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BOUNDS FOR THE SPECTRAL RADIUS OF NONNEGATIVE MATRICES

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ABSTRACT. We propose lower and upper bounds for the spectral radius of nonnegative matrices, and necessary and sufficient conditions to achieve these bounds for irreducible nonnegative matrices.

1. Introduction

Let A be an $n \times n$ nonnegative matrix. Denote by $\rho(A)$ the spectral radius of A. Due to Perron-Frobenius theorem, the spectral radius of a nonnegative matrix is the largest eigenvalue of this matrix. The algebraic and combinatorial properties of nonnegative matrices have been the focus of a good deal of work (see [4], for example).

THEOREM 1 (FROBENIUS). ([1]) *Let A be an $n \times n$ nonnegative matrix and $x = (x_1, x_2, \dots, x_n)^t$ be a positive vector. Then*

$$\min_{1 \le i \le n} \frac{(Ax)_i}{x_i} \le \rho(A) \le \max_{1 \le i \le n} \frac{(Ax)_i}{x_i} . \tag{1}$$

Moreover, if A is irreducible, then any equality holds in (1) *if and only if x is an eigenvector corresponding to $\rho(A)$.*

We call an $n \times n$ nonnegative matrix $A = (a_{ij})$, all of whose row sums d_1, $d_2, \dots, d_n$ are positive, *almost regular*, if there is a positive number r such that, if $a_{ij} > 0$, $d_i d_j = r^2$. Note that [2] and [3] deal with only symmetric matrices, while we consider general nonnegative matrices which are not necessarily symmetric.

We will prove the following theorem.

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THEOREM 2. *Let $A = (a_{ij})$ be an $n \times n$ nonnegative matrix with positive row sums $d_1, d_2, \ldots, d_n$. Then*

$$\min_{1\le i\le n} \frac{d_i}{\sqrt{\sum_{j=1}^{n} \frac{a_{ij}}{d_j}}} \le \min_{1\le i\le n} \frac{\sum_{j=1}^{n} a_{ij}\sqrt{d_j}}{\sqrt{d_i}} \le \rho(A)$$

$$\le \max_{1\le i\le n} \frac{\sum_{j=1}^{n} a_{ij}\sqrt{d_j}}{\sqrt{d_i}} \le \max_{1\le i\le n} \sqrt{\sum_{j=1}^{n} a_{ij} d_j}\,. \tag{2}$$

Moreover, if A is irreducible, then $\rho(A)$ is equal to any of the four items in (2) if and only if A is almost regular.

Let $D = (V, E)$ be a digraph with vertex set $V = \{1, 2, \ldots, n\}$ and arc set E. Loops are permitted but no multiple arcs. For each $u \in V$, the outdegree of u is $d_u = \left|\{v : (u, v) \in E\}\right|$. The adjacency matrix of D is the $n \times n$ matrix $A(D) = (a_{ij})$, where $a_{ij} = 1$ if $(i, j) \in E$ and 0 otherwise. It is well known that D is strongly connected if and only if $A(D)$ is irreducible. The spectral radius of D, denoted by $\rho(D)$, is defined to be the spectral radius of $A(D)$.

An immediate corollary of Theorem 2 is given as follows.

COROLLARY 1. *Let D be a digraph of order n with positive outdegree sequence $d_1, d_2, \ldots, d_n$. Then*

$$\min_{1\le i\le n} \frac{d_i}{\sqrt{\sum_{j:(i,j)\in E} \frac{1}{d_j}}} \le \min_{1\le i\le n} \frac{\sum_{j:(i,j)\in E} \sqrt{d_j}}{\sqrt{d_i}} \le \rho(D)$$

$$\le \max_{1\le i\le n} \frac{\sum_{j:(i,j)\in E} \sqrt{d_j}}{\sqrt{d_i}} \le \max_{1\le i\le n} \sqrt{\sum_{j:(i,j)\in E} d_j}\,. \tag{3}$$

Moreover, if D is strongly connected, then $\rho(D)$ is equal to any of the four items in (3) if and only if there is a positive number r such that whenever $(i, j) \in E$, $d_i d_j = r^2$.

2. Proof of Theorem 2

Setting $x = \left(\sqrt{d_1}, \sqrt{d_2}, \ldots, \sqrt{d_n}\right)^t$ in (1), we obtain

$$\min_{1\le i\le n} \frac{\sum_{j=1}^{n} a_{ij}\sqrt{d_j}}{\sqrt{d_i}} \le \rho(A) \le \max_{1\le i\le n} \frac{\sum_{j=1}^{n} a_{ij}\sqrt{d_j}}{\sqrt{d_i}}. \tag{4}$$

Observe that, for positive z, $\frac{1}{z^2}$ is a strictly convex function. Hence, if $\alpha_1, \alpha_2, \ldots, \alpha_t$ are positive numbers summing to 1, and $z_1, z_2, \ldots, z_t$ are positive numbers,

$$\left(\frac{1}{\sum_{k=1}^{t} \alpha_k z_k}\right)^2 \le \sum_{k=1}^{t} \alpha_k \frac{1}{z_k^2},$$

i.e.,

$$\left(\sum_{k=1}^{t} \alpha_k z_k\right)^2 \ge \frac{1}{\sum_{k=1}^{t} \alpha_k \frac{1}{z_k^2}}, \tag{5}$$

with equality if and only if all z_k are equal. For $a_{ij} > 0$, let $\alpha_k = \frac{a_{ij}}{d_i}$, $z_k = \sqrt{d_i d_j}$; from (5) we have

$$\left(\frac{\sum_{j=1}^{n} a_{ij}\sqrt{d_i d_j}}{d_i}\right)^2 \ge \frac{d_i^2}{\sum_{j=1}^{n} \frac{a_{ij}}{d_j}}. \tag{6}$$

