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ON GENERATING AND CONCRETENESS IN QUANTUM LOGICS

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ABSTRACT. States on quantum logics are fully determined by their values on the set of generators. Finitely generated concrete logics are finite. There exists an infinite logic with a finite set of generators, the sublogics of which are only concrete logics.

We show that states on quantum logics are fully determined by their values on the set of generators. Finitely generated concrete (= set representable) logics are finite. However, we construct an infinite logic with a finite set of generators, the sublogics of which are only concrete logics.

A (*quantum*) *logic* is a partially ordered set L with the least and the greatest elements, 0 and 1, and with the orthocomplementation $': L \to L$ such that for any $a, b \in L$

(i) $a \leq b \Leftrightarrow b' \leq a'$,
(ii) $(a')' = a$,
(iii) if $a \leq b'$, then $a \vee b$ exists in L,
(iv) if $a \leq b$, then $b = a \vee (b \wedge a')$.

A subset M of a logic L is called a *sublogic* if

(i) $0 \in M$,
(ii) $a \in M \Rightarrow a' \in M$,
(iii) $a, b \in M$ and $a \leq b'$ (in L) $\Rightarrow a \vee b \in M$.

A *block* of L is a maximal Boolean subalgebra of L. Two elements $a, b \in L$ are said to be *compatible* if there are three elements $c, d, e \in L$ such that $c \leq e'$, $d \leq e'$, $c \leq d'$ and $a = c \vee e$, $b = d \vee e$.

A *state* s on L is a mapping $s: L \to \langle 0, 1 \rangle$ such that $s(1) = 1$, and $s(a \vee b) = s(a) + s(b)$ provided $a \leq b'$. A state is said to be *two-valued* if $s(L) = \{0, 1\}$. Let us denote by $S(L)$ the set of all states on L and by $S_2(L)$ the set of all two-valued states on L.

A logic L need not have a set representation (consider the lattice of all projectors in a Hilbert space). When it has, we call it *concrete*. There is a simple characterization of concrete logics.

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Proposition 1. *A logic L is concrete if and only if for every pair $a, b \in L$, $a \not\leq b$ there exists a two-valued state s such that $s(a) = 1$ and $s(b) = 0$.*

P r o o f. Follows the standard Boolean set-representation technique — see e.g. [3] or [6].

Let us now state our results.

Proposition 2. *Let L be a logic and let G be a subset of L. Then there exists the least sublogic of L containing G.*

P r o o f. The intersection of sublogics is again a sublogic.

Definition 3. *If $G \subset L$, then the least sublogic of L containing G is said to be the logic generated by G.*

Proposition 4. *Let P be the logic generated by the set $G \subset L$. Suppose $s, t \in S(P)$ and $s(g) = t(g)$ for each $g \in G$. Then $s(a) = t(a)$ for every $a \in P$.*

P r o o f. Denote $L_1 = \{a \in P \mid \text{if } s, t \in S(P) \text{ and } s(g) = t(g) \text{ for each } g \in G, \text{ then } s(a) = t(a)\}$. Obviously, $G \subset L_1$. Since L_1 is a sublogic of P, the assertion follows from the minimality of P.

Proposition 5. *Let P be a concrete logic generated by a finite set G. Then P is finite.*

P r o o f. Suppose P is not finite. Then, as a consequence of Proposition 1, $S_2(P)$ is infinite. Contrariwise, due to Proposition 4, every state on P is fully determined by its values on the elements of the generating set S, and thus $\operatorname{card} S_2(L) \leq 2^{\operatorname{card} G}$.

It should be noticed that the generating in logics follows quite different patterns from the generating in orthomodular lattices ([1], [4]). Our final example shows some properties of generating in logics, which may be rather unexpected.

E x a m p l e 6. By means of the *Greechie technique* ([2], [5]) we construct a logic L with the following properties:

(i) L is infinite,
(ii) L is finit ly gen rated
(iii) L is not oncrete
(iv) any propcr ubl 'c cf L i concrete.

Consıder the hypergr ph in Fıg. 1.

