.

Let us define analogously $W^{(t,y)} = (Q_1^{(t,y)}, Q_2^{(t,y)}, Q_3^{(t,y)})$, the $(4 \times 4 \times 3)$-parallelepiped of the elements t, $d + t$, $2d + t$, $3d + t$, for all $t \in \{1, \dots, d\}$, $y \in \{p, r, s, t, u\}$.

We can continue in the construction in the same way as for $n = 3d$ (i.e. we can replace each member of the latin cube B by an appropriate $W^{(t,y)}$ as in the Construction B and use a similar switching as in the Construction C) to get a latin $(4d \times 4d \times 3d)$-parallelepiped which cannot be extended to a latin cube of order $4d$.

The case $n = 5d$ can be proved in the same way as the cases $n = 3d$, $4d$, concluding the proof of the theorem.

Note that in [5] we have proved that there exists a latin $(n \times n \times (n-2))$-parallelepiped that cannot be extended to a latin cube of order n if and only if $n \geq 5$. That is why we conjecture that the above theorem hold if and only if $n \geq 2d + 1$, for every $d \geq 2$, i.e. each latin $(n \times n \times (n-d))$-parallelepiped can be extended to a latin cube of order n whenever $n \leq 2d$, but there exists a latin $(n \times n \times (n-d))$-parallelepiped that cannot be extended to a latin cube whenever $n \geq 2d + 1$.

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Received December 5, 1988

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