Math. Slovaca, **53** (2003), No. 2, 107–128

Mathematica Slovaca

CHIRAL HYPERMAPS WITH FEW HYPERFACES

ANTONIO BREDA D'AZEVEDO* — ROMAN NEDELA**

(*Communicated by Martin Škoviera*)

ABSTRACT. A hypermap $\mathcal{H}$ is a cellular embedding of a 3-valent graph into a closed surface cells of which are 3-colored (adjacent cells have different colours). The vertices of $\mathcal{H}$ are called flags of $\mathcal{H}$ and let us denote by F the set of flags. An automorphism of the underlying graph which extends to a colour preserving self-homeomorphism of the surface is called an automorphism of the hypermap. If the surface is orientable, the automorphisms of $\mathcal{H}$ split into two classes, orientation preserving and orientation reversing automorphisms. The size of the subgroup of orientation preserving automorphisms is bounded by $|F|/2$ and if the equality is reached, we say that the hypermap is orientably regular. An automorphism of $\mathcal{H}$ reversing the global orientation of the surface is called mirror symmetry. Orientably regular hypermap admitting no mirror symmetries is called chiral. Hence chiral hypermaps have maximum number of orientation preserving symmetries but they are not "mirror symmetric".

The aim of presented paper is to classify chiral hypermaps with at most four hyperfaces. As these have metacycle oriented monodromy groups, we start first with a construction of an infinite family of chiral hypermaps from metacycle groups.

1. Introduction

By an *oriented hypermap* we mean a triple $\mathcal{Q} = (D, R, L)$ where D is a set of darts and R, L are two permutations generating a permutation group $\mathrm{Mon}(\mathcal{Q}) = \langle R, L\rangle$, called the *monodromy group*, acting transitively on D. Every oriented hypermap corresponds to a certain topological map on an orientable surface S (cellular decomposition of S). Geometric representations of hypermaps are briefly described in the following section. It is well known that every

2000 Mathematics Subject Classification: Primary 05C10, 05C25; Secondary 57M15, 14J50.
Keywords: graph, map, hypermap, group, surface, mirror symmetry.

The first author was supported in part by UI&D "Matemática e aplicações".
The second author was supported in part by Slovak Academy of Science.

(topological) map on an orientable surface can be described by an oriented hypermap (D, R, L) with L being involutory. Thus rotary hypermaps generalize in a natural way the notion of oriented maps.

An *automorphism* of an oriented hypermap $\mathcal{Q} = (D, R, L)$ is a permutation ψ of D commuting with both R and L. It is straightforward that $|\operatorname{Aut} \mathcal{Q}| \leq |D|$ and if equality holds the action is regular. For instance, spherical regular oriented maps are the five Platonic solids, cycles and their duals. Regular oriented maps on torus were characterized by Coxeter and Moser in [12]. Classification of regular oriented maps up to genus 7 was completed by Garbe [15]. Recently, with the help of computer programme, Conder and Dobcsányi extended the classification to maps with genus at most 15.

While automorphisms of an oriented hypermap give rise to orientation preserving self-homeomorphisms of the supporting surface there are external symmetries of oriented hypermaps coming from self-homeomorphisms changing the global orientation of the underlying surface. We call such symmetries of oriented hypermaps mirror symmetries (alternatively they are called reflections or inversions). More precisely, a permutation ψ of D will be called a *mirror symmetry* of an oriented hypermap $\mathcal{Q} = (D, R, L)$ if $\psi R = R^{-1}\psi$ and $\psi L = L^{-1}\psi$. An oriented hypermap will be called *mirror asymmetric* if it admits no mirror symmetry. Regular mirror asymmetric oriented hypermap will be called *chiral* hypermap.

Chiral hypermaps and their invariants are the main objective of study of the presented paper. This investigation continues in [7] and [6]. Chiral maps and hypermaps are relatively rare. For instance, examining the list of all regular oriented maps with small genus (see [15], [10]) we see that apart from Coxeter chiral maps on torus there are no chiral maps on surfaces of genus at most 6.

Although the phenomenon of "chirality" was known a long time ago, no general investigation of this phenomenon was done. It was a little surprise for us that one can measure of how far a given hypermap deviates from being mirror symmetric by a group which we call the *chirality group of a hypermap.* Theoretical aspects of this new invariant of hypermaps are studied in [6]. The order of the group is called the *chirality index.* In Section 3 we give the definition and formula allowing to compute the index.

Our approach to the problem of classification of chiral hypermaps is based on a consequence of the Hurwitz bound stating that, except the sphere and torus, the size of the group of orientation preserving automorphisms is bounded by $84(g-1)$, and consequently, the number of chiral hypermaps is bounded provided the genus g of the supporting surface is fixed. Thus we concentrate our attention to the classification of chiral hypermaps of small genera. Lot of them have (up to duality) least number of hyperfaces, thus as a first step towards the classification we deal with chiral hypermaps with at most four hyperfaces, the main objective of study of the presented paper. The classification problem is then treated in [7], where a complete classification of chiral hypermaps up to genus four is given.

In order to build up a self-contained theory together with some necessary algebraic machinery we have decided to include in Section 2 some "well-known" and so called "trivial" observations and explain relations between main notions in terms of both geometric and algebraic representations of hypermaps. Further information on hypermaps as well as undefined terms can be found in the following literature [1], [3], [4], [5], [11], [16], [17], [19], [21], [24], [25].

2. Preliminaries

2.1. Hypermaps.

A *topological hypermap* $\mathcal{H}$ is a cellular embedding of a connected trivalent graph $\mathcal{G}$ into a compact surface S, without boundary and not necessarily orientable, such that the cells (i.e. the connected components of $S\backslash\mathcal{G}$) are 3-colored (say by black, grey and white colours) with adjacent cells having different colours. Numbering the colours 0, 1 and 2, and labeling the edges of $\mathcal{G}$ with the missing adjacent cell number, we can define 3 fixed points free involutory permutations r_i, $i = 0, 1, 2$, on the set F of vertices of $\mathcal{G}$; each r_i switches the pairs of vertices connected by i-edges (edges labeled i). The elements of F are called *flags* and the group G generated by r_0, r_1 and r_2 will be called the *monodromy group*[1] $\mathrm{Mon}(\mathcal{H})$ of the hypermap $\mathcal{H}$. The cells of $\mathcal{H}$ colored 0, 1 and 2 are called the *hypervertices*, *hyperedges* and *hyperfaces*, respectively. Since the graph $\mathcal{G}$ is connected, the monodromy group acts transitively on F and orbits of $\langle r_0, r_1\rangle$, $\langle r_1, r_2\rangle$ or $\langle r_0, r_2\rangle$ on F determine hypervertices, hyperedges and hyperfaces, respectively.

Given a topological hypermap $\mathcal{H}$, we can derive virtually six topological hypermaps on the same surface by permuting the three colours 0, 1, 2 of their cells; in fact, for each permutation $\sigma \in S_3 = S_{\{0,1,2\}}$ we define the σ*-dual* $D_\sigma\mathcal{H}$ to be the hypermap on the same surface, with the same underlying trivalent graph $\mathcal{G}$, whose hypervertices, hyperedges and hyperfaces are the cells colored 0σ, 1σ and 2σ, respectively.

If the surface S is orientable, then we can, and as a rule we will, fix an orientation, for instance the counter-clockwise orientation. The subgroup G^+ generated by r_1r_2 and r_2r_0 acts on F with two orbits F^+ and F^-. Let $D = F^+$ be the orbit such that r_1r_2 and r_2r_0 locally act on D like counter-clockwise rotations around hypervertices and hyperedges, respectively.

[1] This group has been called the monodromy group of $\mathcal{H}$ ([16], [21]), the connection group of $\mathcal{H}$ ([25]) and the Ω-group of $\mathcal{H}$ ([1]).

