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JOINT DISTRIBUTIONS AND COMPATIBILITY OF OBSERVABLES IN QUANTUM LOGICS

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In the paper presented the joint probability distribution in the Urbanik sense on a logic will be studied. A relation between the existence of the joint probability distribution and the existence of compatible observables will be shown.

Let L be a poset with the first and the last element 0 and 1, respectively, with the orthocomplementation $\perp: L \to L$, for which we have (i) $(a^\perp)^\perp = a$ for all $a \in L$, (ii) if $a < b$, then $b^\perp < a^\perp$ (iii) $a \vee a^\perp = 1$ for all $a \in L$. If $a < b^\perp$, then a, b are said to be orthogonal and we write $a \perp b$. Further we assume that if $a_i \perp a_j$, $i \neq j$, then $\bigvee_i a_i$ exists in L; and if $a < b$, then there is $c \perp a$ such that $b = a \vee c$. A poset L satisfying the above axioms is called a logic.

We say that $a, b \in L$ are compatible written $(a \leftrightarrow b)$ if there exist mutually orthogonal elements a_1, b_1, $c \in L$, such that $a = a_1 \vee c$, $b = b_1 \vee c$.

An observable is a map $x: B(R^1) \to L$ such that (i) $x(R^1) = L$, (ii) if $E \cap F = \emptyset$ then $x(E) \perp x(F)$, (iii) $x\left(\bigcup_i E_i\right) = \bigvee_i x(E_i)$ if $E_i \cap E_j = \emptyset$, $i \neq j$. If f is a Borel function and x is an observable, then $f \circ x: E \to x(f^{-1}(E))$, $E \in B(R^1)$ is an observable. Two observables x, y are compatible (written $x \leftrightarrow y$) if $x(E) \leftrightarrow y(F)$ for $E, F \in B(R^1)$. The spectrum $\sigma(x)$ of an observable x is the smallest closed subset A of R^1 such that $x(A) = 1$. An observable x is bounded if $\sigma(x)$ is a bounded set.

A state is a map $m: L \to [0, 1]$ such that (i) $m(1) = 1$, (ii) $m\left(\bigvee_i a_i\right) = \sum_i m(a_i)$ if $a_i \perp a_j$, $i \neq j$. A system M of states of L is called (i) quite full if the statement $m(b) = 1$, whenever $m(a) = 1$, $m \in M$ implies $a < b$, (ii) full if $a < b$ iff $m(a) \leqslant m(b)$ for all $m \in M$. Gudder [2] showed that if M is quite full, then M is full. We call the probability measure $m^x(\cdot) = m(x(\cdot))$ on $B(R^1)$ the distribution of x in the state m. The mean of x in the state m if it exists is

$$E_x^m = \int_{R^1} \lambda m^x(\mathrm{d}\lambda).$$

The sum of bounded observables has been studied by Gudder [2, 3, 4]. In [2, 3] there is given the definition of the sum of unbounded observables.

Dvurečenskij and Pulmannová [1] showed that this definition does not include the important case of a logic $L(H)$, (H Hilbert space, $3 \leqslant \dim H \leqslant \aleph_0$). In the following we shall suppose that a couple (L, M) is a sum quantum logic in the sense of Dvurečenskij—Pulmannová [1].

Definition 1. *We shall say that on a sum logic (L, M) the observables $x_1, \ldots, x_k$ are regular if*

$$M_{x_1, \ldots, x_k} = \{m \in M\colon E^m_{x_i} < \infty,\ i = 1, \ldots, k\} \textit{ is a full system.}$$

The set of all regular systems $x = (x_1, \ldots, x_k)$ of observables will be denoted by O_k. All systems of bounded observables are regular [1]. Let $x = (x_1, \ldots, x_k)$ be a system of observables on a sum logic (L, M) and let $x \in O_k$. Then the observable $\sum_{i=1}^{k} \alpha_i x_i$ exists for all $a \in R^k$, $a = (\alpha_1, \ldots, \alpha_k)$. We shall use the following notation. For $a, b \in R^k$, (a, b) will denote the inner product in R^k and $a \in R^k$ and $x \in O_k$, $(a, x) = \sum_{i=1}^{k} \alpha_i x_i$ if $a = (\alpha_1, \ldots, \alpha_k)$.

