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# A CONSTRUCTION OF GEODETIC BLOCKS

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## 1. Introduction

Geodetic graphs were first defined by Ore [5] as graphs in which every pair of vertices is connected by a unique shortest path. Since a graph is geodetic iff each of its blocks is geodetic (see Stemple and Watkins [13]), it is sufficient to study the geodetic blocks only. The geodetic blocks of diameter two have been studied by Lee [4], Stemple [11], Zelinka [14]. For geodetic blocks of higher diameters there are available some general constructions only (see for example [1, 2, 6, 7, 8, 9, 12, 15]).

In this paper we present one construction of new geodetic blocks $\bar{G}$ or $\bar{G}(s)$ from a known geodetic block $G$ if a geodetic block $G$ can be decomposed into two edge-disjoint geodetic subgraphs $G_1$, $G_2$ with two special properties. This construction consists of replacing certain vertices by new edges. The construction unifies the construction described by Bosák [1] ($g(m, s)$-graphs), Plesník [8] $WPd$-graphs), and Stemple [12] ($K_n^i$-graphs). We used this construction to study a special class of geodetic block which are homeomorphic to $\mathscr{G}(p+1, 2p)$-graphs (see [4]), namely to study $G(p, 2+s)$-graphs.

## 2. Definitions and preliminary results

We use the general notation and terminology of Harary [3]. The graphs considered are simple undirected graphs. If $G$ is a graph, then $V(G)$ and $E(G)$ denote its *vertex set* and *edge set*, respectively. The *distance* between vertices $u$, $v \in V(G)$ is denoted by $\varrho_G(u, v)$. A shortest $u-v$ *path* in $G$ is called a *geodesic* and it is denoted by $\Gamma_G[u, v]$. Any subpath of a geodesic is also a geodesic. If $S$ is a path, $|S|$ will mean the length of $S$. Clearly, if $\Gamma_G[u, v]$ exists, then $\varrho_G(u, v) = |\Gamma_G[u, v]|$. The supremum of all distances in $G$ is the *diameter* of $G$, $d(G)$. If $v \in V(G)$, then we put $V_G^i(v) = \{u \in V(G) | \varrho_G(u, v) = i\}$.

A *clique* is defined as a maximal complete subgraph $U_k$ of order $k \geq 3$, that is, a complete subgraph on at least three vertices which is contained in no larger complete subgraph.

**Theorem A** (see Stemple [11, Theorem 5.5]). *If $G$ is a geodetic block of diameter two and $U_k$, $U_j$ are cliques of $G$, then $k=j$.*

Now, if $G$ is a geodetic block of diameter two and $G$ contains a clique $U_k$, we call $k$ the clique size of $G$. If $G$ contains no clique, we let $k=2$ be the clique size.

**Theorem B** (see Stemple [11, Theorem 5.11]). *Let $G$ have clique size $k \geqslant 3$ and assume that $G$ contains a clique $H$ with the property that for each vertex $v_i \in V(H)$, $i=1, 2, \ldots, k$; there exists a clique $H_i$, where*

$$V(H) \cap V(H_i) = \{v_i\} \quad \text{and} \quad V_G^1(v_i) \subseteq V(H) \cup V(H_i).$$

*Then $G_1 = G - H$ is geodetic of diameter two with clique size $k-1$. If $G_1$ contains cliques (i.e., $k \geqslant 4$), then each clique in $G_1$ is at distance two from every other clique.*

## 3. The construction of $\bar{G}$ and $\bar{G}(s)$

By a *decomposition* of a graph $G$ we mean a set of edge-disjoint subgraphs $G_1, G_2, \ldots, G_n$ of $G$ which together contain the set of edges of $G$; it is denoted by $(G_1, G_2, \ldots, G_n)$.

Let $G$ be a geodetic block and $G_1$, $G_2$ be its geodetic subgraphs (not necessarily blocks) which form a decomposition $(G_1, G_2)$ of $G$. Then $(G_1, G_2)$ is said to be a *g-decomposition* of $G$.