By contracting each 2-edge to a single point we arrive at a 4-valent graph embedding on S called the *topological oriented hypermap* and denoted by $\mathcal{H}^+$. The vertices of the underlying graph, which correspond to the 2-edges of $\mathcal{G}$ contracted to single points form the set D of *darts* of $\mathcal{H}^+$, while the 2-edges of $\mathcal{G}$ can be considered to be the darts of $\mathcal{H}$. It is known (see [11], [17], [13]) that one can represent $\mathcal{H}^+$ by means of two permutations R and L cyclically permuting counter-clockwise the darts around hypervertices and hyperedges, respectively. The group they generate is called the *oriented monodromy group* of $\mathcal{H}$, and will be denoted by $\mathrm{Mon}(\mathcal{H}^+)$. By identifying the set of darts with F^+, the permutations R and L are the same as r_1r_2 and r_2r_0 acting on F^+, respectively. For this reason we will say that R and L are, respectively, the *dart components* of r_1r_2 and r_2r_0, with respect to the chosen orientation. We notice that the group generated by the dart components R_b, L_b of r_1r_2 and r_2r_0 with respect to one orientation may be not "*monodromy isomorphic*" to the group generated by the dart components R_w, L_w of r_1r_2 and r_2r_0 with respect to the other orientation, that is, the assignment $R_b \longrightarrow R_w$, $L_b \longrightarrow L_w$ may not extend to an isomorphism. The oriented monodromy group $\mathrm{Mon}(\mathcal{H}^+)$ can be therefore orientation "dependent". For this reason we write $\mathcal{H}^+$, if the dart components of r_1r_2 and r_2r_0 is taken with respect to the fixed orientation and write $\mathcal{H}^-$ if the dart components of r_1r_2 and r_2r_0 is taken in respect to the counter-fixed orientation.

Topologically speaking, these two oriented hypermaps $\mathcal{H}^+$ and $\mathcal{H}^-$ are not isomorphic if $\mathcal{H}$ is not "mirror symmetric". This is what happens, for example, with the hypermap $\mathcal{H}$ corresponding to the embedding of the Fano plane illustrated in Figure 1 with the corresponding oriented hypermap $\mathcal{H}^+$ shown in Figure 2(a). One can check that this picture is not mirror symmetric, that is, its mirror image $\mathcal{H}^-$ (Figure 2(b)) is not isomorphic to $\mathcal{H}^+$.

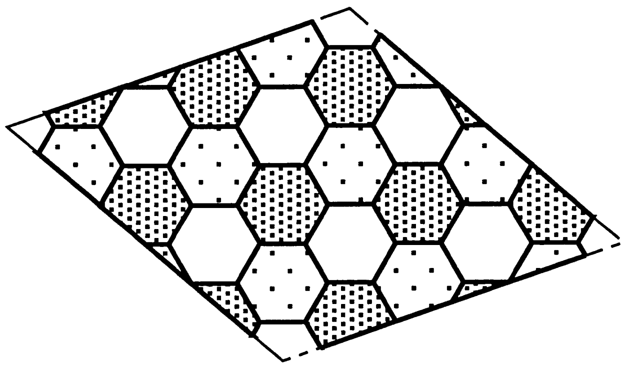

FIGURE 1. The Fano plane embedded in Torus.

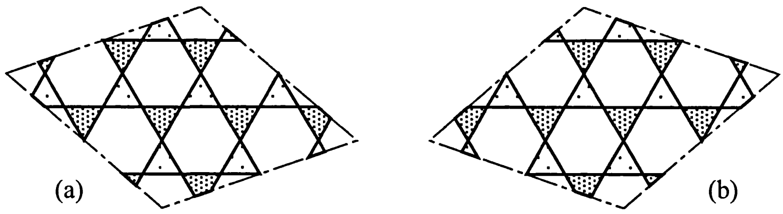

FIGURE 2. The oriented Fano plane and its mirror image.

In this paper we will adopt the bipartite map representation of a hypermap as defined by Walsh [24]. By shrinking hypervertices and hyperedges of a hypermap $\mathcal{H}$ to points (the first to black points and the second to white points), and joining a black vertex to a white vertex by an edge if the respective hypervertex and hyperedge are incident, we are led to a bipartite map on the same surface in which one monochromatic set of vertices (the black vertices) represents the hypervertices and the other monochromatic set of vertices (the white vertices) represents the hyperedges. This map is usually called the *Walsh bipartite map* associated to $\mathcal{H}$.

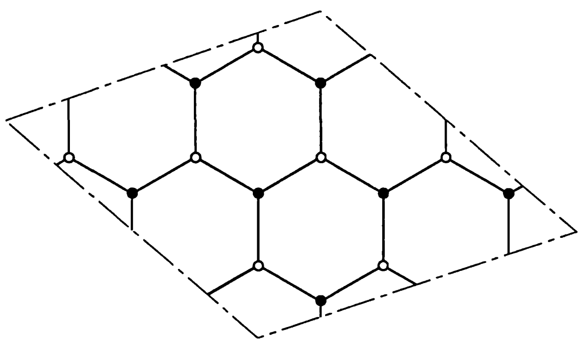

FIGURE 3. The Walsh map of the Fano plane embedding.

2.2. Algebraic representations of hypermaps.

In the proceeding section we saw that a hypermap can be represented by means of three fixed point free involutions acting transitively on its flags. The goal is that up to isomorphism, any geometric realization of the hypermap can be derived from its algebraic description. Invariants of hypermaps as the orientability, the Euler characteristic, the automorphism group and others have their algebraic counterparts in the category of (algebraic) hypermaps (oriented (algebraic) hypermaps). It is often convenient to consider hypermaps as algebraic objects, and when needed, to go back to geometry. In what follows we briefly introduce two kinds of algebraic representations of hypermaps as well as some important notions, invariants and relations between them.

Algebraically, a *hypermap* $\mathcal{H}$ is a 4-tuple $\mathcal{H} = (F, r_0, r_1, r_2)$, where F is a finite set and r_0, r_1 and r_2 are fixed point free involutory permutations of F such that the group generated by r_0, r_1 and r_2, that is, the *monodromy group* $\mathrm{Mon}(\mathcal{H})$ of $\mathcal{H}$, acts transitively on F. Similarly, an *oriented hypermap* is a 3-tuple $\mathcal{Q} = (D, R, L)$, where D is a finite set of darts and R, L are permutations of D such that the *monodromy group* $\mathrm{Mon}(\mathcal{Q}) = \langle R, L\rangle$ acts transitively on D.

It may happen that the even word subgroup $\langle r_1r_2, r_2r_0\rangle$ of $\mathrm{Mon}(\mathcal{H})$ may act on F with one or two orbits (which we denote by F^+ and F^-). In the later case the hypermap $\mathcal{H}$ is called *orientable*. If $\mathcal{H}$ is orientable, then we can derive two associated oriented hypermaps $\mathcal{H}^+ = \left(F^+, r_1r_2\,|_{F^+}, r_2r_0\,|_{F^+}\right)$ and $\mathcal{H}^- = \left(F^-, r_1r_2\,|_{F^-}, r_2r_0\,|_{F^-}\right)$.

Given an oriented hypermap $\mathcal{Q} = (D, R, L)$, we can define an orientable hypermap $\mathcal{H} = (F, r_0, r_1, r_2)$ such that $\mathcal{Q}$ comes as one of its associated oriented hypermaps, say $\mathcal{H}^+$, in the following way. Let C_2 be the set $\{1, -1\}$, identify D with the cartesian product $D\times 1$ and define $F = D\times C_2$. In the topological point of view the number 1 represents the chosen orientation. Then the permutations r_0, r_1, r_2 are defined as follows: for any $(d,i) \in F$, $r_0\colon (d,i) \mapsto (dL^{-i}, -i)$, $r_1\colon (d,i) \mapsto (dR^i, -i)$, $r_2\colon (d,i) \mapsto (d, -i)$.

With the above notation the orbits of the subgroups $\langle r_1, r_2\rangle$, $\langle r_2, r_0\rangle$ and $\langle r_0, r_1\rangle$ on F, or alternatively, the cycles of R, L and RL on D, are called *hypervertices*, *hyperedges* and *hyperfaces*, respectively. The least common multiples l, m, n of the lengths of the cycles of r_1r_2, r_2r_0 and r_0r_1 on F, or alternatively, of the cycles of R, L and RL on D, respectively, determine the *type* (l, m, n) of the hypermap $\mathcal{H}$, or of the oriented hypermap $\mathcal{Q}$. It is straightforward that both $\mathcal{H}$ and $\mathcal{Q}$ have always the same type.