Similarly, note that z^2 is a strictly convex function. Then, if $\alpha_1, \alpha_2, \ldots, \alpha_t$ are positive numbers summing to 1, and $z_1, z_2, \ldots, z_t$ are positive numbers,

$$\left(\sum_{k=1}^{t} \alpha_k z_k\right)^2 \le \sum_{k=1}^{t} \alpha_k z_k^2, \tag{7}$$

with equality if and only if all z_k are equal. For $a_{ij} > 0$, let $\alpha_k = \frac{a_{ij}}{d_i}$, $z_k = \sqrt{d_i d_j}$; from (7) we have

$$\left(\frac{\sum_{j=1}^{n} a_{ij}\sqrt{d_i d_j}}{d_i}\right)^2 \le \sum_{j=1}^{n} a_{ij} d_j. \tag{8}$$

Combining (4), (6) and (8), we have (2).

In the following, we suppose A is irreducible. To prove the second part of Theorem 2, note that by the stipulation of the equality cases in (5) and (7), we need prove that any equality in (4) holds if and only if A is almost regular.

If A is almost regular, then for some $r > 0$, $d_i d_j = r^2$ for every (i,j) such that $a_{ij} > 0$. Then for any $i = 1, 2, \ldots, n$ we have

$$\sum_{j=1}^{n} a_{ij}\sqrt{d_j} = \sum_{j=1,\, a_{ij}>0}^{n} a_{ij}\sqrt{\frac{r^2}{d_i}} = \frac{r}{\sqrt{d_i}} \sum_{j=1,\, a_{ij}>0}^{n} a_{ij} = \frac{r}{\sqrt{d_i}} d_i = r\sqrt{d_i}\,,$$

i.e.,

$$A\left(\sqrt{d_1}, \sqrt{d_2}, \ldots, \sqrt{d_n}\right)^t = r\left(\sqrt{d_1}, \sqrt{d_2}, \ldots, \sqrt{d_n}\right)^t.$$

It follows that $\left(\sqrt{d_1}, \sqrt{d_2}, \ldots, \sqrt{d_n}\right)^t$ is a positive eigenvector of A corresponding the eigenvalue r. Note that for any positive eigenvector of a nonnegative matrix, the corresponding eigenvalue is the spectral radius of that matrix. Hence $\rho(A) = r$ and the equality in (4) holds.

Conversely, suppose some equality in (4) holds. Then

$$A\left(\sqrt{d_1}, \sqrt{d_2}, \ldots, \sqrt{d_n}\right)^t = \rho(A)\left(\sqrt{d_1}, \sqrt{d_2}, \ldots, \sqrt{d_n}\right)^t,$$

i.e.,

$$\rho(A)\sqrt{d_i} = \sum_{j=1}^{n} a_{ij}\sqrt{d_j} \qquad \text{for all} \quad i\,. \tag{9}$$

We are going to show A is almost regular. If all d_i are equal, we are done. Otherwise, let $\delta = \min_{1\le i\le n} d_i$ and $\Delta = \max_{1\le i\le n} d_i$. Choose k and l such that $d_k = \delta$ and $d_l = \Delta$.

Claim 1. *If $a_{km} > 0$, then $d_m = \Delta$.*

P r o o f. Otherwise suppose $d_m < \Delta$; from (9) we have

$$\rho(A) = \sum_{j=1}^{n} a_{kj}\sqrt{\frac{d_j}{d_k}} < \sum_{j=1}^{n} a_{kj}\sqrt{\frac{\Delta}{\delta}} = \sqrt{\delta\Delta}\,.$$

However, from (9) we also have

$$\rho(A) = \sum_{j=1}^{n} a_{lj}\sqrt{\frac{d_j}{d_l}} \ge \sum_{j=1}^{n} a_{lj}\sqrt{\frac{\delta}{\Delta}} = \sqrt{\delta\Delta}\,,$$

which is a contradiction. Thus Claim 1 holds. □

By similar argument as in Claim 1, we have

CLAIM 2. *If* $a_{lm} > 0$, *then* $d_m = \delta$.

Note that A is irreducible. For any (i,j) with $a_{ij} > 0$, there exist $i_1, i_2, \dots$ $\dots, i_{t-1}, i_t$ such that $a_{i_1 i_2} \cdots a_{i_{t-1} i_t} a_{ij} > 0$ with $i_1 = k$ and $i_t = i$ for some $1 \le t \le n$, where $i_r \ne i_{r+1}$ for $1 \le r \le t-1$. Then Claims 1 and 2 imply that whenever $a_{ij} > 0$, then $d_i = \delta$ and $d_j = \Delta$ or vice versa. Hence whenever $a_{ij} > 0$, $d_i d_j = \delta\Delta$, and A is almost regular.

REFERENCES

[1] BERMAN, A.—PLEMMONS, R. J.: *Nonnegative Matrices in the Mathematical Sciences*, Academic Press, New York, 1979.

[2] ZHOU BO—LIU BOLIAN: *On almost regular matrices*, Utilitas Math. **54** (1998), 151–155.

[3] HOFFMAN, A. J.—WOLFE, P.—HOFMEISTER, M.: *A note on almost regular matrices*, Linear Algebra Appl. **226-228** (1995), 105–108.

[4] MINC, H.: *Nonnegative Matrices*, Academic Press, New York, 1988.

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