*We shall say that a g-decomposition $(G_1, G_2)$ of a geodetic block $G$ has the property $P(1)$ if for any two vertices $u$, $v \in V(G_1)$ $[u, v \in V(G_2)]$ every $u-v$ geodesic of $G$ belongs to $G_1$ [to $G_2$] with the exception of $u$, $v \in V(G_1) \cap V(G_2)$ where either $\Gamma_G[u, v] = \Gamma_{G_1}[u, v]$ or $\Gamma_G[u, v] = \Gamma_{G_2}[u, v]$. In other words, $G_1$ and $G_2$ are geodetically closed in $G$ with the exception of vertices of $V(G_1) \cap V(G_2)$.*

*Further, we say that $(G_1, G_2)$ has the property $P(2)$ if for any two vertices $u$, $v \in V(G_1) \cap V(G_2)$ we have*

$$|\Gamma_{G_1}[u, v]| - |\Gamma_{G_2}[u, v]| \equiv 1 \text{ (mod 2)}.$$

A *g-decomposition* with the properties $P(1)$ and $P(2)$ is called a *$\bar{g}$-decomposition.*

Let $(G_1, G_2)$ be a *$\bar{g}$-decomposition* of a geodetic block $G$. From $G$ we construct a graph $\bar{G}$ as follows. Let $v$ be any vertex from $V(G_1) \cap V(G_2)$. Then we replace $v$ by two vertices $v^1$, $v^2$ and join $v^1$ with $v^2$ by an edge. Further we join $v^1$ (or $v^2$) with each vertex of $V_{G_1}^1(v)$ ($V_{G_2}^1(v)$, respectively). We shall denote this construction by $G \to \bar{G}$ and we claim that $\bar{G}$ is geodetic.

In Fig. 1 we have illustrated the construction of $\bar{G}$ by taking $K_5$ as $G$ and two cycles $C_5$ as $G_1$, $G_2$ ($G_1 = [u, v, x, y, z, u]$, $G_2 = [u, x, z, v, y, u]$). It is obvious that $(G_1, G_2)$ is a *$\bar{g}$-decomposition* of $K_5$ and $\bar{G}$ is the Petersen graph.

**Lemma 1.** *Let* $x \in V(G_1)$, $y \in V(G_2)$ *and* $x, y \notin V(G_1) \cap V(G_2)$. *Let* $(G_1, G_2)$ *be a* $\bar{g}$*-decomposition of a geodetic block* $G$. *Then there exists exactly one vertex* $v \in V(G_1) \cap V(G_2)$ *with*

$$\Gamma_G[x, y] = \Gamma_G[x, v] + \Gamma_G[v, y]$$

*and*

$$\Gamma_G[x, v] \subset G_1, \quad \Gamma_G[v, y] \subset G_2.$$

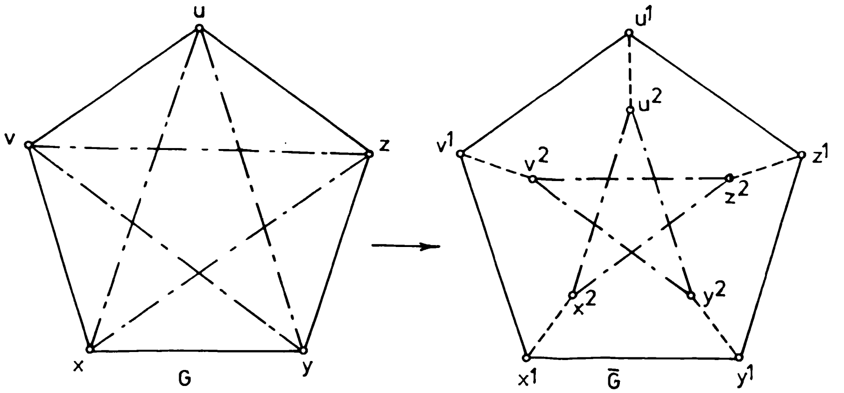



Fig. 1

Proof. Let $\Gamma_G[x, y] = [x = v_0, e_1, v_1, \ldots, e_n, v_n = y]$. Let $v_i$, $[v_j]$ be the first [last] vertex of $\Gamma_G[x, y]$ which is in $V(G_1) \cap V(G_2)$, too. From the property $P(1)$ we have:

$$\Gamma_G[v_i, v_j] = \Gamma_{G_1}[v_i, v_j] \subset G_1 \quad \text{or} \quad \Gamma_G[v_i, v_j] = \Gamma_{G_2}[v_i, v_j] \subset G_2.$$

Let $\Gamma_G[v_i, v_j] = \Gamma_{G_1}[v_i, v_j]$; then $\Gamma_G[x, v_i] + \Gamma_G[v_i, v_j] = \Gamma_G[x, v_j] \subset G_1$ and $v_j$ is the desired vertex. If $\Gamma_G[v_i, v_j] = \Gamma_{G_2}[v_i, v_j] \subset G_2$, then we proceed similarly.