If $\mathcal{H}_1 = (F_1, r_0, r_1, r_2)$ and $\mathcal{H}_2 = (F_2, s_0, s_1, s_2)$ are two hypermaps, a *covering* (or epimorphism) $\psi\colon \mathcal{H}_1 \to \mathcal{H}_2$ is a (surjective) function $\psi\colon F_1 \to F_2$ such that $(\forall\omega \in F_1)(\forall i \in \{0,1,2\})((\omega r_i)\psi = \omega\psi s_i)$. We say that $\mathcal{H}_1$ *covers* $\mathcal{H}_2$ if there is a covering $\psi\colon \mathcal{H}_1 \to \mathcal{H}_2$. As $\mathrm{Mon}(\mathcal{H}_2)$ acts faithfully on F_2, if $\mathcal{H}_1$ covers $\mathcal{H}_2$, then $\mathrm{Mon}(\mathcal{H}_1)$ also *monodromy covers* $\mathrm{Mon}(\mathcal{H}_2)$, that is, the assignment $r_i \mapsto s_i$, for $i = 0,1,2$, extends to an epimorphism $\psi^*\colon \mathrm{Mon}(\mathcal{H}_1) \to \mathrm{Mon}(\mathcal{H}_2)$.

An *isomorphism* $\psi\colon \mathcal{H}_1 \to \mathcal{H}_2$ is an injective covering and an *automorphism* of $\mathcal{H} = (F, r_0, r_1, r_2)$ is a permutation of F commuting with r_0, r_1 and r_2.

In a similar way one can define the notions of covering, isomorphism and automorphism for oriented hypermaps. For instance, if $\mathcal{Q} = (D, R, L)$, a permutation $\phi \in \mathrm{S}_D$ is an *automorphism* of $\mathcal{Q}$ if $\phi R = R\phi$ and $\phi L = L\phi$.

The group of all automorphisms of a hypermap $\mathcal{H}$, denoted by $\mathrm{Aut}\,\mathcal{H}$, acts naturally on F; this action is semi-regular, that is, for any $\psi \in \mathrm{Aut}\,\mathcal{H}$ the equality $w\psi = w$ for some $w \in F$ implies $\psi = 1$. As a consequence we get $|\mathrm{Aut}\,\mathcal{H}| \leq |F|$. If $|\mathrm{Aut}\,\mathcal{H}| = |F|$, that is, if $\mathrm{Aut}\,\mathcal{H}$ acts transitively on F, we say that $\mathcal{H}$ is *regular*.

Let us assume that $\mathcal{H}$ is an orientable hypermap. An automorphism $\psi \in \mathrm{Aut}\,\mathcal{H}$ *preserves the orientation* if ψ preserves the two sets $D = F^+$ and F^-, that is, for all $w \in F^\zeta$, $w\psi \in F^\zeta$, where $\zeta \in \{+,-\}$. The subgroup $\mathrm{Aut}^+\,\mathcal{H}$ of the automorphisms preserving orientation acts semi-regularly on D, as well as on F^-, and satisfies $|\mathrm{Aut}^+\,\mathcal{H}| \leq |D| = |F^-|$. If $\mathrm{Aut}^+\,\mathcal{H}$ acts transitively on D, we say either that $\mathcal{H}$ is *orientably regular*, or that the oriented hypermap $\mathcal{H}^+$ is *regular*.

2.3. Chiral hypermaps.

In what follows all considered hypermaps will be orientable. As already mentioned above, the two oriented hypermaps $\mathcal{H}^+$ and $\mathcal{H}^-$ associated with a given hypermap $\mathcal{H}$, may or may not be "isomorphic". More precisely, an (orientable) hypermap $\mathcal{H}$ is *mirror symmetric* if there exists an automorphism of $\mathcal{H}$ taking F^+ to F^-. If $\mathcal{H}$ is not mirror symmetric, we say it is *mirror asymmetric.* We say that an oriented hypermap $\mathcal{Q} = (D,R,L)$ is *mirror symmetric* if $\mathcal{Q} \cong (D,R^{-1},L^{-1})$; if $\mathcal{Q}$ is not mirror symmetric, we say it is *mirror asymmetric*. A hypermap that is orientably regular but not regular is called *chiral*. The following two lemmas give alternative characterizations of the above defined notions.

Lemma 1. *Let $\mathcal{H}$ be a hypermap with* $\mathrm{Mon}(\mathcal{H}) = \langle r_0, r_1, r_2\rangle$ *and* $\mathrm{Mon}(\mathcal{H}^+) = \langle R, L\rangle$ *(with respect to some fixed orientation). Then the following statements are equivalent:*

i) *$\mathcal{H}$ is mirror symmetric.*
ii) $\mathcal{H}^+ \cong \mathcal{H}^-$.
iii) *$\mathcal{H}^+$ is mirror symmetric.*

Lemma 2. *Let $\mathcal{H}$ be an orientable hypermap with* $\mathrm{Mon}(\mathcal{H}) = \langle r_0, r_1, r_2\rangle$ *and* $\mathrm{Mon}(\mathcal{H}^+) = \langle R, L\rangle$. *Then*

i) *$\mathcal{H}$ is regular if and only if $\mathcal{H}^+$ is regular and mirror symmetric.*
ii) *$\mathcal{H}$ is orientably regular if and only if $\mathcal{H}^+$ is regular.*
iii) *$\mathcal{H}$ is chiral if and only if $\mathcal{H}^+$ is regular and mirror asymmetric.*

If G is a group generated by three involutions r_0, r_1 and r_2, then the 4-tuple (G, r_0, r_1, r_2), where G acts on itself by right multiplication, is a hypermap $\mathcal{H}$. Each element $g \in G$ can be seen as an automorphism of $\mathcal{H}$ by letting it acting on G by "left" multiplication[2], that is,

$$(\forall\, w \in G)\big(w \cdot g = g^{-1}w\big)\,.$$

Hence $\operatorname{Aut}\mathcal{H} \cong G$ and $\mathcal{H}$ is therefore a regular hypermap with the monodromy group G.

Reciprocally, if $\mathcal{H} = (F, r_0, r_1, r_2)$ is a regular hypermap, then, as $\operatorname{Aut}\mathcal{H}$ acts transitively on F and its elements commute with the elements of $\operatorname{Mon}(\mathcal{H})$, the stabilizer $\operatorname{Stab}(\alpha)$ of any flag $\alpha \in F$ in $\operatorname{Mon}(\mathcal{H})$ is trivial. As $|F| \le |\operatorname{Mon}(\mathcal{H})|$, we have $|F| = |\operatorname{Mon}(\mathcal{H})|$. Then the action of $\operatorname{Mon}(\mathcal{H})$ is equivalent to the action of $\operatorname{Mon}(\mathcal{H})$ on itself by right multiplication, and so $\mathcal{H} \cong \bar{\mathcal{H}} = \big(\operatorname{Mon}(\mathcal{H}), \bar{r}_0, \bar{r}_1, \bar{r}_2\big)$, where $\bar{r}_i$ is the permutation of $G = \operatorname{Mon}(\mathcal{H})$ given by right multiplication by r_i. It is clear, that any permutation of G given by left multiplication of an element of G is an automorphism of $\bar{\mathcal{H}}$. Hence $\operatorname{Aut}\mathcal{H} \cong \operatorname{Mon}(\mathcal{H})$ and the following lemma holds.

Lemma 3. *Let $\mathcal{H} = (F, r_0, r_1, r_2)$ be a hypermap. Then $|\operatorname{Aut}\mathcal{H}| \le |F| \le |\operatorname{Mon}(\mathcal{H})|$ and the following statements are equivalent:*

i) *$\mathcal{H}$ is regular.*
ii) *The stabilizer of any $\alpha \in F$ in the monodromy group of $\mathcal{H}$ is trivial.*
iii) $\mathcal{H} \cong \big(\operatorname{Mon}(\mathcal{H}), \bar{r}_0, \bar{r}_1, \bar{r}_2\big)$.
iv) $\operatorname{Aut}\mathcal{H} \cong \operatorname{Mon}(\mathcal{H})$.

