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SEQUENTIAL CAUCHY SPACES

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Two general theories of convergence have evoled during the past two decades; one is based on filters and the other on sequences. Despite many obvious resemblances, there has been surprisingly little interaction between these two schools, each having developed more or less independently of the other. That the interrelationship between these two theories deserve further investigation is indicated by a recent paper by R. Frič, K. McKennon, and G. Richardson. These authors have shown in [4] that the notion of sequential envelope introduced by J. Novák (see [9] and [10]) can be characterized by adapting the c-embedding technique developed by E. Binz and others for filter spaces to the realm of sequential spaces.

This paper, like [4], applies methods and concepts developed in filter convergence spaces to the study of sequential spaces. We introduce the notion of a "sequential Cauchy space", the sequential analogue of the (filter) Cauchy space which has been studied by various authors (see [5], [8], [12], and [13]) and used to obtain completions of the uniform convergence spaces of C. H. Cook and H. R. Fischer [1]. It is shown that a UL^* space (in the sense of A. Goetz, [6]) induces a sequential Cauchy space in a natural way. A completion theory is developed for a class of sequential Cauchy spaces based on a function space approach similar to that used by R. Gazik and D. C. Kent for (filter) Cauchy spaces in [5]. This leads, in turn, to a completion theory for a more general class of UL spaces than that studied by R. Frič in [3].

0.

Let X be a set. Sequences with range in X will be denoted by small Greek letters α, β, ξ, ζ, η, etc. The kth term of the sequence ζ is denoted by $\zeta(k)$. The small Latin letters s, t, u, v, w, denote increasing mappings of N into N. Such maps will be called sequences of indices. The composition $\eta\circ s$ is the subsequence of η which has $\eta(s(k))$ as kth term.

Given sequences α and β, then $\alpha \circ s$ and $\beta \circ s$ are corresponding subsesequences. The term *corresponding subsequence* is distinguished from *common subsequence.* The sequences α and β are said to have a common subsequence if there exist s and t such that $\alpha \circ s = \beta \circ t$. By $\langle x \rangle$ is meant the constant sequence whose kth term is x for all indices k. Let α and β be sequences. The *conjunction* of α and β, denoted by $\alpha \wedge \beta$, is defined to be the sequence η, where $\eta(2k-1) = \alpha(k)$, and $\eta(2k) = \beta(k)$ for all k in N.

Sequences of functions will usually be denoted by capital Latin letters. Let F be a sequence of real-valued functions with domain X and ζ be a sequence on X. Then $\lim_{n,k} F(n)\,(\zeta(k)) = a$ means that given $\varepsilon > 0$ there is a natural m such that n, $k > m$ implies $|F(n)(\zeta(k)) - a| < \varepsilon$.

1.

The concept of a *UL* space was introduced by A. Goetz in 1962 (See [6]), and has been subsequently studied by various authors.

Definition 1.1. *A UL space is a pair (X, n) where X is a set and n an equivalence relation between sequences with ranges in X subject to the following conditions:*
(i) $\langle x \rangle n \langle y \rangle$ iff $x = y$;
(ii) If $\eta n \xi$, then for each s it is true that $(\eta \circ s) n (\xi \circ s)$.
A UL space (X, n) is called a UL space if it satisfies the additional condition:*
($*$) *If for each sequence of indices s there is a sequence of indices t such that $(\xi \circ s \circ t)\ n\ (\eta \circ s \circ t)$, then $\xi n \eta$.*
A UL space is called a $UL^{\square}$ space if it satisfies the additional condition (cf. [11]):
($\square$) *If for each sequence of indices s there are sequences of indices t and u such that $(\xi \circ s \circ t)\ n\ (\eta \circ s \circ u)$, then $\xi n \eta$.*
The relation n of the UL space (X, n) is called a nearness relation; sequences η and ξ are said to be "near" if $\eta n \xi$.

Definition 1.2. *A convergence structure on a set X is a relation between a certain set of sequences on X and the points of X subject to the following conditions:*
(1) *$\langle x \rangle \to x$ for each x in X;*
(2) *$\xi \to x$ implies $\xi \circ s \to \xi$ for each s;*
(3) *$\xi \to x$ and $\xi \to y$ implies $x = y$.*
The pair $(X, \to)$ is called a convergence space. A convergence space is said to be a $*$ convergence space if it satisfies the following additional condition:
(4) *If for each s there exists t such that $\zeta \circ s \circ t \to x$, then $\zeta \to x$.*

A *UL* space (X, n) induces a convergence space $(X, \to)$ by: $\xi \to x$ iff $\xi n \langle x \rangle$. Goetz, [7], defines the notion of a Cauchy sequence in a *UL* space (X, n). A sequence ζ is said to be *n-Cauchy* if $\zeta n(\zeta \circ s)$ for every sequence of indices s. It is easy to see that every convergent sequence in a *UL* space is n-Cauchy; it is also clear that a subsequence of an n-Cauchy sequence is n-Cauchy. Furthermore, a sequence which is near an n-Cauchy sequence is itself n-Cauchy.