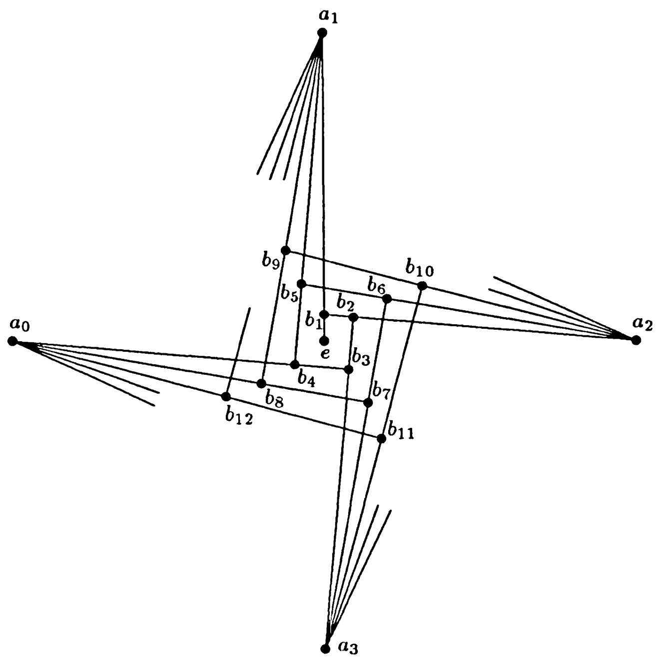

Fig. 1

Since it contains no "loops" of order 3, it is a representation of a logic L. Obviously, L is infinite. Let P be the logic generated by the set $G = \{a_0, a_1, a_2, a_3, e\}$. Then $b_1 = (e \vee a_1)'$ is an element of P, and, by an induction, also $b_{i+1} = (b_i \vee a_{i(\mathrm{mod}\,4)})'$ are elements of P. Hence $P = L$. We have shown that L is an infinite logic with a finite set of generators. According to Proposition 5, L is not concrete.

Let us investigate proper sublogics of L. When a sublogic contains all elements of the set G, it is the whole logic L. It follows that there is only a limited number of sublogics of L. In fact, there are six classes of them:

(C1) a "horizontal sum" ([3], [5]) of a finite or countable set of Boolean algebras 2^2;
(C2) the logic depicted in Fig. 2a) or its sublogic containing only a finite number of blocks;
(C3) a horizontal sum of a logic from the class (C1) and a logic from the class (C2);
(C4) a horizontal sum of two logics from the class (C2);
(C5) the logic depicted in Fig. 2b) or its sublogic containing a finite number of pairs of blocks;
(C6) the logic depicted in Fig. 2c) or its sublogic containing a finite number of triads of blocks.

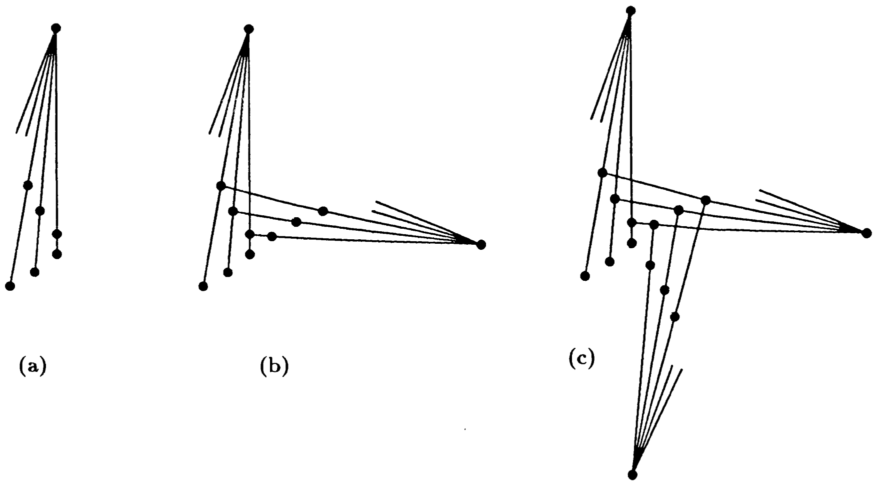

Fig. 2

It is an easy consequence of Proposition 1 that all these sublogics are concrete. As an example we construct a "separating" family of states for the class (C6). We denote the atoms in agreement with Fig.1 (the element a_0 is omitted). A state is fully described by its evaluations on all atoms. For the reason of a simpler description we denote $e = b_0$. Now the two-valued states required by Proposition 1 are given in Tab.1.

Table 1. "Separating" states for the class (C6).

	a_1	a_2	a_3	b_{4j+1}	b_{4j+2}	b_{4j+3}	b_{4j}
s_1	1	1	1	0	0	0	0
s_2	1	0	0	0	1	0	0
s_3	1	1	0	0	0	1	0
s_4	0	1	1	0	0	0	1
s_5	0	0	1	1	0	0	0
s_6	0	0	0	1	0	1	0
s_7	0	0	0	0	1	0	1
t_i	0	0	0	δ_{ij}	$1-\delta_{ij}$	δ_{ij}	$1-\delta_{ij}$

$i = 0, 1, 2, 3, \dots$

$j = 0, 1, 2, 3, \dots$

$\delta_{i,j} = 1$ if $i = j$, $\delta_{ij} = 0$ otherwise

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