Corollary 4. *Let $\mathcal{H}$ be a hypermap and σ be any permutation of the symmetric group S_3. Then,*

i) *$\mathcal{H}$ is regular if and only if $D_\sigma(\mathcal{H})$ is regular,*
ii) *$\mathcal{H}$ is chiral if and only if $D_\sigma(\mathcal{H})$ is chiral.*

As was already mentioned any orientable hypermap $\mathcal{H}$ determines two permutations R, L generating the monodromy group $\operatorname{Mon}(\mathcal{H}^+)$. Moreover, if $\mathcal{H}$ is orientably regular, then $\operatorname{Mon}(\mathcal{H}^+)$ can be seen as the set of darts (see Lemma 5ii). Vice-versa, given a triple (G, R, L), where $G = \langle R, L\rangle$, the hypermap $\mathcal{H} = (G \times C_2, r_0, r_1, r_2)$, where $C_2 = \{1, -1\}$, $G = G \times 1$ and

$$r_0\colon (g,i) \mapsto (gL^{-i}, -i)\,,$$
$$r_1\colon (g,i) \mapsto (gR^{i}, -i)\,,$$
$$r_2\colon (g,i) \mapsto (g, -i)\,,$$

is necessarily orientable and it has oriented monodromy group $\operatorname{Mon}(\mathcal{H}^+) = G$. Each element of G can be seen as an automorphism preserving orientation by

[2] If we fix the counter-clockwise orientation, then $r_1 r_2$ seen as an element of the monodromy group counter-clockwise permutes the darts around hypervertices, but seen as an automorphism it is a clockwise rotation around the vertex containing flag 1.

acting on the cartesian set $G \times C_2$ by "left" multiplication,

$$(\forall\, b \in G)\big((b,i)\cdot g = (b,i)\cdot(g,1) = (g^{-1}b,i)\big)\,.$$

Then $\mathrm{Aut}^+ \mathcal{H} \cong G$, and consequently, $\mathcal{H}$ is orientably regular. Hence hypermaps on orientable surfaces can be alternatively described by means of two permutations R, L generating the monodromy group of the associated oriented map. Now, replacing (in Lemma 3) F with $D = F^+$, $\mathrm{Mon}(\mathcal{H}) < \mathrm{S}_F$ with $\mathrm{Mon}(\mathcal{H}^+) < \mathrm{S}_D$, $\mathrm{Aut}\,\mathcal{H}$ with $\mathrm{Aut}^+ \mathcal{H}$ and regular with orientably regular we have:

LEMMA 5. *Let $\mathcal{H} = (F, r_0, r_1, r_2)$ be a hypermap. Then $|\mathrm{Aut}^+ \mathcal{H}| \leq |D| \leq |\mathrm{Mon}(\mathcal{H}^+)|$ and the following statements are equivalent*

i) *$\mathcal{H}$ is orientably regular.*

ii) *The stabilizer of any $\delta \in D$ in the monodromy group of $\mathcal{H}^+$ is trivial.*

iii) *$\mathcal{H}^+ \cong (\mathrm{Mon}(\mathcal{H}^+), R, L)$, where $R = r_1 r_2|_{F^+}$ and $L = r_2 r_0|_{F^+}$.*

iv) $\mathrm{Aut}^+ \mathcal{H} \cong \mathrm{Mon}(\mathcal{H}^+)$.

If G and K are two groups generated by r_0, r_1, r_2 and s_0, s_1, s_2, respectively, then the regular hypermap $\mathcal{H}_1 = (G, \bar{r}_0, \bar{r}_1, \bar{r}_2)$ covers the regular hypermap $\mathcal{H}_2 = (K, \bar{s}_0, \bar{s}_1, \bar{s}_2)$ if and only if the assignment $r_i \mapsto s_i$, $i = 0,1,2$, determines an epimorphism $\varrho^*\colon G \to K$. This epimorphism ϱ^* is itself a covering $\varrho\colon \mathcal{H}_1 \to \mathcal{H}_2$.

Let $\Delta = \Delta(\infty,\infty,\infty)$ be the free product $C_2 * C_2 * C_2$,

$$\Delta = \langle R_0, R_1, R_2 \mid R_0^2 = R_1^2 = R_2^2 = 1\rangle\,.$$

This group gives rise to an infinite regular hypermap (the universal hypermap [18]) with monodromy group Δ. If $\mathcal{H} = (F, r_0, r_1, r_2)$ is any hypermap, then $\varrho\colon R_i \mapsto r_i$ determines an epimorphism $\varrho\colon \Delta \to \mathrm{Mon}(\mathcal{H})$. Δ acts naturally on F by $x \cdot d = x\, d\varrho$. Fix $\alpha \in F$ and let H be the stabilizer of α in Δ. Since $\mathrm{Mon}(\mathcal{H})$ acts transitively and faithfully on F, then $\mathrm{Ker}(\varrho) = H_\Delta$, the core of H in Δ, and so $\mathrm{Mon}(\mathcal{H}) \cong \Delta/H_\Delta$. As the action of Δ on F is equivalent to the action of Δ on the right cosets $\Delta/_r H$ of H in Δ, then $\mathcal{H}$ is isomorphic to

$$(\Delta/_r H,\ H_\Delta R_0,\ H_\Delta R_1,\ H_\Delta R_2)\,.$$

That is, the flags of any hypermap $\mathcal{H}$ can be seen as right cosets Hd, $d \in \Delta$, of some subgroup H in Δ, called *hypermap-subgroup* of $\mathcal{H}$, while the monodromy group of $\mathcal{H}$ can be seen as the quotient group Δ/H_Δ acting on the right cosets Hd by right multiplication. In this view $\mathcal{H}$ is orientable if and only if $H < \Delta^+$, where Δ^+ is the even subgroup of index 2 in Δ generated by R_1R_2 and R_2R_0. The hypermap-subgroups are not unique, but they are all conjugate. But, because conjugate hypermap-subgroups give rise to isomorphic hypermaps, they are unique up to a conjugacy, or in other words, up to an isomorphism.

Similar correspondence between subgroups of Δ^+ and oriented hypermaps can be established. In particular, the darts of the oriented hypermap $\mathcal{H}^+$ correspond to right cosets (darts) Hd, $d \in \Delta^+$, while the monodromy group $\mathrm{Mon}(\mathcal{H}^+)$ corresponds to Δ^+/H_{Δ^+}.

As a consequence of the above discussion we get the following lemma.

LEMMA 6. *Let $\mathcal{H}$ be a hypermap with the respective hypermap subgroup $H \leq \Delta$. Then $\mathcal{H}$ is orientably regular if and only if $H \lhd \Delta^+$, regular if and only if $H \lhd \Delta$, and chiral if and only if $H \lhd \Delta^+$ but $H \not\lhd \Delta$.*

Given two hypermaps $\mathcal{H}_1$ and $\mathcal{H}_2$, $\mathcal{H}_1$ covers $\mathcal{H}_2$ if and only if there exist hypermap-subgroups H_1 and H_2 of $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively, such that $H_1 < H_2$. The covering is given by $\Delta/_r H_1 \longrightarrow \Delta/_r H_2$, $H_1 d \longrightarrow H_2 d$.

It is easy to see that the conjugation by R_2 inverts the generators R_1R_2, R_2R_0 of Δ^+. Hence, an orientable regular map $\mathcal{H}$ with the hypermap subgroup H is regular (or not chiral) if $H = H^{R_2}$. In [6] the *chirality group* of a hypermap $\mathcal{H}$ is defined by setting $X(\mathcal{H}) = H/H_\Delta$. Since $\mathcal{H}$ is mirror symmetric if and only if $X(\mathcal{H})$ is trivial, the chirality group can be considered to be an algebraic measure of how much an orientable regular map deviates from being regular. The chirality index $\kappa(\mathcal{H}) = |X(\mathcal{H})|$ is the size of the chirality group. It is proved in [6] that the chirality group of a hypermap can be viewed as a subgroup of $\mathrm{Mon}(\mathcal{H}^+)$ thus the chirality index is a divisor of the number of darts of $\mathcal{H}$.

For more information on chirality groups the reader is referred to [6]. To calculate the chirality index of a given hypermap $\mathcal{H}$, the following lemma will be useful.