Definition 1.3. *A Cauchy space is a pair (X, L) where X is a set and L a collection of sequences with range in X which satisfies the following conditions:*
(1) $\langle x \rangle \in L$ *for each* x *in* X;
(2) $\zeta \in L$ *implies* $\zeta \circ s \in L$ *for each* s;
(3) *If* ζ *and* η *are sequences in* L *with a common subsequence, then* $\zeta \wedge \eta \in L$;
(4) *If* $\eta \wedge \langle x \rangle \in L$ *and* $\eta \wedge \langle y \rangle \in L$, *then* $x = y$.
If a Cauchy space (X, L) *satisfies the additional condition*
(5) $\eta \in L$ *whenever (a) for each s there is t such that* $\eta \circ s \circ t \in L$, *and (b) if* $\eta \circ s$ *and* $\eta \circ t$ *are two subsequences of* η *in L, then* $(\eta \circ s) \wedge (\eta \circ t) \in L$,
then (X, L) *is said to be a* $*$ *Cauchy space. If* $\zeta \in L$, *then* ζ *will be said to be a Cauchy sequence. We shall use the abbreviation C.S. for Cauchy space.*

The effect of condition (5) in the definition of a $*$ Cauchy space can be brought out by considering the real line with its usual matric. Every bounded sequence of real numbers has a Cauchy subsequence. Hence, every bounded sequence of real numbers satisfies condition *(a)*. Yet every bounded sequence of real numbers is not Cauchy in the usual sense because *(b)* is lacking; e.g. consider the sequence 0, 1, 0, 1, 0, 1, ...

A C.S. (X, L) induces a convergence space in the following natural way: $\eta \to x$ iff $\eta \wedge \langle x \rangle \in L$. Moreover, if (X, L) is a *C.S., then the induced convergence space is a $*$ convergence space.

A C.S. is said to be *complete* if every Cauchy sequence converges in the induced convergence space.

The propositions that follow involve *UL* spaces, C.S. 's, and their interrelationships; the straightforward proofs are omitted.

Proposition 1.4. *Let (X, n) be a UL* space. If ζ and η are n-Cauchy sequences and $\zeta n \eta$, then $\zeta \wedge \eta$ is also an n-Cauchy sequence, and $\zeta n(\zeta \wedge \eta)$.*

Proposition 1.5. *Let L be the set of all n-Cauchy sequences in a UL* space (X, n). Then (X, L) is a *C.S.*

Proposition 1.6. *Let (X, L) be a C.S. Define a relation $\sim$ between sequences in L as $\eta \sim \xi$ iff $\eta \wedge \xi \in L$. Then $\sim$ is an equivalence relation on L.*

We will call $\sim$ the *intrinsic equivalence relation* on L. We say that a *UL* space (X, n) *intrinsically induces* a C.S. (X, L) if L is the set of all n-Cauchy sequences and the partition of L into equivalence classes by n is the same as the partition of L by the intrinsic equivalence relation. It follows from the definition of $\sim$ that a *UL* space (X, n) intrinsically induces a C.S. (X, L) iff whenever ζ and η are n-Cauchy sequences such that $\zeta n \eta$, then $\zeta \wedge \eta$ is n-Cauchy.

Remark. The sequences which converge to x in the convergence space induced by the C.S. (X, L) are those in the equivalence class $[\langle x \rangle]$ with respect to the intrinsic equivalence relation. The sequences which converge to x in the convergence space induced by the *UL* space (X, n) are those sequences on X which are in the equivalence class $[\langle x \rangle]$ with respect to n. Moreover, as all n-convergent sequences are n-Cauchy it follows that if (X, n) intrinsically induces (X, L), then the convergence space induced by (X, n) is the same as the convergence space induced by (X, L).

Proposition 1.7. *A UL* space intrinsically induces a *C.S.*

We will say that a *UL* space (X, n_1) is *finer* than a *UL* space (X, n_2) iff $\zeta n_1 \eta$ implies $\zeta n_2 \eta$. Conversely, (X, n_2) will ba said to be *coarser* than (X, n_1).

Proposition 1.8. *Let (X, L) be a C.S. Define a relation n_L between sequences with ranges in X as follows*

$$\zeta n_L \eta \quad \text{iff} \quad \zeta = \eta \quad \text{or} \quad \zeta \wedge \eta \in L.$$

Then (X, n_L) is a UL space. Moreover, (X, n_L) is the finest UL space intrinsically inducing (X, L).

Let (X, n) be a *UL* space. Goetz [6] denotes by (X, n^*) the finest UL^* space coarser than (X, n). The relation n^* is defined by $\zeta n^* \eta$ iff for each s there exists t such that $(\zeta \circ s \circ t)\ n\ (\eta \circ s \circ t)$.

Let (X, L) be a C.S. The $*$ *modification* L^* is obtained by adding to L all sequences ζ which satisfy the following two properties:

(1) For each s there exists t such that $\zeta \circ s \circ t \in L$;

(2) If u and v are such that $\zeta \circ u \in L$ and $\zeta \circ v \in L$, then $(\zeta \circ v) \wedge (\zeta \circ u) \in L$.

It can be proved that (X, L^*) is a *C.S.

Proposition 1.9. *Let (X, n) be a UL space which intrinsically induces (X, L). Then the $*$ modification of L is the set of all n^*-Cauchy sequences iff each n^*-Cauchy sequence contains an n-Cauchy subsequence.*

Proposition 1.10. *Let (X, L) be a *C.S. Then (X, n_L^*) is the finest UL* space inducing (X, L) intrinsically.*

2.

A real-valued function f on a Cauchy space (X, L) will be said to be *continuous* if whenever $\zeta \to x$ in the induced convergence space, it follows that $\lim_k f(\zeta(k)) = f(x)$. The collection of continuous functions on a Cauchy space will be denoted by $C(X, L)$. The continuous functions are completely determined by the induced convergence space. Hence all Cauchy spaces inducing a given convergence space have the same set of continuous functions.