Q.E.D.

**Corollary 1.** *Let* $(G_1, G_2)$ *be a* $\bar{g}$*-decomposition of a geodetic block* $G$. *Let* $x \in V(G_1)$, $y \in V(G_2)$, $x, y \notin V(G_1) \cap V(G_2)$. *Let* $\Gamma_{\bar{G}}[y, y]$ *be a geodesic from* $x$ *to* $y$ *in the graph* $\bar{G}$. *Then there exists exactly one edge* $[v^1, v^2] \subset \Gamma_{\bar{G}}[x, y]$ *with* $[v^1, v^2] \notin E(G)$. ($[v^1, v^2]$ *is a new edge corresponding to a vertex* $v$).

**Theorem 1.** *If* $G$ *is a geodetic block, then* $\bar{G}$ *is a geodetic block, too.*

Proof. It is sufficient to prove that for any two distinct vertices $u$, $v$ of $\bar{G}$ there exists exactly one geodesic between them. Suppose, on the contrary, that there are two distinct geodesics $\bar{\Gamma}_1$, $\bar{\Gamma}_2$ between $u$ and $v$. We can suppose that $\bar{\Gamma}_1$ and $\bar{\Gamma}_2$ are internally disjoint (otherwise there are internally disjoint subpaths $P_1$ of $\bar{\Gamma}_1$ and $P_2$

of $\bar{\Gamma}_2$ and we can take $P_1$ and $P_2$ for $\bar{\Gamma}_1$ and $\bar{\Gamma}_2$, respectively). We shall consider the following cases:

*Case 1(a).* Both $u$ and $v$ belong to $V(G_1) - V(G_2)$. There cannot be two distinct shortest $u - v$ paths, because of the property $P(1)$.

*Case 1(b).* Both $u$ and $v$ belong to $V(G_2) - V(G_1)$. Then there cannot be two distinct shortest $u - v$ paths, because of the property $P(1)$.

*Case 2.* One of the vertices $u$ and $v$, say $v$, belongs to $V(G_1)[V(G_2)]$ and the other $u = w^i$, $i = 1, 2$. ($w^1$, $w^2$ are new vertices corresponding to a vertex $w$.) From the property $P(1)$ it follows that there cannot be two distinct shortest paths between $u$ and $w$. Hence, there cannot be two distinct $u - w^i$ geodesics.

*Case 3.* One of the vertices $u$ and $v$, say $v$, belongs to $V(G_1)$ and the other $u$ to $V(G_2)$, $u, v \notin V(G_1) \cap V(G_2)$. By *Corollary 1* both $\bar{\Gamma}_1$ and $\bar{\Gamma}_2$ contain exactly one new edge $e_1$ and $e_2$, respectively. From $|\bar{\Gamma}_1| = |\bar{\Gamma}_2|$ it follows for corresponding $u - v$ geodesics $\Gamma_1$ and $\Gamma_2$ in the graph $G$ that $|\Gamma_1| = |\bar{\Gamma}_1| - 1 = |\bar{\Gamma}_2| - 1 = |\Gamma_2|$. But this is not possible because $G$ is geodetic.

*Case 4.* Let $u = u^i$, $i \in \{1, 2\}$; $v = v^j$, $j \in \{1, 2\}$. Then by the property $P(2)$ $u^1$, $u^2$, $v^1$, $v^2$ belong to an odd cycle $C = \Gamma_{G_1}[u, v] + [v^1, v^2] + \Gamma_{G_2}[v, u] + [u^2, u^1]$. Hence the $u^i - v^j$ geodesic is the shorter part of $C$. It is obvious that there cannot be two distinct geodesics.

Q.E.D.