Definition 2. *We say that $x \in O_k$ has joint distribution of type 2 if there is a measure μ_m^x on $B(R^k)$ such that*

$$\mu_m^x(\omega\colon (a, \omega) \in E) = m^{(a, x)}(E)$$

for all $a \in R^k$ and $E \in B(R^1)$.
By the Cramer-Wold theorem, if the joint distribution exists, it is unique. Joint distributions of this type were introduced by Urbanik [5] and they were studied by Urbanik [5, 6], Gudder [2, 3] and Varadarajan [7].

By $\hat{\mu}_m^x$ we will denote the characteristic function of μ_m^x and byx $\hat{m}^{(a, x)}$ we will denote the characteristic function of the measure $m^{(a, x)}(\cdot)$. By Definition 2 we have

$$\hat{\mu}_m^x(t) = \hat{m}^{(t, x)}(1), \tag{1}$$

where $t \in R^k$. Given $x \in O_k$, we shall denote by $M(x)$ the set of all states m for which μ_m^x exists. Let y be a system of compatible observables. The observables y_i, $i = 1, \ldots, k$ are compatible if and only if there is an observable x and Borel functions f_i, $i = 1, \ldots, k$, such that $y_i = f_i \circ u$ [4].
If $y \in O_k$, then $y_1 + \ldots + y_k = (f_1 + \ldots + f_k)\ u$, [1]. $M(x) = M$ if and only if x consists of compatible observables [2, 3, 7].

Definition 3. *Let $x \in O_k$. We say that x fulfils the probabilistic commutation condition if there exists a system $y \in O_k$ consisting of compatible obsevables such that*

$$\mu_m^x = \mu_m^y \quad \textit{for all} \quad m \in M(x).$$

Let $a = (\alpha_1, \ldots, \alpha_k)$, $\alpha_i \in R^1$, $\alpha_i \neq 0$, $b = (\beta_1, \ldots, \beta_k)$, $\beta_i \in R^1$, $i = 1, \ldots, k$, $x = (x_1, \ldots, x_k)$, $x \in O_k$. We shall use the notation

$$ax + b = (\alpha_1 x_1 + \beta_1, \ldots, \alpha_k x_k + \beta_k),$$

where $\alpha_i x_i + \beta_i$ will denote the observables $y_i = f_i \circ x_i$ and $f_i(u) = \alpha_i u + \beta_i$, $u \in R^1$. Since $E_m^{f_j\, x} = \int_{R^1} f_j(u)\, m^x(\mathrm{d}u)$ [4] we have: if $x \in O_k$, then $ax + b \in O_k$. Let $t = (t_1, \ldots, t_k)$, $t_i \in R^1$, $i = 1, \ldots, k$, $at = (\alpha_1 t_1, \ldots, \alpha_k t_k)$, $(t, ax + b) = (at, x) + (t, b)$. It is evident that $\hat{m}(\tau)^{(t, ax+b)} = e^{i\tau(t,b)} \hat{m}(\tau)^{(at, x)}$ for $m \in M$. For every $m \in M(x)$ we have (1), consequently $\hat{\mu}_m^{ax+b}(t) = e^{i(t,b)} \hat{\mu}_m^x(at)$. We have the following lemma.

Lemma 1. *If $a = (\alpha_1, \ldots, \alpha_k) \in R^k$, $\alpha_j \neq 0$, $j = 1, \ldots, k$ and $b \in R^k$, then $x \in O_k$ if and only if $ax + b \in O_k$ and $M(x) = M(ax + b)$.*

Lemma 2. *If $a = (\alpha_1, \ldots, \alpha_k) \in R^k$, $\alpha_j \neq 0$, $j = 1, \ldots, k$, $b \in R^k$ and $x \in O_k$, then x and $ax + b$ fulfil or do not fulfil the probabilistic commutation condition simultaneously.*

The proof is obvious.