A real-valued function on (X, L) is said to be *Cauchy-continuous* if for all sequences $\zeta \in L$, it is true that $\lim_k f(\zeta(k))$ exists. We will denote the set of Cauchy-continuous functions by $\hat{C}(X, L)$. Note that a Cauchy-continuous function is continuous since if $\zeta \to x$ in the induced convergence space, then $\zeta \wedge \langle x \rangle \in L$. Hence it follows $\lim_k f(\zeta(k)) = f(x)$. Therefore $\hat{C}(X, L)$ is contained in $C(X, L)$. The Cauchy-continuous functions are not completely determined by the induced convergence space. However, if the C.S. is complete, then $C(X, L) = \hat{C}(X, L)$. In general, the collection $\hat{C}(X, L)$ is an intrinsic property of the Cauchy space (X, L). A complete C.S. (X, L) can be considered as a convergence space and hence convergence notions are well-defined for (X, L). For instance, we say that a complete *C.S. (X, L) is *sequentially regular,* which means that $\zeta \to x$ iff $\lim_k f(\zeta(k)) = f(x)$ for each $f \in C(X, L)$ (i.e. the convergence $\to$ is projectively generated by $C(X, L)$), and *sequentially complete* if ζ converges in (X, L) whenever $\lim_k f(\zeta(k))$ exists for each $f \in C(X, L)$.

Let M be the set of all sequences F with range in $\hat{C}(X, L)$ such that $\lim_{n,k} F(n)(\zeta(k))$ exists for all $\zeta \in L$. We call M the *continuous Cauchy structure* on $\hat{C}(X, L)$.

Let (X_1, L_1) and (X_2, L_2) be C.S. 's. A mapping $\Phi: X_1 \to X_2$ is said to be *Cauchy-continuous* if $\eta \in L_1$ implies $\Phi(\eta) \in L_2$, where $(\Phi(\eta))(k) = \Phi(\eta(k))$.

Theorem 2.1. *Let (X, L) be a C.S. Then $(\hat{C}(X, L), M)$ is a complete *Cauchy space.*

Proof. The verification that $(\hat{C}(X, L), M)$ is a *C.S. is routine and is left to the reader. Completeness will be shown. Let $F \in M$. Then $F \to f$ in the induced convergence space iff $F \wedge \langle f \rangle \in M$. But $F \wedge \langle f \rangle \in M$ iff $\lim_{n,k} F(n)(\zeta(k)) =$

$= \lim_k f(\zeta(k))$ for all $\zeta \in L$.

Let $F \in M$. Define a function f on X by $f(x) = \lim_n F(n)(x)$ for all $x \in X$. Then as $\langle x \rangle \in L$ for all $x \in X$, $f(x)$ is defined for all $x \in X$. Let $\zeta \in L$. Since $F \in M$ it follows that $\lim_{n,k} F(n)(\zeta(k)) = L_1$ exists. Hence given $\varepsilon > 0$, there exists a pair of natural numbers (n_0, k_0) such that $(n, k) \geqq (n_0, k_0)$ implies $|F(n)(\zeta(k)) - L_1| < \varepsilon/2$. Given $k \geqq k_0$ there exists $n_1(k) \geqq n_0$ such that $|f(\zeta(k)) - F(n_1)(\zeta(k))| < \varepsilon/2$.

Hence

$$|f(\zeta(k))-L_1| \leqq |f(\zeta(k))-F(n_1)(\zeta(k))| + \\ + |F(n_1)(\zeta(k))-L_1| < \varepsilon/2+\varepsilon/2=\varepsilon.$$

Therefore $\lim_k f(\zeta(k))=L_1$.

Note that if ζ and ξ are Cauchy sequences belonging to the same equivalence class in the C.S. (X, L), i.e., $\zeta \wedge \xi \in L$, then for all $f \in \hat{C}(X, L)$ it follows that $\lim_k f(\zeta(k)) = \lim_k f(\xi(k))$. A C.S. (X, L) is said to be *Cauchy-separated* if the Cauchy-continuous functions separate the Cauchy equivalence classes; i.e., given Cauchy sequences ζ, η such that $\zeta \wedge \eta \notin L$, then there exists $f \in \hat{C}(X, L)$ such that $\lim_k f(\zeta(k)) \neq \lim_k f(\eta(k))$. Cauchy-separated implies that the Cauchy-continuous functions separate points.

Let (X, L) be a C.S. The C.S. $(\hat{C}(X, L), M)$ determines a collection L_M of sequences on X. A sequence ζ is defined to be in L_M iff $\lim_{n,k} F(n)(\zeta(k))$ exists for all $F \in M$. It is clear that $L \subset L_M$. Moreover, if $\hat{C}(X, L)$ separates points, then (X, L_M) will be a C.S.

Proposition 2.2. *If* (X, L) *is a C.S. such that* $L=L_M$, *then* (X, L) *is Cauchy-separated.*

Proof. Suppose ζ and η are Cauchy sequences belonging to different Cauchy classes. Then $\zeta \wedge \eta \notin L=L_M$. Hence there exists $F \in M$ such that $\lim_{n,k} F(n)(\zeta(k)) \neq \lim_{n,k} F(n)(\eta(k))$. As $(\hat{C}(X, L), M)$ is complete, it follows that $F \to f \in \hat{C}(X, L)$. Hence

$$\lim_k f(\zeta(k)) = \lim_{n,k} F(n)(\zeta(k)) \neq$$

$$\neq \lim_{n,k} F(n)(\eta(k)) = \lim_k f(\eta(k)).$$

We designate by $(\hat{C}^2(X, L), M^2)$ the continuous Cauchy structure formed on the set of real-valued Cauchy-continuous functions on the complete C.S. $(\hat{C}(X, L), M)$. Then X can be mapped into $(\hat{C}^2(X, L), M^2)$ by the canonical map i, where $i(x)(f)=f(x)$ for all $f \in \hat{C}(X, L)$. Note that if $\zeta \in L$ and $F \in M$, then $\lim_{n,k} i(\zeta(k))(F(n)) = \lim_{n,k} F(n)(\zeta(k))$ exists. Hence $i(\zeta) \in M^2$, and i is Cauchy-continuous.