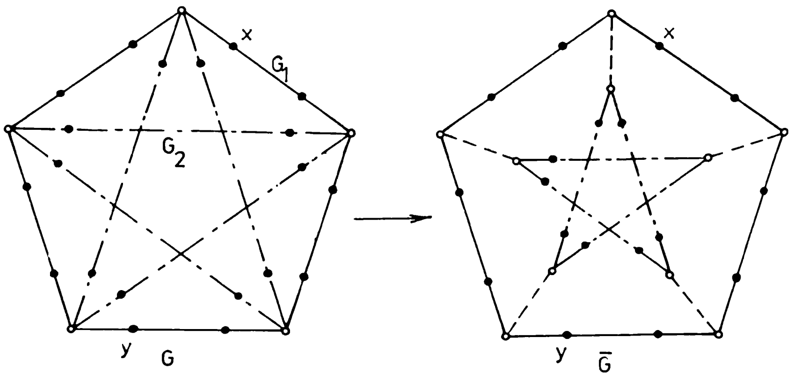



Fig. 2

Note 1. The property $P(1)$ cannot be omitted (see Fig. 2). There $G$ is a $K_5^i$-*graph* (see [12]) and $G_1$, $G_2$ are odd cycles which are presented differently. Both vertices $x$ and $y$ belong to $V(G_1)$ but the $x - y$ geodesic does not belong to $G_1$. Then there are two shortest paths between $x$ and $y$ in $\bar{G}$.

Note 2. The property $P(2)$ cannot be omitted (see Fig. 3). There $G$ is $K_4$ and $|\Gamma_{G_1}[v, w]| = 1$, $|\Gamma_{G_2}[v, w]| = 3$. $\bar{G}$ is not geodetic because there are two distinct shortest paths from $v^2$ to $w^2$.

Note 3. If $G$ is of diameter $d$, then the diameter of $\bar{G}$ is $d+1$ if there exists a pair of vertices $x, y \notin V(G_1) \cap V(G_2)$, $x \in V(G_1)$, $y \in V(G_2)$ with $\varrho_G(x, y) = d$; otherwise the diameter of $\bar{G}$ is $d$.

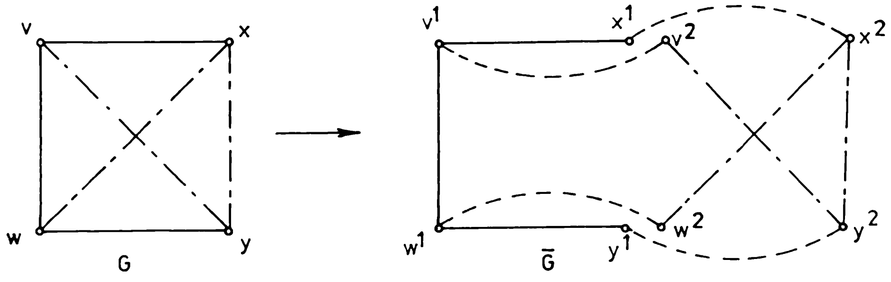



Fig. 3

**Problem.** *It would be interesting to find a geodetic block $G$ of diameter $d$ for which $d(\bar{G}) = d$.*

Note 4. The construction cannot be extended for a *$\bar{g}$-decomposition* with more than two subgraphs. A counterexample is in Fig. 4(c). Subgraphs $G_1$, $G_2$, $G_3$ are presented differently. There are two distinct shortest paths from $a$ to $u^2$.

Now, let $(G_1, G_2)$ be a *$\bar{g}$-decomposition* of a geodetic block $G$. From $G$ we shall construct a graph $\bar{G}(s)[G \to \bar{G}(s)]$ which is a generalization of the graph $\bar{G}$ described above and is obtained as follows: Every vertex $v \in V(G_1) \cap V(G_2)$ is replaced by a path of length $s$, that is $v \to P[v^1, \ldots, v^{s+1}]$ and $v^1$ [or $v^{s+1}$] is joined with each vertex of $V^1_{G_1}(v)$ [$V^1_{G_2}(v)$, respectively]. For an illustration, we have a graph $G$ and its graph $\bar{G}(s)$ in Fig. 4(a) and 4(b), respectively.

**Theorem 2.** *If $G$ is a geodetic block, then $\bar{G}(s)$ is a geodetic block, too.*

Proof. The proof is similar to that for $\bar{G}$. Let $u$, $v$ be two distinct vertices of $\bar{G}(s)$. We shall show that there is exactly one shortest $u-v$ path in $\bar{G}(s)$. We shall consider the following cases:

*Case 1.* Both $u$ and $v$ belong to $V(G_1) - V(G_2)$ [or $V(G_2) - V(G_1)$]. Then the assertion is obvious.