LEMMA 7. *Let $\mathcal{H}$ be an orientable regular hypermap. Then* $\kappa(\mathcal{H}) = \frac{|\mathrm{Mon}(\mathcal{H})|}{2|\mathrm{Mon}(\mathcal{H}^+)|}$.

Proof. $\kappa(\mathcal{H}) = |H/H_\Delta| = \frac{|\Delta^+/H_\Delta|}{|\Delta^+/H|} = \frac{1}{2}\frac{|\mathrm{Mon}(\mathcal{H})|}{|\mathrm{Mon}(\mathcal{H}^+)|}$. □

3. Chiral hypermaps from metacyclic groups

Let us denote by $G = G(m,n,r,s) = \langle a,b \mid a^n = 1,\ b^m = a^s,\ bab^{-1} = a^r \rangle$ with s and r satisfying $(r-1)s = 0 \pmod{n}$ and $r^m = 1 \pmod{n}$, the metacyclic group with parameters m, n, r and s. This group has order mn. The aim of this section is to prove the following theorem.

THEOREM 8. *The hypermap $\mathcal{Q} = (G, ab, b^{-1})$ is a chiral hypermap (with m hyperfaces of valency n) if and only if $r^2 \neq 1 \pmod{n}$.*

Proof. We have already observed that $\mathcal{Q}$ is an oriented regular hypermap. By [20], $\langle a\rangle$ is normal in G and the factor $G/\langle a\rangle$ is cyclic of order m, then $\mathcal{Q}$ has m hyperfaces, each of valency n. It remains to show that $\mathcal{H}$ is mirror asymmetric. Assume it is mirror symmetric. Then $b^{-1}a^{-1}b = a^{-r}$ and $bab^{-1} = a^r$. Combining the above two relations we end with $a^{r^2} = a$, and consequently, $r^2 = 1 \pmod{n}$. On the other hand, assuming $r^2 = 1 \pmod{n}$, the Substitution Test ([20]) implies that the assignment $b \mapsto b^{-1}$, $ab \mapsto (ab)^{-1}$ extends to a mirror symmetry. □

COROLLARY 9. *If $r^2 \neq 1 \pmod{n}$, then the hypermap $\mathcal{Q} = (G, ab, b^{-1})$ is chiral with chirality group $X(\mathcal{Q}) = \langle a^{r^2-1}\rangle$ and chirality index $\frac{n}{(n,r^2-1)}$.*

Proof. By Theorem 8, $\mathcal{Q}$ is chiral. Now the greatest regular hypermap covered by $\mathcal{Q}$ has monodromy group

$$\langle a,b \mid a,b \mid a^n = 1,\ b^m = a^s,\ bab^{-1} = a^r,\ b^{-1}a^{-1}b = a^{-1}\rangle \cong G/\langle a^{r^2-1}\rangle .$$

Then $X(\mathcal{Q}) = \langle a^{r^2-1}\rangle$ and hence $\mathcal{Q}$ has chirality index $\kappa = |X(\mathcal{Q})| = \frac{n}{(n,r^2-1)}$. □

4. Chiral hypermaps with at most four hyperfaces

The aim of this section is to give a classification of all chiral hypermaps with at most four hyperfaces. In what follows we shall use the following notation for some numerical invariants of orientable regular hypermaps: N — the negative Euler characteristic; l, m and n — the valency of a hypervertex, hyperedge and hyperface, respectively; V, E and F — the number of hypervertices, hyperedges and hyperfaces, respectively; κ — the chirality index.

THEOREM 10. *If $\mathcal{H}$ is orientably regular with 1 hyperface, then $\mathcal{H}$ is regular.*

Proof. Without any loss of generality, we consider the dual $D(\mathcal{H})$ with one hypervertex. If $\mathcal{H}$ has one hypervertex of valency n, then $\mathrm{Mon}(\mathcal{H}^+)$ is a cyclic group $C_n = \langle R\rangle$. Then $L = R^s$ for some $s \in \{0,1,\dots,n-1\}$, and so $\mathrm{Mon}(\mathcal{H}^+) = \langle R, L \mid R^n = R^sL^{-1} = 1\rangle$. Then $\mathrm{Mon}(\mathcal{H})$ has presentation

$$\begin{aligned}
&\langle x,y,z \mid x^2 = y^2 = z^2 = (yz)^n = 1,\ zx = (yz)^s\rangle \\
&= \langle x,y,z \mid x^2 = y^2 = z^2 = (yz)^n = 1,\ x = z(yz)^s\rangle \\
&= \langle y,z \mid y^2 = z^2 = (yz)^n = 1\rangle = D_n .
\end{aligned}$$

As $|\mathrm{Mon}(\mathcal{H})| = 2|\mathrm{Mon}(\mathcal{H}^+)|$, then $\kappa(\mathcal{H}) = 1$. □

THEOREM 11. *If $\mathcal{H}$ is orientably regular with 2 hyperfaces, then $\mathcal{H}$ is regular.*

Proof. Let F_1 and F_2 be the two hyperfaces of $\mathcal{H}$. Then F_1 and F_2 must appear around each hypervertex and around each hyperedge. Let a be the permutation that cyclically permutes the darts counter-clockwise around F_1 and b the permutation that cyclically permutes the darts counter-clockwise around a hypervertex incident with F_1. Then b^2 must be a power a^s for some $s \in \{0,\dots,n-1\}$ and $(ab)^2$ must also be a power a^t for some $t \in \{0,\dots,n-1\}$. Let $G_{s,t}$ be the group with presentation

$$\langle a,b \mid a^n = 1,\ b^2 = a^s,\ (ab)^2 = a^t \rangle .$$

As $b^2 = a^s$, the subgroup $K = \langle a \rangle$ has index 2 in $G_{s,t}$, and so $|G_{s,t}| \le 2n$ is finite. Then $\mathrm{Mon}(\mathcal{H}^+) = G_{s,t}$ for some s and t, and so $\mathrm{Mon}(\mathcal{H})$ has presentation

$$\langle x,y,z \mid x^2 = y^2 = z^2 = (yz)^n = 1,\ (zx)^2 = (yz)^s,\ (yx)^2 = (yz)^z \rangle .$$

Changing variables $a = yz$, $b = zx$, $z = z$ in the above presentation, it changes to

$$\langle a,b,z \mid z^2 = a^n = 1,\ b^z = b^{-1},\ a^z = a^{-1},\ b^2 = a^s,\ (ab)^2 = a^t \rangle ,$$

which shows that $\mathrm{Mon}(\mathcal{H})$ is a split extension of $G_{s,t}$ by $C_2 = \langle z \mid z^2 = 1 \rangle$. Hence $\kappa(\mathcal{H}) = 1$ and thus $\mathcal{H}$ is regular. □

As one may expect, the difficulty in dealing with chiral hypermaps sharply increases with the number k of hyperfaces. So our strategy will be based on an observation that there is a natural homomorphism $\Phi\colon \mathrm{Aut}^+\mathcal{H} \to S_k$ whose kernel is formed by automorphisms fixing point-wise all the k hyperfaces of $\mathcal{H}$. In what follows we examine the structure of $\mathrm{Mon}(H^+) \cong \mathrm{Aut}^+ H$ via the induced action in the symmetric group S_k for $k = 3$ and 4.

THEOREM 12. *If a hypermap $\mathcal{H}$ is chiral with 3 hyperfaces of valency n, then $n \ge 7$ and its oriented monodromy group is the metacyclic group $\mathrm{Mon}(\mathcal{H}^+) = \langle a,b \mid a^n = 1,\ b^3 = a^s,\ bab^{-1} = a^r \rangle$ for some $s \in \{0,\dots,n-1\}$ and $r \in \{2,\dots,n-1\}$ satisfying $(r-1)s = 0 \pmod n$ and $r^3 = 1 \pmod n$.*

Vice-versa, the group G with the above presentation defines an oriented hypermap (G, ab, b^{-1}) (where b and ab acts on G by right multiplication) which is chiral and has 3 hyperfaces. Moreover, different solutions (r,s) correspond to different (non-isomorphic) hypermaps.