Theorem 2.3. *The canonical map* $i:(X, L) \to (\hat{C}^2(X, L), M^2)$ *is a Cauchy embedding iff* $L = L_M$.

Proof. Suppose $L = L_M$. Then since $\hat{C}(X, L)$ separates points, it follows that i is one-to-one. The canonical mapping i is always Cauchy-continuous. Let $\zeta \in M^2$ be such that the range of ζ is contained in $i(X)$. Let $\eta = i^{-1}(\zeta)$ and $F \in M$. Then $\lim_{k,n} \zeta(k)(F(n))$ exists. Since $\lim_{k,n} F(n)(\eta(k)) = \lim_{k,n} i(\eta(k))(F(n)) =$ $= \lim_{k,n} \zeta(k(F(n))$ exists, it follows that $\eta \in L_M$. Therefore, if $L = L_M$, i^{-1} is Cauchy--continuous, and (X, L) is Cauchy embedded in $(\hat{C}^2(X, L), M^2)$.

Suppose (X, L) is Cauchy embedded in $(\hat{C}^2(X, L), M^2)$. Let $\zeta \in L_M$. Then for all $F \in M$, it follows that $\lim_{n,k} F(n)(\zeta(k)) = \lim_{n,k} i(\zeta(k))(F(n))$ exists. Hence $i(\zeta) \in M^2$. Since i^{-1} is Cauchy-continuous, it follows that $\zeta \in L$. Therefore, we have $L \supset L_M$. As $L \subset L_M$ is always true, it follows that $L = L_M$.

Definition 2.4. *A C.S.* (X, L) *is said to be* $\hat{C}$*-embedded if* $i:(X, L) \to (\hat{C}^2(X, L), M^2)$ *is a Cauchy embedding.*

Notation 2.5. *Let* (X, L) *be a C.S. The collection of all sequences* ζ *on* X *such that* $\lim_n f(\zeta(n))$ *exists for all* $f \in C(X, L)$ *will be denoted by* L_F.

Notation 2.6. *Let* (X, L) *be a C.S. Denote by* c *the set of all sequences* F *on* $C(X, L)$ *such that* $\lim_{n,k} F(n)(\zeta(k))$ *exists whenever* $\zeta \to x$ *in the induced convergence space. We call* c *the continuous convergence structure on* $C(X, L)$.

Notation 2.7. *Let* (X, L) *be a C.S. The collection of all sequences* ζ *on* X *such that* $\lim_{n,k} F(n)(\zeta(k))$ *exists whenever* $F \in c$ *will be denoted by* L_c.

If the continuous functions separate points, then (X, L_F) and (X, L_c) are both C.S. 's. It is always true that $L_c \subset L_F$. In [4] it was shown that, when (X, L) induces a sequentially regular convergence space, then $L_c = L_F$. In [14] it was implicitly shown that $(C(X, L), c)$ is a complete sequentially regular *C.S. We will denote convergence in the induced convergence space by $T \xrightarrow{c} f$; this convergence is called *continuous convergences.* $(C(X, L), c)$ is the Cauchy space determined by the continuous convergence structure. The straightforward proof of the following theorem is omitted.

Theorem 2.8. *Let* $F \xrightarrow{c} f$ *and* $\zeta \in L_c$. *Then* $\lim_{n,k} F(n)(\zeta(k)) = \lim_k f(\zeta(k))$.

Notation 2.9. *Let (X, L) be a C.S. Then $L_{\hat{F}}$ will denote the set of all sequences ζ on X such that $\lim_k f(\zeta(k))$ exists for all $f \in \hat{C}(X, L)$.*

It is always true that $L_M \subset L_{\hat{F}}$. If $\hat{C}(X, L)$ separates points, then $(X, L_{\hat{F}})$ is a C.S.

Theorem 2.10. *A C.S. (X, L) is $\hat{C}$-embedded iff $L = L_{\hat{F}}$.*

Proof. Assume that $L = L_{\hat{F}}$. It is always true that $L \subset L_M \subset L_{\hat{F}}$. So when $L = L_{\hat{F}}$, it follows that $L = L_M$. Hence, by Theorem 2.3, (X, L) is $\hat{C}$-embedded.