*Case 2.* One of the vertices $u$ and $v$, say $v$, belongs to $V(G_1)$ [$V(G_2)$] and the other $u = w^i$, $i \in \{1, 2, \ldots, s+1\}$. From the property $P(1)$ it follows that there cannot be two distinct shortest paths between $u$ and $w$ in $G$. Hence, there cannot be two distinct $u - w^i$ geodesics.

*Case 3.* One of the vertices $u$ and $v$, say $v$, belongs to $V(G_1)$ and the other $u$ to $V(G_2)$; $u, v \notin V(G_1) \cap V(G_2)$. From Lemma 1 it follows that there is exactly one new path $P[w^1, w^2, \ldots, w^{s+1}]$ which lies on the $u-v$ geodesic in $\bar{G}(s)$. Then the existence of two distinct $u-v$ geodesics in $\bar{G}(s)$ results in the existence of two distinct $u-v$ geodesics in $G$.

*Case 4.* Let $u=u^i$, $i\in\{1, 2, \dots, s+1\}$, $v=v^j$, $j\in\{1, 2, \dots, s+1\}$. Then by the property $P(2)u^i$, $v^j$ belong to the odd cycle

$$C=\Gamma_{G_1}[u^1, v^1]+P[v^1, \dots, v^{s+1}]+\Gamma_{G_2}[v^{s+1}, u^{s+1}]+P[u^{s+1}, \dots, u^1].$$

Hence the $u^i-v^j$ geodesic is the shorter part of $C$.

Q.E.D.

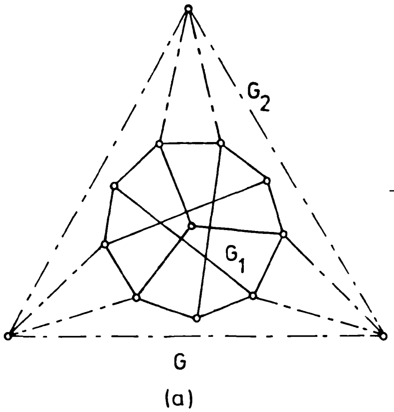



Fig. 4a

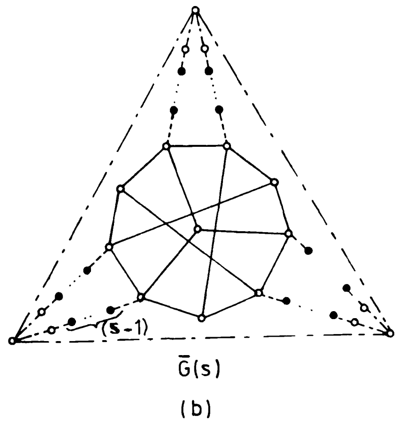



Fig. 4b

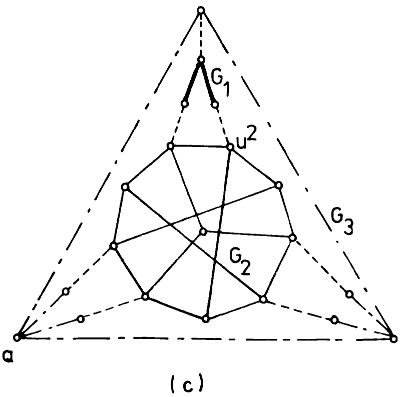



Fig. 4c

Note 5. If we take $G$ to be $K_5$ and $G_1$, $G_2$ are both $C_5$, then $\bar{G}(s)$ is the graph *WPd* of Plesník [8], where $d=s+1$, $s\geqslant 1$.

Note 6. If we take $G$ to be $K_{m+1}$ and $G_1=K_m$, $G_2=K_{1,m}$, then $\bar{G}(s)$ is the graph $g(m, s)$ of Bosák [1] and taking $G_2$ to be a homeomorph of $K_{1,m}$ successively over each vertex of $K_{m+1}$, then $\bar{G}(s(v))$ is the graph $K^i_{m+1}$ of Stemple [12], where $s = s(v)$ is a mapping from $V(K_{m+1})$ to the set of nonnegative integers.

Note 7. If $s \geq 2$, then $\bar{G}(s)$ must contain vertices of degree two, but if $s = 1$, then there exists a geodetic block $\bar{G}(1)$ without vertices of degree two. It is the Petersen graph in Fig. 1.

*Question. Is there a geodetic block $\bar{G}(1)$ without vertices of degree two different from the Petersen graph?*

It is obvious that if such a graph exists, then $G$ is without vertices of degree two and for each $v \in V(G_1) \cap V(G_2)$ both $deg_{G_1}\ v \geq 2$ and $deg_{G_2}\ v \geq 2$ are true.