Proof. We stretch first that it is clear from the presentation of the oriented monodromy group that if $\mathcal{H}_1$ and $\mathcal{H}_2$ are two chiral hypermaps with 3 hyperfaces of valency n with oriented monodromy groups corresponding to different solutions (r,s), then $\mathcal{H}_1$ and $\mathcal{H}_2$ are not isomorphic.

Let F_1, F_2 and F_3 be the hyperfaces of $\mathcal{H}$. They will be distributed around hypervertices and hyperedges as pictured below, where $F, F' \in \{F_1, F_2, F_3\}$. We cannot have $F = F' = F_1$ otherwise we would have 2 hyperfaces only. Then we have three cases:

(I) $F = F_3$, $F' = F_1$.
(II) $F = F' = F_3$.
(III) $F = F_1$, $F' = F_3$.

By taking the dual $D_{(0\,1)}$ that transpose hypervertices with hyperedges we see that Case (III) is equivalent to Case (I). Thus, it is sufficient to consider only the first two cases.

Let a be the permutation that cyclically permutes the darts counter-clockwise around F_1 and b the permutation that cyclically permutes the darts counter-clockwise around the hypervertex v.

Case (I).
In this case $a^n = 1$, $b^3 = a^s$, $(ab)^2 = a^t$ and $b^{-1}ab^{-1} = a^r$ for some $s, r, t \in \{0, \dots, n-1\}$. Let $G_{s,r,t}$ be the group with presentation

$$\langle a, b \mid a^n = 1,\ b^3 = a^s,\ (ab)^2 = a^t,\ b^{-1}ab^{-1} = a^r \rangle$$

and let K be the subgroup generated by a. From the relations $(ab)^2 = a^t$, $b^{-1}ab^{-1} = a^r$ and $b^3 = a^s$ we get $ba = a^{t-1}b^{-1}$, $b^{-1}a = a^r b$ and $b^{-1} = a^{-s}b^2$, respectively. Then $Kb^{-1} = Kb^2$, $Kb^3 = K$, $Kba = Kb^{-1} = Kb^2$, $Kb^2a = Kb^{-1}a = Kb$, and so $G_{s,r,t}$ is partitioned into 3 cosets K, Kb and Kb^2. Hence $G_{s,r,t}$ is finite ($|G_{s,r,t}| = 3\,\mathrm{ord}(a)$), and consequently, $\mathrm{Mon}(\mathcal{H}^+) = G_{s,r,t}$ for some s, r and t. But by the Substitution Test ([20]) the function $a \mapsto a^{-1}$, $b \mapsto b^{-1}$ extends to an automorphism of $G_{s,r,t}$. By Lemma 2, $\kappa(\mathcal{H}) = 1$, and so Case (I) gives rise only to regular hypermaps.

Case (II).
In this case $a^n = 1$, $b^3 = a^s$, $(ab)^3 = a^t$ and $bab^{-1} = a^r$ for some $s, r, t \in \{0, \dots, n-1\}$. Let $G_{s,r,t}$ be the group with presentation

$$\langle a, b \mid a^n = 1,\ b^3 = a^s,\ (ab)^3 = a^t,\ bab^{-1} = a^r \rangle$$

and let K be the subgroup generated by a. By $bab^{-1} = a^r$ we get $ba = a^r b$ and $b^2a = ba^r b$, and the relation $b^3 = a^s$ implies $b^{-1} = a^{-s}b^2$. Then $Kb^{-1} = Kb^2$, $Kb^3 = K$, $Kba = Kb$, $Kb^2a = Kba^rb = Kb^2$, and so $G_{s,r,t}$ decomposes in just 3 cosets K, Kb and Kb^2. In other words, $G_{s,r,t}$ is finite ($|G_{s,r,t}| = 3\,\mathrm{ord}(a)$), hence $\mathrm{Mon}(\mathcal{H}^+) = G_{s,r,t}$ for some s, r, t. Combining relations $b^3 = a^s \iff b^2 = a^s b^{-1}$ and $bab^{-1} = a \iff bab = a^r b^2$ we derive

$$a^t = (ab)^3 = aba^r b^2 = aba^{r+1}a^s b^{-1} = a(bab^{-1})^{r+s+1} = a^{r^2+rs+r+1}.$$

Notice that this shows that the relation $(ab)^3 = a^t$ is redundant. So $t = r^2 + rs + r + 1 \pmod{n}$. From $bab^{-1} = a^r$ we have

$$ba^s b^{-1} = a^{rs} \iff a^{(r-1)s} = 1\,.$$

Thus $(r-1)s = 0 \pmod{n}$. The same relation gives

$$ba^t b^{-1} = a^{rt} \iff b(ab)^3 b^{-1} = a^{rt} \iff (ba)^3 = a^{rt} \iff a^t = a^{rt}\,.$$

Consequently, $t(r-1) = 0 \pmod{n}$, which is equivalent to $r^3 = 1 \pmod{n}$ since $t(r-1) = (r-1)(r^2+r+1) + (r-1)rs = r^3 - 1 \pmod{n}$.

Assume $r = 1$. Then $ab = ba$ and

$$(ab)^3 = a^{s+2} = b^3 a^2 \iff a = 1\,.$$

This implies that $G_{s,r,t} = C_3$ is cycle and $\mathcal{H}$ has only one hypervertex. By Theorem 10, $\mathcal{H}$ is regular. If $r = 0$, then $r^3 = 1 \pmod{n}$ is equivalent to $1 = 0 \pmod{n}$. The only possibility is $n = 1$ in which case we would have again a cyclic group $G_{s,r,t}$ of order 3 and a regular hypermap. Hence $r \in \{2, \dots, n-1\}$.

Finally, the congruences $(r-1)s = 0 \pmod{n}$ and $r^3 = 1 \pmod{n}$, where $s \in \{0, \dots, n-1\}$ and $r \in \{2, \dots, n-1\}$, have no solutions for $n = 2, 3, 4, 5, 6$. Thus $n > 6$.

The other implication follows from Theorem 8. □

COROLLARY 13. *If $\mathcal{H}$ is a chiral hypermap of type (l, m, n) with 3 hyperfaces, then l and m must be equal to $0 \pmod 3$.*

P r o o f . In the proof of Theorem 12, we saw that we must have 3 distinct hyperfaces around both a hypervertex and a hyperedge (Case (III)). This implies that $l = 0 \pmod 3$ and $m = 0 \pmod 3$. □

COROLLARY 14. *There is no chiral map with 3 faces.*

P r o o f . This comes directly from Theorem 12, and Corollary 13. □

Up to a mirror image and a permutation among i-cells ($i \in \{0, 1, 2\}$), the following list shows all the chiral hypermaps (excluding mirror images) with 3 hyperfaces of valency $n < 27$:

N	r	s	l	m	n	V	E	F	darts	κ
4	2	0	3	3	7	7	7	3	21	7
10	3	0	3	3	13	13	13	3	39	13
12	4	0	3	9	9	9	3	3	27	3
16	7	0	3	3	19	19	19	3	57	19
18	4	3	9	9	9	3	3	3	27	3
18	9	0	3	6	14	14	7	3	42	7
18	4	0	3	3	21	21	21	3	63	7
30	7	0	3	18	18	18	3	3	54	3
36	7	6	9	6	18	6	9	3	54	3
36	3	0	3	6	26	26	13	3	78	13
36	3	13	3	6	26	26	13	3	78	13
42	7	3	18	9	18	3	6	3	54	3
46	4	7	9	9	21	7	7	3	63	7
46	4	14	9	9	21	7	7	3	63	7

THEOREM 15. *If $\mathcal{H}$ is a chiral hypermap with 4 hyperfaces of valency n, then $n \geq 5$ and its oriented monodromy group is the metacyclic group* $\mathrm{Mon}(\mathcal{H}^+) = \langle a,b \mid a^n = 1,\ b^4 = a^r,\ bab^{-1} = a^t\rangle$ *for some* $r \in \{0,\dots,n-1\}$ *and* $t \in \{2,\dots,n-1\}$ *satisfying* $(n,t) = 1$, $t^4 = 1 \pmod n$ *but* $t^2 \neq 1 \pmod n$, *and* $r(t-1) = 0 \pmod n$.