Now assume (X, L) is $\hat{C}$-embedded. Since $(\hat{C}(X, L), M)$ is a complete C.S., it follows that M^2 is the same Cauchy structure as the one determined by the continuous convergence structure on $\hat{C}(\hat{C}(X, L), M)$. Thus it follows that $(\hat{C}^2(X, L), M^2)$ is sequentially regular. Let f be a continuous real-valued function on the C.S. $(\hat{C}^2(X, L), M^2)$. Then define a function $\bar{\bar{f}}(x) = f(i(x))$ for each $x \in X$. As (X, L) is Cauchy embedded in the complete C.S. $(\hat{C}^2(X, L), M^2)$, it follows that $\bar{\bar{f}}(x)$ is Cauchy-continuous. In particular, if $\zeta \in L_{\hat{F}}$ it follows that $i(\zeta) \in (M^2)_F$. But, as $(\hat{C}^2(X, L), M^2)$ is sequentially regular, $i(\zeta) \in (M^2)_c$. Hence by Proposition 3.1 in [14], $i(\zeta)$ converges in $(\hat{C}^2(X, L), M^2)$, and so $i(\zeta) \in M^2$. Hence $\zeta \in L$, and $L_{\hat{F}} \subset L$. As $L \subset L_{\hat{F}}$ is always true, it follows $L = L_{\hat{F}}$.

Definition 2.11. *A C.S. (X, L) is Cauchy-regular if, whenever a sequence ζ on X has no Cauchy subsequence, there exists a Cauchy-continuous function f such that $\lim_n f(\zeta(n))$ does not exist.*

Theorem 2.12. *A C.S. (X, L) is $\hat{C}$-embedded iff (X, L) is a Cauchy-separated, Cauchy-regular *C.S.*

Proof. If (X, L) is $\hat{C}$-embedded, then $L = L_{\hat{F}}$. The C.S. $(X, L_{\hat{F}})$ has the desired properties.

Now suppose that (X, L) satisfies the three conditions. Let $\zeta \in L_{\hat{F}}$. Then, since every subsequence of ζ is in $L_{\hat{F}}$, it follows that every subsequence of ζ has a Cauchy subsequence. Moreover, as (X, L) is Cauchy-separated, it follows that all Cauchy subsequences of ζ belong to the same Cauchy class. Then $\zeta \in L$ since (X, L) is a *C.S. and so $L_{\hat{F}} \subset L$. The inclusion $L \subset L_{\hat{F}}$ is always true. Hence $L = L_{\hat{F}}$, and (X, L) is $\hat{C}$-embedded.

A complete sequentially regular *C.S. which is sequentially complete is clearly Cauchy-separated Cauchy-regular *C.S. and hence by Theorem 2.12 it is $\hat{C}$-embedded. It was proved in [4] that $(\hat{C}^2(X, L), M^2)$ is sequentially complete. Since it is also sequentially regular, it is $\hat{C}$-embedded.

Theorem 2.13. *If (X, L_c) is a C.S., then (X, L_c) is $\hat{C}$-embedded.*

Proof. The sets $\hat{C}(X, L_c)$ and $C(X, L)$ are identical. Denote by M the continuous Cauchy structure on $\hat{C}(X, L_c)$. A simple computation shows that $(\hat{C}(X, L_c), M) = (C(X, L), c)$. Thus $(L_c)_M = L_c$, and, by Theorem 2.3, (X, L_c) is $\hat{C}$-embedded.

Theorem 2.14. *The pair (X, L_c) is a C.S. iff $C(X, L)$ separates points.*

Theorem 2.15. *Let (X, L) be a C.S. If $C(X, L)$ separates points, then $L_c = L_F$.*

Proof. Since $C(X, L)$ separates points, (X, L_c) is a C.S. Theorem 2.13 implies that (X, L_c) is C-embedded. Hence,

$$L_c = (L_c)_{\hat{F}} = (L_c)_F = L_F.$$

Theorem 2.16. *If (X, L) induces a sequentially regular $*$ convergence space, then (X, L_c) induces the same convergence space.*

Proof. A sequentially regular $*$ convergence space is projectively generated by the real-valued continuous functions. This is the same convergence induced by the C.S. (X, L_F). Since (X, L) is sequentially regular, $L_c = L_F$.

Proposition 2.17. *For any sequentially regular $*$ convergence, there exist a $\hat{C}$-embedded C.S. which induces it.*

Proof. The set L_c can be determined from the convergence space. The C.S. (X, L_c) is a C-embedded C.S. inducing this convergence space.

Note that all the C.S. 's which induce a given convergence space have the same L_c and L_F.

Definition 2.18. *We will say that a C.S. (X, L_1) is finer than a C.S. (X, L_2) iff $L_1 \subset L_2$.*

Theorem 2.19. *Let $(X, \rightarrow)$ be a sequentially regular $*$ convergence space. Then there is a finest $\hat{C}$-embedded C.S. inducing $(X, \rightarrow)$.*

Proof. Proposition 2.17 states the existence of a $\hat{C}$-embedded C.S. which induces $(X, \rightarrow)$. Let (X, L) be any $\hat{C}$-embedded C.S. inducing $(X, \rightarrow)$. Then $L_c \subset L_M = L$, and (X, L_c) induces $(X, \rightarrow)$. Since (X, L_c) will be the same for all C.S. 's which induce this convergence, the result follows.

Proposition 2.20. *Let $F \xrightarrow{M} f$ and $\zeta \in L_M$. Then $\lim_{n,k} F(n)(\zeta(k)) = \lim_k f(\zeta(k))$.*

Theorem 2.21. *Let (X, L) be a C.S. If (X, L_M) is a C.S., then it is $\hat{C}$-embedded.*

Proof. Note that $\hat{C}(X, L) = \hat{C}(X, L_M)$. Since $(L_M)_M = L_M$, Theorem 2.3 implies that (X, L_M) is $\hat{C}$-embedded.

Theorem 2.22. *If (X, L_M) is a C.S., then $L_M = L_{\hat{F}}$.*

Proof. Note that $\hat{C}(X, L) = \hat{C}(X, L_M)$. Theorem 2.10 now gives the result.