## 4. An application to geodetic graphs of diameter two

Stemple [11] proved that for any geodetic graph $G$ of diameter two, there exist integers $n$ and $m$ satisfying the properties that $G$ contains exactly $n \cdot m + 1$ vertices, and every vertex in $G$ has degree $n$ or $m$. For fixed $n$ and $m$ denote by $\mathcal{G}(m, n)$ the class of all geodetic graphs of diameter two satisfying the above properties. Lee [4, Theorem 1] used orthogonal *Latin squares* to construct the class $\mathcal{G}(p+1, 2p)$ for any prime power $p$, $p \geq 3$, (for $p = 2$ such a graph is given in Fig. 4(a)) as follows:

From $p-1$ orthogonal Latin squares of order $p$ we first construct a $[p^2 \times (p+1)]$ array $\mathbf{A} = (a_{ij})$ of integers, $1 \leq a_{ij} \leq p$ [10, Theorem 1.3].

A graph $G \in \mathcal{G}(p+1, 2p)$ can be constructed by the following steps:

(i) take vertex disjoint $(p+1)$-cliques $H_1, H_2, \ldots, H_{p+1}$, and label the vertices of each $H_r$ as $u_{r,0}, u_{r,1}, \ldots, u_{r,p}$ for $r = 1, \ldots, p+1$;

(ii) join every pair $[u_{i,0}, u_{j,0}]$ with an edge for $i \neq j$, in this way we make a new clique $H$;

(iii) take new vertices $v_1, v_2, \ldots, v_{p^2}$, not on any $H_r$ and join $v_t$ and $u_{i,a_{ti}}$ with an edge for all $t = 1, \ldots, p^2$ and $i = 1, \ldots, p+1$.

It can be verified that $G$ is geodetic [4, Theorem 1] of diameter two with the following properties:

I. $G$ has clique size $p+1$.

II. For every vertex $u_{i,0} \in V(H)$, $i = 1, \ldots, p+1$, there exists a clique $H_i$ where $V(H) \cap V(H_i) = \{u_{i,0}\}$ and $V^1_G(u_{i,0}) \subseteq V(H) \cup V(H_i)$.

For example, if $p = 3$, then two orthogonal Latin squares of order 3 and its $[9 \times 4]$ array $\mathbf{A}$ are:

$$\mathbf{A}_1 = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{bmatrix} \qquad \mathbf{A}_2 = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 \\ 1 & 3 & 3 & 3 \\ 2 & 1 & 2 & 3 \\ 2 & 2 & 3 & 1 \\ 2 & 3 & 1 & 2 \\ 3 & 1 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 3 & 3 & 2 & 1 \end{bmatrix}$$

The corresponding graph $G \in \mathscr{G}(4, 6)$ is in Fig. 5(a).
From I, II and Theorem B it follows:

**Lemma 2.** *Let $p$ be a prime power, $p \geq 2$, and $G \in \mathscr{G}(p+1, 2p)$. Then $G_1 = G - H$ is a geodetic graph of diameter two with clique size $p$ and $G_1 \in \mathscr{G}(p+1, 2p-1)$.*

For $p = 2$, the graph $G$ is in Fig. 4(a). The outer triangle is $H$ and $G - H$ is the Petersen graph. The graph in Fig. 5(a) has $p = 3$ and the complete 4-graph with darkened edges is $H$.

**Lemma 3.** *Let $G \in \mathscr{G}(p+1, 2p)$, $G_1 = G - H$, $G_2$ be a subgraph of $G$ consisting of the subgraph $H$ and the edges $[u_{1,0}, u_{i,r}]$ for every $i = 1, \ldots, p+1$, $r = 1, \ldots, p$. Then $(G_1, G_2)$ is a $\bar{g}$-decomposition of $G$.*