Theorem 1 is the generalization of the (L, M) Urbanik—Theorem [6, Theorem 1]. K. Urbanik considers a situation in a Hilbert space H. In the proof of Theorem [6] the spectral theorem is used. In this paper instead of the spectral theorem we introduce a system consisting of compatiable observables directly.

Theorem 1. *Let $x \in O_k$ and x consists of one side bounded observables with a purely point spectrum. Then x fulfils the probabilistic commutation condition.*

Proof. If $M(x)$ is empty, then our assertion is obvious. We assume that $M(x)$ is non-empty. Let E_j be the spectrum of x_j, $j = 1, \ldots, k$. By Lemma 2 we may assume that E_j contains positive numbers only. The probability measure $m^{x_j}(\cdot)$ is concentrated on the set E_j for every $m \in M$. Let $E = E_1 \times E_2 \times \ldots \times E_k$. Gudder [2] showed that if M is quite full, then if $l \in L$, $l \neq 0$, there exists $m \in M$ such that $m(l) = 1$. Consequently, for any $a \in R^k$ the probability measure $m^{(a, x)}(\cdot)$ is concentrated on the set $(a, E) = ((a, e)\colon e \in E)\, m \in M$ [3].

Let F be the subset of R^k consisting of all elements $a = (\alpha_1, \ldots, \alpha_k)$ with linearly independent coordinates $\alpha_1, \ldots, \alpha_k$ over the denomerable field generated by the set $\bigcup_{j=1}^{k} E_j$. It is clear that F is dense in R^k. Moreover, for $a \in F$ the mapping $e \to (a, e)$ from E onto (a, E) is one-to-one. Let $(a, e) \in (a, E)$. By Definition 2 we have

$$\mu_m^x(\omega\colon (a, \omega) = (a, e)) = m^{(a, x)}(\{(a, e)\}).$$

However, for every $m \in M(x)$ the joint probability distribution μ_m^x is concentrated on the set E [2] and for every $a \in F$ the mapping $e \to (a, e)$ is one-to-one, and we have the formula

(2) $$\mu^x_m(\{e\}) = m^{(a,x)}(\{(a,e)\})(m\in M(x),\ e\in E,\ a\in F).$$

Since F is dense in R^k we can find an element $b\in F$ with positive coordinates. Let $y=(b,x)$, y is the observable with a purely point, positive spectrum. We shall define z_j which is concentrated on the set E_j: let $e\in E$, $e=\{\varepsilon_1,\dots,\varepsilon_k\}$,

$$z_j(\{\varepsilon_j\}) = \bigvee_{\substack{e\in E\\ e=\{\dots,\varepsilon_j,\dots\}}} y(\{(b,e)\}),\qquad \varepsilon_j\in E_j.$$

This z_j is an observable. Indeed if $\varepsilon_j\neq\varepsilon_j'$, then $y(\{(b,e)\})\perp y(\{(b,e')\})$ and $e=\{\dots,\varepsilon_j,\dots\}$ $e'=\{\dots,\varepsilon_j',\dots\}$

$$\bigvee_{\substack{e\in E\\ e=\{\dots,\varepsilon_j,\dots\}}} y(\{(b,e)\})\perp \bigvee_{\substack{e'\in E\\ e'=\{\dots,\varepsilon_j',\dots\}}} y(\{(b,e')\}),\quad z_j\Big(\bigcup_k B_k\Big)=\bigvee_k z_j(B_k)$$

where $B_l\cap B_m=\emptyset$, $l\neq m$, $B_k\in B(R^1)$, $k=1,\dots,z_j(R^1)=1$.