Corollary 2.23. *If $\hat{C}(X, L)$ separates points, then $L_M = L_{\hat{F}}$.*

Theorem 2.24. *If the canonical mapping $i: (X, L) \rightarrow (C^2(X, L), c^2)$ is a Cauchy embedding, then $L = L_{\hat{F}} = L_F$ and the C.S. 's $(\hat{C}^2(X, L), M^2)$ and $(C^2(X, L), c^2)$ are the same.*

Proof. The sequence $\zeta \in L$ implies that $i(\zeta)$ belongs to c^2 since i was assumed to be a Cauchy embedding. Since $(C^2(X, L), c^2)$ is complete, it follows that $i(\zeta)$ converges. Hence for all real-valued continuous functions f on the C.S. $(C^2(X, L), c^2)$ it is true that $\lim_k f(i(\zeta(k)))$ exists. Define a function $\bar{\bar{f}}(x) = f(i(x))$ for all $x \in X$. When f is a continuous function on $(C^2(X, L)c^2)$ it follows that $\bar{f}(x) \in \hat{C}(X, L)$. Assume this is the case and $\eta \in L_{\hat{F}}$. Then $\lim \bar{f}(\eta(k)) = \lim f(i(\eta))$ exists. Since the sequentially regular space $(C^2(X, L), c^2)$ is sequentially complete ([14]), $i(\eta)$ converges in $(C^2(X, L), c^2)$. Hence $i(\eta) \in c^2$. Since the mapping i is a Cauchy embedding it follows that $\eta \in L$. Therefore $L_{\hat{F}} \subset L$. The opposite inclusion $L \subset L_F$ is always true. Thus $L_{\hat{F}} = L$.

Now let $\zeta \in L$. Then since i is a Cauchy embedding it follows that $i(\zeta) \in c^2$. Let $f \in C(X, L)$. Then $\langle f \rangle \in c$. Therefore it follows that $\lim_k i(\zeta(k))(f) = \lim_k f(\zeta(k))$ exists. Therefore $\zeta \in L_F$. Thus $L \subset L_F$. But L_F is always contained in $L_{\hat{F}}$ and $L_{\hat{F}}$ has been shown to be L. Therefore it follows that $L = L_{\hat{F}} = L_F$. This observation implies $C(X, L) = \hat{C}(X, L)$. Since (X, L) is sequentially regular we have $L_c = L_F = L$. Therefore the C.S. 's $(\hat{C}(X, L), M)$ and $(C(X, L)\ c)$ are the same, and this fact implies the desired conclusion.

3.

Let (X, L) be a Cauchy space, and let Cl_X^α be the α-th iteration of the closure operator relative to the convergence space induced by (X, L), where α is an ordinal number not exceedind ω_1. It is well-known that $Cl_X^{\omega_1}$ is closure operator of the topological modification of the induced convergence structure; in other words, this closure is idempotent.

In this section, we define the *natural completion* (X', L') of a $\hat{C}$-embedded Cauchy space (X, L) and show that this completion is unique relative to this class.

D finition 3.1. *Let (X, L) be a $\hat{C}$-embedded Cauchy space, and let $X' = Cl^{\omega_1} i(X)$, where the closure is relative to $(\hat{C}^2(X, L), M^2)$. Let L' be the Cauchy structure which X' inherits from $(\hat{C}^2(X, L), M^2)$. Since X' is closed, (X', L') is a complete C.S. and is called the natural completion of (X, L).*

Note that the natural completion of a $\hat{C}$-embedded C.S. is also $\hat{C}$-embedded.

Proposition 3.2. *Let $\Phi: (X, L) \to (X_1, L_1)$ be a Cauchy-continuous map. Let $f \in \hat{C}(X_1, L_1)$. Then $f \circ \Phi \in \hat{C}(X, L)$.*

Theorem 3.3. *Let $\Phi: (X, L) \to (X_1, L_1)$ be a Cauchy-continuous map. Define $\Phi_1: \hat{C}(X_1, L_1) \to \hat{C}(X, L)$ by $\Phi_1(f) = f \circ \Phi$. Then $\Phi_1: (\hat{C}(X_1, L_1), M_1) \to (\hat{C}(X, L), M)$ is Cauchy-continuous.*

Proof. It follows by the previous proposition that $\Phi_1: C(X_1, L_1) \to \hat{C}(X, L)$. Hence it suffices to show Φ_1 is Cauchy-continuous. Let $F \in M_1$ and $\zeta \in L$. Then the

$\lim_{n,k} \Phi_1(F(n))(\zeta(k)) = \lim_{n,k} F(n)(\Phi(\zeta(k)))$ exists since $\Phi(\zeta)\in L_1$. Hence $\Phi_1(F)\in M$.

Let $\Phi:(X, L)\to(X_1, L_1)$ be Cauchy-continuous. We have seen that

$$\Phi_1:(\hat{C}(X_1, L_1), M_1)\to(\hat{C}(X, L), M)$$

defined by $\Phi_1(f)(x) = f(\Phi(x))$ is Cauchy-continuous. Hence it follows that

$$\Phi_2:(\hat{C}^2(X, L), M^2)\to(\hat{C}^2(X_1, L_1), M_1^2)$$

defined by $\Phi_2(g)(f)=g(\Phi_1(f))$, i.e., $\Phi_2(g)=g\circ\Phi_1$ is Cauchy-continuous.