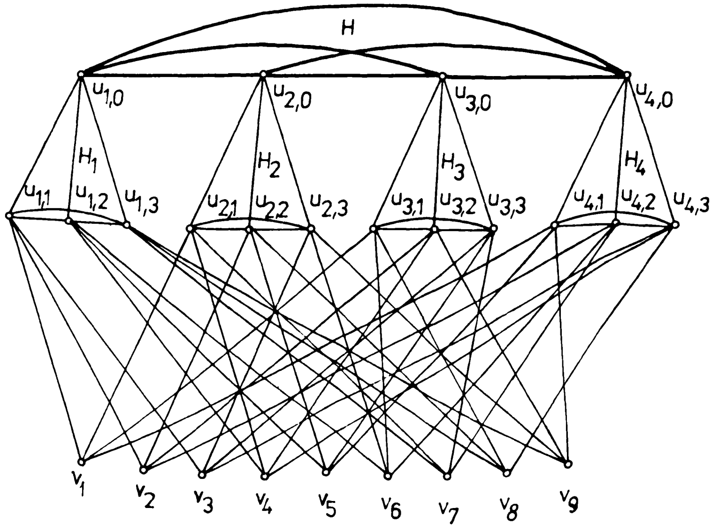
Proof. $G_1$ is geodetic, because of Lemma 2. From the definition of $G_2$ it follows that $G_2$ is a geodetic graph, too. Now, we shall prove the property $P(1)$. If $x, y \in V(G_1)$ and $\varrho_{G_1}(x, y) = 1$, then $\varrho_G(x, y) = 1$, too. If $x, y \in V(G_1)$ and $\varrho_{G_1}(x, y) = 2$, then $\varrho_G(x, y) = 2$, too, since from $\varrho_G(x, y) = 1$ it follows that $\varrho_{G_2}(x, y) = 1$. Then, by the definition of $G_2$, at least one of the vertices $x$ and $y$ belongs to $\{u_{1,0}, \ldots, u_{p+1,0}\}$ and this is a contradiction to the assumption $x, y \in V(G_1)$. Therefore, the graph $G_1$ is geodetically closed. If $x, y \in V(G_2)$, $x, y \notin V(G_1)$, then $x, y \in \{u_{1,0}, \ldots, u_{p+1,0}\}$ and $\varrho_{G_2}(x, y) = \varrho_G(x, y) = 1$. Hence, the property $P(1)$ is proved. Now, we shall prove the property $P(2)$. $V(G_1) \cap V(G_2) = \bigcup_{r-1}^{p+1} \{u_{r,1}, \ldots, u_{r,p}\}$. Let $x, y \in V(G_1) \cap V(G_2)$. We distinguish two cases:

A. $x = u_{r,j}$, $y = u_{r,i}$, $r = 1, \ldots, p+1$; $j \neq i$, $j$, $i \in \{1, \ldots, p\}$; then $\varrho_{G_1}(x, y) = \varrho_{G_1}(u_{r,j}, u_{r,i}) = 1$ and $\varrho_{G_2}(x, y) = \varrho_{G_2}(u_{r,j}, u_{r,i}) = 2$.

B. $x = u_{m,i}$, $y = u_{s,j}$, $m \neq s$, $m$, $s \in \{1, \ldots, p+1\}$; $i$, $j \in \{1, \ldots, p\}$; then $\varrho_{G_1}(x, y) = \varrho_{G_1}(u_{m,i}, u_{s,j}) = 2$ and $\varrho_{G_2}(x, y) = \varrho_{G_2}(u_{m,i}, u_{s,}\ ) = 3$. Hence, the property $P(2)$ is proved.

Q.E.D.

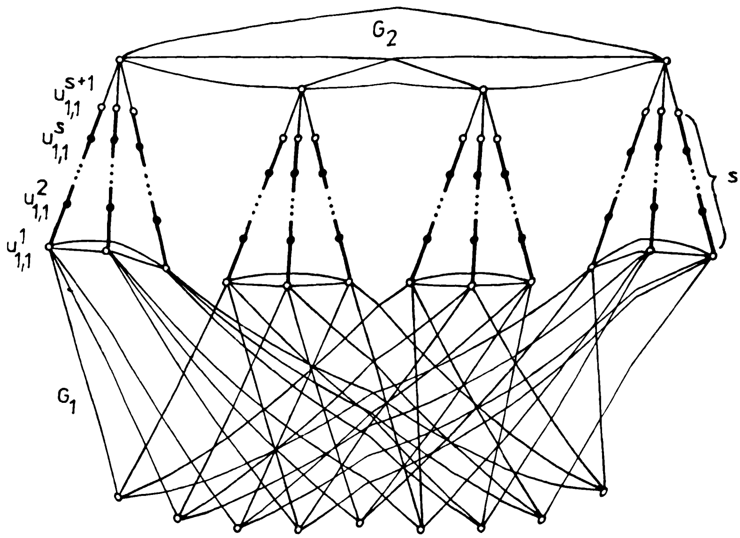
**Theorem 3.** *For every prime power $p \geq 2$, every integer $s \geq 1$ and every $G \in \mathscr{G}(p+1, 2p)$, $\bar{G}(s)$ is a geodetic graph of diameter $2+s$. (We shall denote it by $G(p, 2+s)$.)*




