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ALMOST λ-CONVEX AND ALMOST WRIGHT-CONVEX FUNCTIONS

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ABSTRACT. Let X be a linear space, Δ be a nonempty convex subset of X and λ be a fixed number in $(0,1)$. It is shown that if a function $f\colon \Delta \to \mathbb{R}$ is almost λ-convex, then there exists a λ-convex function g which is equal to f almost everywhere. It generalizes the classical result of Kuczma obtained for Jensen-convex functions. A similar result for Wright-convex function is also proved.

1. Introduction

This paper is devoted to almost λ-convex and almost Wright-convex functions. This subject is related to the following problem raised by P. Erdös [2] in 1960: Suppose that a function $f\colon \mathbb{R} \to \mathbb{R}$ satisfies the additivity relation

$$f(x+y) = f(x) + f(y)$$

for almost all pairs $(x,y) \in \mathbb{R}^2$. Does there exist an additive function g such that $f(x) = g(x)$ almost everywhere in $\mathbb{R}$?

An answer to this question in the affirmative was given by N. G. de Bruijn [1] and W. B. Jurkat [4]. An analogous theorem for Jensen-convex function was obtained by M. Kuczma [6] (cf. also [7] and the references given there) and for convex function by R. Ger [3]. Set valued versions of these theorems are present by E. Sadowska in [10]. In this paper we prove similar results for λ-convex and Wright-convex functions.

Let X be a nonempty set and 2^X denote the family of all subsets of X. A nonempty subfamily $\Im \subset 2^X$ is called *σ-ideal* if and only if it satisfies the

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conditions

$$(A \in \Im \ \wedge \ B \subset A) \implies (B \in \Im),$$

and

$$(\forall n \in \mathbb{N})\,(A_n \in \Im) \implies \bigcup_{n \in N} A_n \in \Im .$$

If additionally $X \notin \Im$, then the σ-ideal $\Im$ is called *proper.*

We say that a σ-ideal $\Im$ defined on a linear space X is *linearly invariant,* if $x - A \in \Im$ for every $x \in X$ and $A \in \Im$. The name *proper linearly invariant σ-ideal* will be abbreviated to p.l.i. σ-ideal in the sequel.

We will say that a *property $P(x)$ is satisfied $\Im$-almost everywhere in X* ($\Im$-(a.e.) in X) if and only if there exists a set $A \in \Im$ such that $P(x)$ holds true for all $x \in X \setminus A$.

Let $\Im_1$ be a σ-ideal in X and $\Im_2$ be a σ-ideal in $X \times X$. We say that the σ-ideals $\Im_1$ and $\Im_2$ are *conjugate* if for every set $A \in \Im_2$ there exists a set $B \in \Im_1$ such that for every $x \in X \setminus B$ the x-section of A (i.e. the set $A_x = \{y \in X : \ (x,y) \in A\}$) belongs to $\Im_1$. Given a p.l.i. σ-ideal $\Im$ we define the family

$$\Omega(\Im) := \{M \subset X^2 : \ \{x \in X : \ M_x \notin \Im\} \in \Im\} .$$

It forms the largest p.l.i. σ-ideal conjugate with $\Im$.

2. Almost λ-convex functions

In what follows X be a (real) linear space and $\Im_1$, $\Im_2$ be two conjugate p.l.i. σ-ideals in X and X^2, respectively. Assume further that

(i) $A \in \Im_1$ implies $\lambda A \in \Im_1$ for all $\lambda \in \mathbb{R}$,

(ii) $\Im_2$ is invariant with respect to the transformations $T \colon X^2 \to X^2$ and $L \colon X^2 \to X^2$ given by formulas

$$T(x,y) := \big(\lambda x + (1-\lambda)y,\, y - x\big), \qquad (x,y) \in X^2 ,$$
$$L(x,y) := (y,x), \qquad (x,y) \in X^2 ,$$

(iii) Δ is a nonempty convex subset of X such that for every $\lambda \in (0,1)$ the set

$$\Delta(x) := \frac{1}{1-\lambda}(x - \Delta) \cap \frac{1}{\lambda}(\Delta - x) \notin \Im_1 , \qquad x \in \Delta .$$