Hence in particular

$$\begin{aligned}\Phi_2(i(x))(f)=i(x)(\Phi_1(f))&=i(x)(f\circ\Phi)\\&=(f\circ\Phi)(x)\\&=i_1(\Phi(x))(f).\end{aligned}$$

In this sense we may think of Φ_2 as a Cauchy-continuous extension of Φ which takes $\hat{C}^2(X, L)$ into $\hat{C}^2(X_1, L_1)$, and $i(X)$ into $i_1(X_1)$.

Theorem 3.4. *Let Φ be a Cauchy-continuous mapping of a $\hat{C}$-embedded C.S. (X, L) into the complete C-emedded C.S. (X_1, L_1). Then there is a unique Cauchy-continuous extension Ψ of Φ which maps (X', L') into (X_1, L_1).*

Proof. By the observations which precede the theorem, $\Phi_2:(\hat{C}^2(X, L), M^2)\to(\hat{C}^2(X_1, L_1), M_1^2)$ is Cauchy-continuous and $\Phi_2(i(X)) \subset i_1(X_1)$. Consequently, $\Phi_2(Cl^{\omega_1}_{X'}i(X)) = \Phi_2(X')\subset Cl^{\omega_1}_{X_1'}(i_1(X_1))=i_1(X_1)$. Thus $\Psi=i_1^{-1}\circ\Phi_2$ is the desired extension. The uniqueness of Ψ is obvious.

4.

The primary purpose of this section is to use the C.S. completion for $\hat{C}$-embedded C.S. to obtain a *UL* space completion for *UL* spaces which intrinsically induce a $\hat{C}$-embedded C.S. It will be shown that every *UL** space which has a completion in the sense of Frič [3] intrincically induces a $\hat{C}$-embedded C.S. and thus has a completion described above.

Lemma 4.1. *Let $\Phi:(X, L_1)\to(Y, L_2)$ be a Cauchy embedding of a C.S. (X, L_1) into a C.S. (Y, L_2). Let (X, n) be any UL space intrinsically inducing (X, L_1). Let m be a relation between sequences with range in Y defined by $\zeta m \eta$ iff any one of the three following conditions hold:*
(1) $\zeta(k), \eta(k)\in\Phi(X)\ \forall k\in N$ and $\Phi^{-1}(\zeta) n \Phi^{-1}(\eta)$;
(2) $\zeta(k)=\eta(k)\ \forall k\in N$;
(3) $\zeta\wedge\eta\in L_2$.
Then (Y, m) is a UL space intrinsically inducing the C.S. (Y, L_2). The mapping Φ

*is a uniform embedding of (X, n) into (Y, m). Moreover, if (X, n) is a UL^* space and (Y, L_2) is a *Cauchy space, then the C.S. induced by (Y, m^*) is (Y, L_2).*

Proof. Clearly m is symmetric and reflective. If $\zeta m \eta$, the for the same reason that two sequences are near, the corresponding subsequences are near. Suppose $\zeta m \eta$ and $\eta m \xi$. If the two pairs are near because they satisfy the same condition (of the three possible conditions), or if the reason that one of the two pairs is near is equality of the sequences, then it follows easily that $\zeta m \xi$. Hence assume without loss of generality that $\Phi^{-1}(\zeta) n \Phi^{-1}(\eta)$ and $\eta \wedge \xi \in L_2$. Hence $\eta \in L_2$, and $\Phi^{-1}(\eta) \in L_1$, as Φ is a Cauchy embedding. But in a UL space (X, n) any sequence near an n-Cauchy sequence is n-Cauchy, and so $\Phi^{-1}(\zeta)$ must also be n-Cauchy. Since the partition of L_1 by n is the same as the intrinsic partition of L_1, it follows that $\Phi^{-1}(\zeta) \wedge \Phi^{-1}(\eta) \in L_1$. Hence the image of $\Phi^{-1}(\zeta) \wedge \Phi^{-1}(\eta)$ by Φ is $\zeta \wedge \eta \in L_2$ since Φ is a Cauchy embedding. Since $\zeta \wedge \eta \in L_2$ and $\eta \wedge \xi \in L_2$ and these two sequences have the common subsequence η, it follows that

$$(\zeta \wedge \eta) \wedge (\eta \wedge \xi) \in L_2.$$

But $(\zeta \wedge \xi)$ is a subsequence of $(\zeta \wedge \eta) \wedge (\eta \wedge \xi)$. Hence $\zeta \wedge \xi \in L_2 \Rightarrow \zeta m \xi$. Therefore (Y, m) is a UL space.

It follows by the definition of m that Φ is a uniform embedding of (X, n) into (Y, m). Since Φ is also a Cauchy embedding, it follows that any m-Cauchy sequence ζ with range in $\Phi(X)$ belongs to L_2.

Let η be an m-Cauchy sequence whose range is not contained entirely in $\Phi(X)$. If there exists an s such that $(\eta \circ s)(1) \neq \eta(1)$, then $(\eta \circ s) m \eta$ implies $(\eta \circ s) \wedge \eta \in L_2$. Hence $\eta \in L_2$. If no such sequence of indices s exists, then η is a constant sequence which belongs to L_2.

Clearly, be definition of m, sequence in L_2 is m-Cauchy. It is easy to see that (Y, m) induces (Y, L_2) intrinsically.

Finally, if (X, n) is a UL^* space and (Y, L_2) is a *C.S., then it follows by Proposition 1.10 that $(Y, ,n^*_{L_2})$ intrinsically induces (Y, L_2). It can be shown that $(Y, n^*_{L_2})$ and (Y, m^*) intrinsically induce the same *C.S.