$G \in \mathcal{G}(4,6)$

(a)

Fig. 5a




$G(3, 2+s)$

(b)

Fig. 5b

Proof. The geodeticity of $G(p, 2+s)$ follows from Lemma 3 and Theorem 2. Using Lemma 1, we evidently have

$$\varrho_{G(s)}(u_{r,0}, v_i)=2+s,$$

for $r=1, \ldots, p+1$; $i=1, \ldots, p^2$. Therefore it is sufficient to prove

$$\varrho_{\bar{G}(s)}(x, y) \leq 2+s$$

for any $x, y \in V(G(p, 2+s))$. This is obvious if $x, y \in V(G_1)$ or $x, y \in V(G_2)$. If $x \in V(G_1)-V(G_2)$ and $y=y^j$, $j=1, \ldots, s+1$, (i.e. the vertex $y^j$ lies on a new path $P[y^1, \ldots, y^{s+1}]$ of length $s$), then $\varrho_{G_1}(x, y^1) \leq 2$ and it follows that

$$\varrho_{G(p, 2+s)}(x, y^j) \leq \varrho_{G_1}(x, y^1)+s \leq 2+s.$$

Similarly, if $x \in V(G_2)-V(G_1)$ and $y=y^j$, then $\varrho_{G_2}(x, y^{s+1}) \leq 2$ and hence

$$\varrho_{G(p, 2+s)}(x, y^j) \leq \varrho_{G_2}(x, y^{s+1})+s \leq 2+s.$$

Finally, if $x=v^i$, $i=1, \ldots, s+1$; $y=w^j$, $j=1, \ldots, s+1$; then there exist vertices $v$, $w \in V(G_1) \cap V(G_2)$ and corresponding paths $P_1=[v^1, \ldots, v^{s+1}]$ and $P_2=[w^1, \ldots, w^{s+1}]$, respectively, with $v^i \in P_1$, $w^j \in P_2$. We distinguish two cases:

A. $\varrho_{G_1}(v, w)=\varrho_{G(p,2+s)}(v^1, w^1)=1$ and $\varrho_{G_2}(v, w)=\varrho_{G(p,2+s)}(v^{s+1}, w^{s+1})=2$. Then $x$, $y$ lie on a cycle

$$C=\Gamma_{G(p, 2+s)}[v^1, w^1]+P_2+\Gamma_{G(p, 2+s)}[w^{s+1}, v^{s+1}]+P_1'$$

(where $P_1'$ is the path reverse to $P_1$) of length $2s+3$. Hence,

$$\varrho_{G(p, 2+s)}(x, y) \leq [|C|/2] \leq 2+s.$$

B. $\varrho_{G_1}(v, w)=\varrho_{G(p,2+s)}(v^1, w^1)=2$ and $\varrho_{G_2}(v, w)=\varrho_{G(p,2+s)}(v^{s+1}, w^{s+1})=3$. Then $x$, $y$ lie on a cycle

$$C'=\Gamma_{G(p, 2+s)}[v^1, w^1]+P_2+\Gamma_{G(p, 2+s)}[w^{s+1}, v^{s+1}]+P_1'$$

of length $2s+5$. Hence,

$$\varrho_{G(p, 2+s)}(x, y) \leq [|C'|/2]=2+s.$$

Q.E.D.

For illustration, the graph $G(3, 2+s)$ is in Fig. 5(b).

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## ОДНА КОНСТРУКЦИЯ ГЕОДЕЗИЧЕСКИХ ГРАФОВ

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Резюме

Неориентированный граф называется геодезическим графом, если для каждых двух вершин существует единственная кратчайшая цепь между ними. Автор дает одну конструкцию этих графов. Эта конструкция состоит в натяжении определенного *$\bar{g}$-разложения* $(G_1, G_2)$ геодезического графа при каждой из вершин $V(G_1) \cap V(G_2)$ на единицу или больше. Эта конструкция объединяет некоторые известные конструкции геодезических графов.