We will show that $z_i\leftrightarrow z_j$, $i\neq j$, $i,j=1,\dots,k$. Let $i<j$, $\varepsilon_i\in E_i$, $\varepsilon_j\in E_j$.

$$z_i(\{\varepsilon_i\}) = \bigvee_{\substack{e\in E\\ e=\{\dots,\varepsilon_i,\dots,\varepsilon_j,\dots\}}} y(\{(b,e)\})\vee \bigvee_{\substack{e\in E\\ e=\{\dots,\varepsilon_i,\dots,\varepsilon_j^{(n_j)}\neq\varepsilon_j,\dots\}}} y(\{(b,e)\})$$

$$z_j(\{\varepsilon_j\}) = \bigvee_{\substack{e\in E\\ e=\{\dots,\varepsilon_i,\dots,\varepsilon_j,\dots\}}} y(\{b,e)\})\vee \bigvee_{\substack{e\in E\\ e=\{\dots,\varepsilon_i^{n_i}\neq\varepsilon_i,\dots,\varepsilon_j,\dots\}}} y(\{(b,e)\})$$

where $\varepsilon_j^{(n_j)}\in E_j$ and $\varepsilon_j^{(n_j)}\neq\varepsilon_j$, $j=1,\dots,k$. Since $z_i\leftrightarrow z_j$ then $a_iz_i\leftrightarrow a_jz_j$, $i,j=1,\dots$ $\dots,k$ [4].

We will prove that $z=(z_1,\dots,z_k)\in O_k$.

$$E^m_{y^2}=\sum_{e\in E}(b,e)^2m^y(\{(b,e)\}).$$

Since $x\in O_k$ then $bx=(b_1x_1,\dots,b_kx_k)\in O_k$ [1]. By Definition 1 we have $E^m_{y^2}<\infty$ for any $m\in M_y\supset M_{bx}=M_x$. We can find $b=b^*\in F$, such that $\varepsilon_j^2\leqslant(b^*,e)^2$, $j=1,\dots,k$.

$$E^m_{z_j^2}=\sum_{e\in E}\varepsilon_j^2m^y(\{(b,e)\})\quad\text{if}\quad E^m_{z_j^2}<\infty$$

If $b=b^*$ and $m\in M_x$, then $E^m_{z_j^2}\leqslant E^m_{y^2}<\infty$, $j=1,\dots,k$. Since M_x is full then $z\in O_k$. We have

$$\sum_{j=1}^k E^m_{a_jz_j}=\sum_{j=1}^k\sum_{\varepsilon_j\in E_j}a_j\varepsilon_jm\Big(\bigvee_{\substack{e\in E\\ e=\{\dots,\varepsilon_j,\dots\}}} y(\{(b,e)\})\Big)=$$

$$= \sum_{e \in E} (a, e)\, m^{y}(\{(b, e)\})$$

for every $m \in M_x$, $b = b^*$ and $a \in R^k$.
The spectrum of (a, z), $\sigma(a, z) = \{(a, e)\colon e \in E\}$ and

$$(a, z)(\{(a, e)\}) = \bigvee_{\substack{e \in E \\ (a,e) = (a,e')}} y(\{(b, e)\})$$

for every $a \in R^k$.

Consequently, by (2) $\hat{m}^{(a,z)}(\tau) = \sum_{e \in E} e^{i\tau(a,e)} \mu_m^x(\{(e)\})$ for every $m \in M(x)$, $a \in R^k$. It is easy to see that $\hat{\mu}_m^x(\tau a) = \hat{m}^{(a,x)}(\tau)$ for every $m \in M(x)$, $\tau \in R^1$, $a \in R^k$ and by the formula $\hat{\mu}_m^x(\tau a) = \Sigma\, e^{i\tau(a,e)} \mu_m^x(\{e\})$ we have $\hat{m}^{(a,z)}(\tau) = \hat{m}^{(a,x)}(\tau)$ for every $m \in M(x)$ and $a \in R^k$. This yields the equation $\mu_m^z = \mu_m^x$ for all $m \in M(x)$, which completes the proof.

Q.E.D.

Remark. Theorem 1 may be proved for the definition of a sum of observables in the sense of Gudder [2, 3, 4].

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СОВМЕСТНОЕ РАСПРЕДЕЛЕНИЕ И СОГЛАСЕ НАВЛЮДАЕМЫХ НА ЛОГИКЕ

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Резюме

К. Урбаник в [6] доказал теорему о существовании для некоторой системы наблюдаемых в пространстве Гильберта, которая имеет одно и тоже совместное распределение. В данной работе этот результат обобщается на логику.