Definition 4.2. *A UL space (X, n) is said to be of type $\hat{C}$ if (X, n) induces a $\hat{C}$-embedded C.S. intrinsically.*

Let (X_k, n_k) be a UL space of type $\hat{C}$, L_k the intrinsically induced Cauchy structure, and $i_k: (X_k, L_k) \to (\hat{C}^2(X_k, L_k), M_k^2)$ the canonical Cauchy embedding. Denote by m_k the UL relation between sequences with range in $\hat{C}^2(X_k, L_k)$ defined by conditions (1), (2), and (3) of Lemma 4.1. Then $(\hat{C}^2(X_k, L_k), m_k)$ intrinsically induces the $\hat{C}$-embedded C.S. $(\hat{C}^2(X_k, L_k), M_k^2)$ and $i_k: (X_k, n_k) \to (\hat{C}^2(X_k, L_k), m_k)$ is a uniform embedding. Furthermore, m_k can be replaced by m_k^* if n_k is a UL^* relation.

Theorem 4.3. *Let (X_k, n_k), $k = 1, 2$, be UL spaces of type $\hat{C}$, L_k and m_k be as* *specified in the preceding paragraph. Let $\Phi: (X_1, n_1) \to (X_2, n_2)$ be a uniformly continuous mapping. Then there is a uniformly continuous mapping Ψ: $(\hat{C}^2(X_1, L_1), m_1) \to (\hat{C}^2(X_2, L_2), m_2)$ such that: (a) for each $x \in X_1$ we have $\Psi(i_1(x)) = i_2(\Phi(x))$, (b) $\Psi(Cl^{\omega_1}i_1(X_1)) \subset Cl^{\omega_1}i_2(X_2)$.*

Proof. Since Φ: $(X_1, L_1) \to (X_2, L_2)$ is Cauchy-continuous, the mapping Ψ: $(\hat{C}^2(X_1, L_1), M_1^2) \to (\hat{C}^2(X_2, L_2), M_2^2)$ defined for $g \in \hat{C}^2(X_1, L_1)$ and $f \in \hat{C}(X_2, L_2)$ by $(\Psi(g))(f) = g(f \circ \Phi)$ is Cauchy-continuous (cf. Section 3). Simple calculations show that Ψ is an $m_1 - m_2$ uniformly continuous mapping satisfying conditions *(a)*, *(b)*.

Definition 4.4. *Let (X, n) be a UL space of type $\hat{C}$, L the intrinsically induced Cauchy structure, i: $(X, L) \to (\hat{C}^2(X, L), M^2)$ the canonical Cauchy embedding, and $(\hat{C}^2(X, L), m)$ the UL space of type $\hat{C}$ defined above. Let $X' = Cl^{\omega_1}i(X)$ in $(\hat{C}^2(X, L), m)$ and let n' be the inherited UL structure on X'. Then (X', n') will be called the natural completion of (X, n).*

Note that (X', n') is a complete *UL* space of type $\hat{C}$.

Theorem 4.5. *Let (X, n) be a UL space of type $\hat{C}$, (X', n') its natural completion, and (X_1, n_1) a complete UL space of type $\hat{C}$. Then each uniformly continuous map Φ: $(X, n) \to (X_1, n_1)$ has a uniquely determined uniformly continuous extension Ψ: $(X', n') \to (X_1, n_1)$.*

The existence of Ψ follows from Theorem 4.3. The uniqueness follows by a standard topological argument.

Let (X, n) be a *UL* space. Denote by $U = U(X, n)$ the set of all uniformly continuous functions on (X, n) and by U_0 a subset of U.

A *UL* space (X, n) is said to be *U_0-regular* if the following condition is satisfied:

(U_0R) if ζ and η are two sequences such that $\zeta \circ s n \eta \circ s$ for all corresponding subsequences, then there is f in U_0 such that $\lim_k |f(\zeta(k)) - f(\eta(k))| = 0$ does not hold.

Remark 4.6. A U_0-regular space (X, n) is U-regular and U_0 separates points. A UL^* space (X, n) is U_0-regular iff n is projectively generated by U_0.

Remark 4.7. Let $\mathcal{F}$ be a collection of real-valued functions on $X \neq \emptyset$ which separates points. Denote by (X, n) the *UL* space projectively generated by $\mathcal{F}$. Then $\mathcal{F} \subset U(X, n)$ and (X, n) is an $\mathcal{F}$-regular $UL^{\square}$ space.

Theorem 4.8. *Let (X, n) be a U_0-regular UL^* space. Then (X, n) is of type $\hat{C}$.*

Proof. From Proposition 1.7 it follows that (X, n) intrinsically induces a *C.S. Denote it by (X, L). By Theorem 2.10 it suffices to prove that $L = L_{\mathcal{F}}$. The inclusion $L \subset L_{\mathcal{F}}$ is trivial. Let $\zeta \notin L$. Then there is s such that $\zeta \circ s \notin \zeta$. Consequently, there is f in $U_0 \subset U(X, n) \subset \hat{C}(X, L)$ such that $\lim_k f(\zeta(k))$ does not exists. Thus $\zeta \notin L_{\mathcal{F}}$, and hence $L_{\mathcal{F}} \subset L$.

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СЕКВЕНЦИАЛЬНЫЕ ПРОСТРАНСТВА КОШИ

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Резюме

Вводятся пространства Коши определенные последовательностями. Используясь методами введеными для общих пространств Коши, строится пополнение этих пространств. Это позволяет получить пополнение для некоторого класса UL-пространств.