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# REMARK ON AN OPERATOR INEQUALITY

DAN KUCEROVSKY

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ABSTRACT. We show that the operator inequality  $\pm i[T,S] \leq ST^2S$  holds if and only if  $\pm i[T,S] \leq TS^2T$ .

## 1. Introduction

It is known ([2]) that in an  $\sigma$ -unital  $C^*$ -algebra, given two strictly positive operators  $\ell$  and k, there is a nonzero function f such that the commutator of  $T := f(\ell + k)$  and S := f(k) satisfies

$$\pm i[T,S] \le ST^2S$$

The purpose of this note is to show that this inequality is equivalent to  $\pm i[T,S] \leq TS^2T\,.$ 

### 2. Theorem

**THEOREM 1.** Suppose that S and T are self-adjoint Hilbert space operators. Then  $\pm i[T, S] \leq ST^2S$  if and only if  $\pm i[T, S] \leq TS^2T$ .

**Remark.** This theorem is trivial if the operators are invertible, but the invertible case does not imply the general case.

Proof. Let Q := iST. It is sufficient to prove that

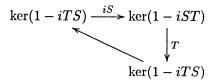
 $(1-Q)(1-Q)^* \ge 1$  and  $(1+Q)(1+Q)^* \ge 1$ if  $(1-Q)^*(1-Q) \ge 1$  and  $(1+Q)^*(1+Q) \ge 1$ .

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It can be verified that the following triangle is well-defined and commutes:



where the diagonal arrow is just the identity map. It follows that S is injective when restricted to the kernel ker(1-iTS), and therefore that dim ker $(1-iST) \ge$ dim ker(1-iTS). Repeating the argument with minor changes we find that, in terms of the operator Q,

$$\dim \ker(1-Q) = \dim \ker(1+Q^*) \quad \text{and} \quad \dim \ker(1+Q) = \dim \ker(1-Q^*),$$

where the dimensions could be a priori infinite. However, the hypothesis implies that the kernel of  $1 \pm Q$  is zero. The above equalities then imply that the partial isometries  $V_{\pm}$  in the polar decomposition of 1 - Q and 1 + Q are actually unitaries, which proves the theorem, since if  $(1-Q)^*(1-Q) \ge 1$ , then  $(1-Q)(1-Q)^* \ge V_V_-^*$ . The same holds for the other choice of sign, so we are done.

It is also possible to prove a one-sided version of the above theorem:

**THEOREM 2.** Suppose that S and T are positive Hilbert space operators. Then  $i[T,S] \leq ST^2S$  if and only if  $i[T,S] \leq TS^2T$ .

The proof of this is omitted.

This paper was motivated by a technical problem in KK-theory that required finding a pair of unbounded Hilbert module operators  $S^{-1}$  and  $T^{-1}$  such that their commutator would be densely defined, and bounded on its domain (as well as satisfying certain other conditions).

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