Vice-versa, the group G with the above presentation defines an oriented hypermap (G, ab, b^{-1}) (where b and ab act on G by right multiplication) which is chiral and has 4 hyperfaces. Moreover, different solutions (r,t) correspond to different (non-isomorphic) hypermaps.

Proof. Let us observe that the second part of the statement is a direct consequence of Theorem 8 and of Corollary 9.

Now let $\mathcal{H}$ be a chiral hypermap with 4 hyperfaces of valency n. Without any loss of generality (that is, up to a colour reassignment) we consider the number of distinct hyperfaces around hyperedges to be greater or equal than the number of distinct hyperfaces surrounding hypervertices.

Let F_1, F_2, F_3 and F_4 be the hyperfaces of $\mathcal{H}$, v a hypervertex incident to F_1 and e a hyperedge incident to both v and F_1. Let $A, B \in \mathrm{Aut}^+\mathcal{H}$ be the one step counter-clockwise rotations about F_1 and v respectively. Then AB is the one step rotation clockwise about e.

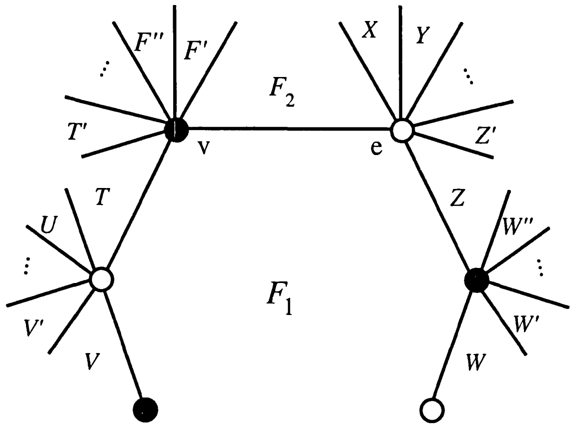

FIGURE 4. The faces surrounding face F_1.

To fix notations, let $a \in \mathrm{Mon}(\mathcal{H}^+)$ be the permutation that cyclically permutes the darts counter-clockwise around F_1 and $b \in \mathrm{Mon}(\mathcal{H}^+)$ the permutation that cyclically permutes the darts counter-clockwise around v. These permutations generate the oriented monodromy group $G = \mathrm{Mon}(\mathcal{H}^+)$. Let also $K = \langle a \rangle < G$.

To discuss the distribution of the hyperfaces around F_1, v and e we will use the transitive action of $\mathrm{Aut}^+ \mathcal{H} = \langle A, B \rangle$ on the 4 hyperfaces of $\mathcal{H}$ and will see A and B as permutations of S_4 rather than as elements of $\mathrm{Aut}^+ \mathcal{H}$, so in what follows we shall use the same letters A, B to denote the respective permutations of the 4-element set of hyperfaces induced by the action of A, B.

Since we must have at least 3 distinct hyperfaces around either v or e, without any lost of generality we may suppose that in the induced action on hyperfaces $B = (F_1, F_2, F_3)$ or $B = (F_1, F_2, F_3, F_4)$, that is for brevity, $B = (1,2,3)$ or $B = (1,2,3,4)$.

Case (I). $B = (1,2,3)$.
Then we have 4 possibilities for AB: $(1,2)(3,4)$, $(1,2,4)$, $(1,2,3,4)$ and $(1,2,4,3)$.

	B	AB	A
(Ia)	$(1,2,3)$	$(1,2)(3,4)$	$(2,3,4)$
(Ib)	$(1,2,3)$	$(1,2,4)$	$(2,4,3)$
(Ic)	$(1,2,3)$	$(1,2,3,4)$	$(3,4)$
(Id)	$(1,2,3)$	$(1,2,4,3)$	$(2,4)$

Case (II). $B = (1,2,3,4)$.
This produces 6 possibilities for AB: $(1,2)$, $(1,2)(3,4)$, $(1,2,3)$, $(1,2,4)$, $(1,2,3,4)$ and $(1,2,4,3)$, displayed in the table below in a different order for convenience.

	B	AB	A
(IIa)	$(1,2,3,4)$	$(1,2)$	$(2,4,3)$
(IIb)	$(1,2,3,4)$	$(1,2)(3,4)$	$(2,4)$
(IIc)	$(1,2,3,4)$	$(1,2,4,3)$	$(2,3,4)$
(IId)	$(1,2,3,4)$	$(1,2,3,4)$	id
(IIe)	$(1,2,3,4)$	$(1,2,3)$	$(3,4)$
(IIf)	$(1,2,3,4)$	$(1,2,4)$	$(2,3)$

Notice that (IIe) transfers to (Ic) by exchanging v with e, while (IIf) is the same as (Id) by a relabeling of hyperfaces. The knowledge on the local distribution of hyperfaces (colours) and the fact that $\mathrm{Mon}(\mathcal{H}^+) = \langle a,b\rangle$ acts regularly on darts of $\mathcal{H}$ allows us in each of the cases below to determine a set of relations on the generators a and b. This set differs from case to case, thus each of them has to be discussed separately.

Case (Ia).
We have $a^n = 1$, $b^3 = a^r$, $(ab)^2 = a^s \iff bab = a^{s-1}$ and $b^{-1}a^2b^{-1} = a^t \iff ba^3b^{-1} = a^{s+t-1} = a^u$ for some $r,s,t \in \{0,\dots,n-1\}$. Since b^2ab^i is not in $K = \langle a\rangle$, K, Kb, Kb^2 and Kb^2a are distinct cosets. From above relation we can also deduce that bab^{-2} and $b^2a^2b^{-1}$ are elements of K, so any coset word Kw can be reduced to one of the above 4 cosets. This means that the oriented monodromy group G has presentation

$$\langle a,b \mid a^n = 1,\ b^3 = a^r,\ (ab)^2 = a^s,\ b^{-1}a^2b^{-1} = a^t\rangle$$

for some $r,s,t \in \{0,\dots,n-1\}$. Then by the Substitution Test the function $a \mapsto a^{-1}$, $b \mapsto b^{-1}$ extends to an automorphism of G, so this case induces regular hypermaps.

Case (Ib).
We have the following relations $a^n = 1$, $b^3 = a^r$, $(ab)^3 = a^s$, $ba^2b = a^t$ and $b^2ab^2 = a^u$, for some $r,s,t,u \in \{0,\dots,n-1\}$. Thus G is partitioned into the 4 cosets K, Kb, Kb^2, Kba with the above relations as defining relations. Similarly as above, $a \mapsto a^{-1}$, $b \mapsto b^{-1}$ extends to an automorphism of G and so this case gives rise to regular hypermaps either.

Case (Ic).
The relations $a^n = 1$, $b^3 = a^r$, $(ab)^4 = a^s$, $b^{-1}ab = a^t$ and $b(a)^2b^{-1} = a^u$ hold in $G = \mathrm{Mon}(\mathcal{H}^+)$ for some $r, s, t, u \in \{0, \dots, n-1\}$. In fact they define G since they reduce any word Kw, $w \in F(a, b)$, to one of K, Kb, Kb^2 and Kba. But then $b^{-1}ab = a^t \iff baba = b^2a^{t+1} = b^{-1}a^{r+t+1}$, and so

$$a^{s-1} = bababab = b^{-1}a^{r+t+1}bab = a^{t(r+t+1)+1}b,$$

that is, $b \in K$ which forces $\mathcal{H}$ to have fewer than 4 hyperfaces, a contradiction.

Case (Id).
We have $a^n = 1$, $b^3 = a^r$, $(ab)^4 = a^s$, $bab^{-1} = a^t$ and $b^{-1}a^2b = a^u$ for some $r, s, t, u \in \{0, \dots, n-1\}$. These relations define G which is partitioned in the 4 cosets K, Kb, Kb^2 and Kb^2a. Similarly as in (Ic) we have $abab = a^{1+r+t}b^{-1}$ derived from $bab^{-1} = a^t$ and $b^3 = a^r$. Then

$$a^{s-1} = bababab = baba^{1+r+t}b^{-1} = ba^{t(1+r+t)+1},$$

that is, $b \in K$, a contradiction.