Fix a number $\lambda \in (0,1)$. We say that a function $f \colon \Delta \to \mathbb{R}$ is:

- λ*-convex* if

$$f\big(\lambda x + (1-\lambda)y\big) \le \lambda f(x) + (1-\lambda) f(y) , \qquad (x,y) \in \Delta^2 ,$$

- $\Im_2$-*(a.e.) λ-convex* if

  $$f(\lambda x + (1-\lambda)y) \le \lambda f(x) + (1-\lambda)f(y)\,, \qquad (x,y) \in \Delta^2 \setminus M\,,$$

  for some $M \in \Im_2$.

The main result in this section generalizes the theorem of Kuczma [6], [7] proved for Jensen-convex (i.e. $\frac{1}{2}$-convex) functions. It reads as follows:

THEOREM 1. *Let X be a linear space, $\Im_1$, $\Im_2$ be two conjugate p.l.i. σ-ideals in X and X^2, respectively, and suppose the assumptions* (i)–(iii) *to be satisfied.*

If a function $f\colon \Delta \to \mathbb{R}$ is $\Im_2$-(a.e.) λ-convex, then there exists a λ-convex function $g\colon \Delta \to \mathbb{R}$ such that

$$f(x) = g(x) \qquad \Im_1\text{-(a.e.) in } \Delta\,.$$

The idea of the proof is similar as in [7] and [3]. We start with the following lemmas.

LEMMA 1. *Let X be a linear space, $\Im_1$, $\Im_2$ be two conjugate p.l.i. σ-ideals in X and X^2, respectively, and suppose the assumptions* (i)–(iii) *to be satisfied.*

If a function $f\colon \Delta \to \mathbb{R}$ is $\Im_2$-(a.e.) λ-convex, then the function $g\colon \Delta \to [-\infty,\infty)$ given by formula

$$g(x) := \Im_1 - \operatorname*{inf\,ess}_{h\in\Delta(x)}\big[\lambda f\big(x-(1-\lambda)h\big) + (1-\lambda)f(x+\lambda h)\big], \qquad x \in \Delta\,,$$

is well defined and $f(x) = g(x)$ $\Im_1$-(a.e.) in Δ.

LEMMA 2. *Under the assumptions and denotations of Lemma* 1

$$g(x) = \Im_1 - \operatorname*{inf\,ess}_{h\in\Delta(x)}\big[\lambda g\big(x-(1-\lambda)h\big) + (1-\lambda)g(x+\lambda h)\big]\,, \qquad x \in \Delta\,.$$

Moreover, the function g is $\Omega(\Im_1)$-(a.e.) λ-convex.

LEMMA 3. *Under the assumptions and denotations of Lemma* 1, *the function g is λ-convex.*

The proofs of the above lemmas are natural modifications of the corresponding lemmas given in [7] and [3]. Therefore we omit them. Lemmas 1 and 3 are used in the proof of Theorem 1 below. Lemma 2 is technical and is needed only to prove Lemma 3.

Proof of Theorem 1. We define the function $g\colon \Delta \to [-\infty,\infty)$ by the formula

$$g(x) := \Im_1 - \operatorname*{inf\,ess}_{h\in\Delta(x)}\big[\lambda f\big(x-(1-\lambda)h\big) + (1-\lambda)f(x+\lambda h)\big]\,, \qquad x \in \Delta\,.$$

By Lemma 3, function g is λ-convex and by Lemma 1, $f(x) = g(x)$ $\Im_1$-(a.e.) in Δ. □

3. Almost Wright-convex functions

We say that a function $f\colon \Delta \to \mathbb{R}$ is:

- *Wright-convex* if

$$f\big(\lambda x + (1-\lambda)y\big) + f\big((1-\lambda)x + \lambda y\big) \le f(x) + f(y)\,,$$

for every $x, y \in \Delta$ and $\lambda \in [0,1]$,

- $\Im_2$-*(a.e.) Wright-convex* if for any $\lambda \in [0,1]$ there exists a set $M(\lambda) \in \Im_2$ such that

$$f\big(\lambda x + (1-\lambda)y\big) + f\big((1-\lambda)x + \lambda y\big) \le f(x) + f(y)\,, \qquad (x,y) \in \Delta^2 \setminus M(\lambda)\,.$$

THEOREM 2. *Let X be a linear space, $\Im_1$, $\Im_2$ be two conjugate p.l.i. σ-ideals in X and X^2, respectively, and suppose the assumptions* (i'), (ii), (iii) (*cf.* [3; p. 65]) *to be satisfied.*

If a function $f\colon \Delta \to \mathbb{R}$ is $\Im_2$-(a.e.) Wright-convex, then there exist a Wright-convex function $g\colon \Delta \to \mathbb{R}$ such that

$$f(x) = g(x) \qquad \Im_1\text{-(a.e.) in } \Delta\,.$$

In the proof of the above theorem we use the following lemma.

LEMMA 4. *Let X be a linear space, $\Im_1$, $\Im_2$ be two conjugate p.l.i. σ-ideals in X and X^2, respectively, and suppose the assumptions* (i'), (ii), (iii) (*cf.* [3; p. 65]) *to be satisfied.*

If a function $f\colon \Delta \to \mathbb{R}$ is $\frac{1}{2}$-convex and $\Im_2$-(a.e.) Wright-convex, then it is Wright-convex.

Proof. Fix $x \in \Delta$ and $\alpha(x) > f(x)$, and put

$$B(x) := \Big\{h \in \Delta(x) :\ \tfrac{f(x-h)+f(x+h)}{2} < \alpha(x)\Big\}\,.$$

The set $B(x)$ does not belong to $\Im_1$. Indeed, otherwise

$$\begin{aligned}\Im_1 - \operatorname*{inf\,ess}_{h\in\Delta(x)} \frac{f(x-h)+f(x+h)}{2} &\ge \inf_{h\in\Delta(x)\setminus B(x)} \frac{f(x-h)+f(x+h)}{2}\\ &\ge \alpha(x) > f(x)\,,\end{aligned}$$

which is impossible (cf. [7; p. 454, Lemma 6]).

Fix $(x,y) \in \Delta^2$ and $\lambda \in [0,1]$, and put $z := \lambda x + (1-\lambda)y$. Let $M(\lambda) \in \Im_2$ be a set such that

$$f\big(\lambda x + (1-\lambda)y\big) + f\big((1-\lambda)x + \lambda y\big) \le f(x) + f(y)\,, \qquad (x,y) \in \Delta^2 \setminus M(\lambda)\,,$$

and $U(\lambda) \in \Im_1$ be a set associated with $M(\lambda)$ according to the conjugacy relation.

Choose

$$h \in B(x) \setminus \big[(U(\lambda) - x) \cup (x - U(\lambda))\big]$$

and

$$k \in B(y) \setminus \Big[\Big(\big[M(\lambda)\big]_{x+h} - y\Big) \cup \Big(y - \big[M(\lambda)\big]_{x-h}\Big)\Big]$$

(which is possible because the sets $B(x)$ and $B(y)$ do not belong to $\Im_1$).

Then, in particular,

$$(x+h, y+k) \notin M(\lambda) \qquad \text{and} \qquad (x-h, y-k) \notin M(\lambda)\,.$$

Let $l := \lambda h + (1-\lambda)k$. Observe that $z - l \in \Delta$ and $z + l \in \Delta$. Similarly, putting $z' := (1-\lambda)x + \lambda y$, $l' := (1-\lambda)h + \lambda k$, we have that the elements $z' - l'$ and $z' + l'$ belong to Δ, too. Finally

$$\begin{aligned}
&f\big(\lambda x + (1-\lambda)y\big) + f\big((1-\lambda)x + \lambda y\big)\\
&\quad\le f\Big(\frac{z-l+z+l}{2}\Big) + f\Big(\frac{z'-l'+z'+l'}{2}\Big)\\
&\quad\le \frac{1}{2}\big[f(z-l) + f(z+l) + f(z'-l') + f(z'+l')\big]\\
&\quad= \frac{1}{2}\big[f\big(\lambda(x-h) + (1-\lambda)(y-k)\big) + f\big(\lambda(x+h) + (1-\lambda)(y+k)\big)\big]\\
&\qquad\quad + \frac{1}{2}\big[f\big((1-\lambda)(x-h) + \lambda(y-k)\big) + f\big((1-\lambda)(x+h) + \lambda(y+k)\big)\big]\\
&\quad\le \frac{f(x-h) + f(x+h)}{2} + \frac{f(y-k) + f(y+k)}{2}\\
&\quad< \alpha(x) + \alpha(y)\,.
\end{aligned}$$