Case (IIa).
We take the dual $B = (1, 2)$, $AB = (1, 2, 3, 4)$ and $A = (2, 3, 4)$ instead. In this settlement we must have $a^n = 1$, $b^2 = a^r$, $(ab)^4 = a^s$, $ba^{-1}bab = a^t$, $ba^3b^{-1} = a^u$ and $baba^{-1}b^{-1} = a^v$ for some $r, s, t, u, v \in \{0, \dots, n-1\}$. The 4 distinct cosets of K in G are K, Kb, Kba and Kba^2. One easily see that $Kba^i = Kb$, Kba, or Kba^2 according as $i = 0, 1, 2 \pmod 3$. As ba^2, ba^2b and $ba^2ba^{-1}b^{-1}$ are not elements of K, then $Kba^2b = Kba^2$. But combining the 4th and 5th relations we get $ba^2bab = a^{t+u}$ and thus $Kba^2b = Kba^{n-1}$. Hence $n = 0 \pmod 3$. Now $Kbab = Ka^vba = Kba$ and $Kbab = Kb^{-1}a^t = Kba^t$, so $t = 1 \pmod 3$. But $a^{-1}ba = b^{-1}a^tb^{-1} = b^{-1}a^{t-r}b$, so powering up by r we have $a^r = b^{-1}a^{2t-2r}b \iff ba^rb^{-1} = a^{2t-2r} \iff a^{3r} = a^{2t}$, so $t = 0 \pmod 3$, which is a contradiction. So case (IIa) gives no chiral hypermap.

Case (IIb).
As in (IIa) we also take the dual $B = (1, 2)(3, 4)$ and $AB = (1, 2, 3, 4)$, which implies that $A = (2, 4)$. Then $a^n = 1$, $b^2 = a^r$, $(ab)^4 = a^s$ and $ba^2b = a^t$ for some $r, s, t \in \{0, \dots, n-1\}$. K, Kb, Kba and $Kbab$ are distinct cosets, so these make up the 4 cosets of K in G. One can easily see that the above relations are enough to reduce any word Kw, $w \in F(a, b)$, to one of the above cosets. Hence they define G. But by the Substitution Test, the function $a \mapsto a^{-1}$, $b \mapsto b^{-1}$ extends to an automorphism of G, and so this case induces regular hypermaps.

Case (IIc).
We have $a^n = 1$, $b^4 = a^r$, $(ab)^4 = a^s$, $bab^2 = a^t$, $b^{-1}ab^{-1} = a^u$, $b^2ab = a^v$ and $b^2a^2b^{-1} = a^x$ for some $r, s, t, u, v, x \in \{0, \dots, n-1\}$. These relations define $G = \mathrm{Mon}(\mathcal{H}^+)$ which is partitioned in cosets K, Kb, Kb^2 and Kb^3. Having in mind that a^r commutes with a and b, b^2ab commutes with a and $b^{-2} = a^{-r}b^2$

then

$$\begin{aligned} a^{2v} &= b^{-1}ab^{-2}ab^{-1} \\ &= a^{-r}b^{-1}ab^2ab^{-1} \\ &= a^{-2r}b^{-1}ab^2ab^3 \\ &= a^{-2r}b^{-1}a(b^2ab)b^2 \\ &= a^{-2r}b^{-1}(b^2ab)ab^2 \\ &= a^{-2r}ba(bab^2) \\ &= a^{-2r}ba^{t+1}\,. \end{aligned}$$

Thus $b \in K$, a contradiction.

Case (IId).

We have $a^n = 1$, $b^4 = a^r$, $(ab)^4 = a^s$, $bab^{-1} = a^t$, and $b^2ab^2 = a^u$ for some $r, s, t, u \in \{0, \dots, n-1\}$. As above, G is partitioned in cosets K, Kb, Kb^2 and Kb^3 and it is straightforward to see that the relations $a^n = 1$, $b^4 = a^r$, $bab^{-1} = a^t$ and $b^2ab^2 = a^u$ define $G = \mathrm{Mon}(\mathcal{H}^+)$. Since $b^2ab^2 = b^2a^{r+1}b^{-2} = a^{t^2(r+1)}$, then $v = t^2(r+1) \pmod n$. Note that from $b^{-1}ab = b^{-2}bab^{-1}b^2 = b^{-3}ba^tb^{-1}b^3 = a^{t^3}$, we compute $(ab)^4 = aba(b^{-1}b^2)abab = a^{t+1}b^2ab^2b^{-1}ab = a^{t+1+u+t^3}$. Hence

$$\mathrm{Mon}(\mathcal{H}^+) = \langle a, b \mid a^n = 1,\ b^4 = a^r,\ bab^{-1} = a^t \rangle$$

for some $r, t \in \{0, \dots, n-1\}$. From $bab^{-1} = a^t$ we derive that $(n, t) = 1$. From $b^{-1}ab = a^{t^3}$ we get $a = a^{t^4}$, and so $t^4 = 1 \pmod n$. By powering both sides of $bab^{-1} = a^t$ by r we get $a^r = a^{rt}$ which forces $r(t-1) = 0 \pmod n$. Finally, G determines a chiral hypermap if and only if the assignment $a \mapsto a^{-1}$ and $b \mapsto b^{-1}$ does not extend to an automorphism of G and, by the Substitution Test, this is equivalent to $b^{-1}ab \neq a^t \iff t^2 \neq 1 \pmod n$. For $n = 1, 2, 3, 4$ the system of equations $(n, t) = 1$, $t^4 = 1 \pmod n$, but $t^2 \neq 1 \pmod n$, and $r(t-1) = 0 \pmod n$ have no solutions $(r, t) \in \{0, \dots, n-1\} \times \{1, \dots, n-1\}$. □

COROLLARY 16. *If $\mathcal{H}$ is a chiral hypermap of type (l, m, n) with 4 hyperfaces, then l and m must be* $0 \pmod 4$.

P r o o f . As proven in the body of Theorem 15, $B = (1\,2\,3\,4) = AB$ which means that any chiral hypermap $\mathcal{H}$ with 4 hyperfaces must be a covering of a hypermap of type $(4, 4, 1)$, and so, if $\mathcal{H}$ has type (l, m, n), then $l = 0 \pmod 4$ and $m = 0 \pmod 4$. □

COROLLARY 17. *There is no chiral map with* 4 *faces.*

Up to a permutation among i-cells and mirror images, all the chiral hypermaps with 4 hyperfaces and valency $n < 30$ are listed in the following table:

N	r	t	l	m	n	V	E	F	darts	κ
6	0	2	4	4	5	5	5	4	20	5
16	0	3	4	4	10	10	10	4	40	5
26	5	3	8	8	10	5	5	4	40	5
22	0	5	4	4	13	13	13	4	52	13
26	0	2	4	4	15	15	15	4	60	5
36	0	7	4	12	15	15	5	4	60	5
46	10	7	12	12	15	5	5	4	60	5
36	0	3	4	8	16	16	8	4	64	2
40	0	5	4	16	16	16	4	4	64	2
48	8	5	8	16	16	8	4	4	64	2
30	0	4	4	4	17	17	17	4	68	17
36	0	3	4	4	20	20	20	4	80	5
36	0	13	4	4	20	20	20	4	80	5
66	5	13	16	16	20	5	5	4	80	5
66	15	13	16	16	20	5	5	4	80	5
56	10	3	8	8	20	10	10	4	80	5
56	10	13	8	8	20	10	10	4	80	5
46	0	7	4	4	25	25	25	4	100	25
48	0	5	4	4	26	26	26	4	104	13
74	13	5	8	8	26	13	13	4	104	13
54	0	12	4	4	29	29	29	4	116	29

COROLLARY 18. *The least number of faces of a chiral map is* 5.

Proof. By Corollary 14, and Corollary 17, no chiral map with at most 4 faces exists. On the other hand, it is known ([12]) that the complete graph $\mathcal{K}_5$ has a chiral embedding $\mathcal{M} = \{4,4\}_{2,1}$ on the Torus with 5 faces and 5 vertices. □

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Received December 14, 2000
Revised May 13, 2002

* *Department of Mathematics*
University Aveiro
Aveiro
PORTUGAL

** *Mathematical Institute*
Slovak Academy of Science
Severná 5
SK-974 00 Banská Bystrica
SLOVAKIA