Letting $\alpha(x)$ and $\alpha(y)$ tend to $f(x)$ and $f(y)$, respectively, we get the inequality

$$f\big(\lambda x + (1-\lambda)y\big) + f\big((1-\lambda)x + \lambda y\big) \le f(x) + f(y)$$

valid for all $x, y \in \Delta$ and all $\lambda \in [0,1]$. This finishes the proof. □

P r o o f o f T h e o r e m 2 . Assume that the function f is $\Im_2$-(a.e.) Wright-convex. Hence, in particular, it is $\Im_2$-(a.e.) $\frac{1}{2}$-convex. From Theorem 1 there exists a $\frac{1}{2}$-convex function $g\colon \Delta \to \mathbb{R}$ equal to the function f $\Im_1$-(a.e.).

In light of the Lemma 4 it is enough to show that the function g is $\bar{\Im}_2$-(a.e.) Wright-convex, where $\bar{\Im}_2$ is a p.l.i. σ-ideal in X^2 conjugate with $\Im_1$.

Let $S \in \Im_1$ be a set such that $f(x) = g(x)$ for all $x \in \Delta \setminus S$. Fix $\lambda \in [0,1]$ and put

$$\begin{aligned}
K(\lambda) &:= \big\{(x,y) \in \Delta^2 :\ \lambda x + (1-\lambda)y \in S\big\}\,,\\
K'(\lambda) &:= \big\{(x,y) \in \Delta^2 :\ (1-\lambda)x + \lambda y \in S\big\}\,,\\
N(\lambda) &:= M(\lambda) \cup (S \times X) \cup (X \times S) \cup K(\lambda) \cup K'(\lambda)\,.
\end{aligned}$$

Obviously, the set $N(\lambda)$ belongs to $\Omega(\Im_1)$. Take $(x,y) \in \Delta^2 \setminus N(\lambda)$. Then $(x,y) \notin M(\lambda)$, $x \notin S$, $y \notin S$, $\lambda x + (1-\lambda)y \notin S$ and $(1-\lambda)x + \lambda y \notin S$. Henceforward

$$\begin{aligned} g\big(\lambda x + (1-\lambda)y\big) + g\big((1-\lambda)x + \lambda y\big) &= f\big(\lambda x + (1-\lambda)y\big) + f\big((1-\lambda)x + \lambda y\big) \\ &\le f(x) + f(y) = g(x) + g(y)\,. \end{aligned}$$

Now, using Lemma 4, we have that the function g is Wright-convex, which ends the proof. □

In [8], C. T. Ng proved that each Wright-convex function defined on a convex subset of $\mathbb{R}^n$ is the sum of an additive function and a convex function (cf. also [9]). Z. Kominek [5] extended this result to functions defined on algebraically open and convex subsets of a linear space. Using this characterization we get the following result.

COROLLARY. *Let Δ be an algebraically open and convex subset of a real linear space X, $\Im_1$, $\Im_2$ be two conjugate p.l.i. σ-ideals in X and X^2, respectively, and suppose the assumptions* (i'), (ii), (iii) (*cf.* [3; p. 65]) *to be satisfied. If a function $f\colon \Delta \to \mathbb{R}$ is $\Im_2$-(a.e.) Wright-convex, then there exist a convex function $h\colon \Delta \to \mathbb{R}$ and an additive function $a\colon \Delta \to \mathbb{R}$ such that*

$$f(x) = h(x) + a(x) \qquad \Im_1\text{-}(a.e.) \text{ in } \Delta\,.$$

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