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RANDOMLY COMPLETE n-PARTITE GRAPHS

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Let G be a graph containing a subgraph H without isolated vertices. In [3] the concept of "randomly H graphs" is defined as follows: We call G a *ramdomly H graph* if any subgraph of G without isolated vertices which is isomorphic to a subgraph of H, can be extended to a subgraph H_1 of G such that H_1 is isomorphic to H. The property of being a randomly H graph is a generalization of the properties of "arbitrarily traceable graphs" (later referred to as "randomly eulerian graphs") introduced by Ore [4] in 1951, "randomly hamiltonian graphs" introduced by Chartrand and Kronk [2] in 1968, and "radomly matchable graphs" introduced by Sumner [5] in 1979. In [5] a historical background of how the concept "randomly" is used in some areas of graph theory is provided.

Every nonempty graph is randomly K_2, while every graph G without isolated vertices is a randomly G graph. Let P_n denote the path of order n. In Figure 1,

S:

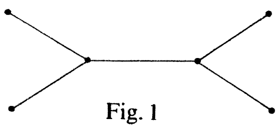

Fig. 1

the double star graph S is randomly P_3, but not randomly P_4. The graph $K(3,4)$ is randomly $K(2,2)$, while the graph $G = K(2,2,3)$ in Figure 2 is not randomly $K(2,2,2)$. This follows since there is no way that the subgraph F, even though it is isomorphic to a subgraph of $K(2,2,2)$, can be extended to a subgraph K of G that is isomorphic to $K(2,2,2)$.

In [3] the authors pointed out the following concerning isolated vertices: In the definition of randomly H graphs, it was stipulated that H and F be without isolated vertices. Suppose that H has order m and that $\mathcal{G}$ is the class of all randomly H graphs. Let $H' = H \cup nK_1$, where n is a positive integer. If, in the definition of randomly H graphs, the requirement that H is without isolated vertices is deleted, then the class $\mathscr{G}'$ of all randomly H' graphs consists of those graphs in $\mathscr{G}$ having order at least $m + n$. Hence it suffices to assume that H is without isolated vertices.

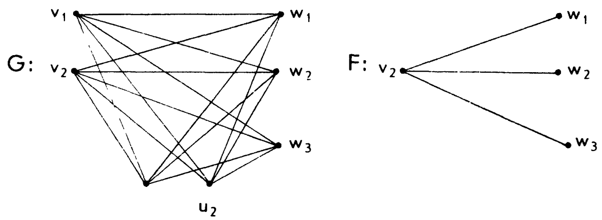

Fig. 2

The authors [3] also point out the following: Since a graph G without isolated vertices is a radomly H graph if and only if $G \cup K_1$ is randomly H, we can assume that every randomly H graph is free of isolated vertices. In order to avoid a situation where only complete graphs would be randomly H for a variety of graphs H, we have required F to be without isolated vertices; otherwise, for example, only complete graphs of order at least 2 would be randomly K_2 (which can be seen by taking F to be the empty graph on two vertices).

In [3] Chartrand, Oellermann, and Ruiz characterized graphs that are randomly H graphs where H is $2K_2$, P_3, $K_n (n \geq 2)$, or C_n. They also characterized grphs G that are randomly H graphs for every subgraphs H of G. In this paper we characterize graphs G that are randomly complete n-partite graphs.

Even though all the terms used in this paper may be found in [1], we will define a few here. The graphs G is said to be *n-partite*, for n an integer at least 2, if it is possible to partition the vertex set V of G into n subsets $V_1, V_2, \ldots, V_n$ such that every edge of G joins a vertex of V_ν to a vertex of V_μ, $\nu \neq \mu$. The sets $V_1, V_2, \ldots, V_n$ are called *partite sets* of the n-partite graph G. If $n = 2$, such graphs are called *bipartite graphs*. A *complete n-partite graph G* is an n-partite graph with partite sets $V_1, V_2, \ldots, V_n$ having the additional property that if u is a vertex of V_ν and v is a vertex of V_μ, $\nu \neq \mu$, then uv is an edge of G. If the set V_ν has order p_ν, $1 \leq \nu \leq n$, then the symbol $K(p_1, p_2, \ldots, p_n)$ is used to denote this complete n-partite graph. The *set of neighbors* of a vertex of a graph G is the set of all vertices of G adjacent to v. We shall use $|G|$ to denote the order of the graph G and $|S|$ to denote the number of elements of the set S. If U is a nonempty subset of the vertex set $V(G)$ of a graph G, then the subgraph $\langle U \rangle$ of G *induced by* U is the graph having vertex set U and whose edge set consists of those edges of G incident with two elements of U. A subgraph H of G is called *induced* if $H = \langle U \rangle$ for some subset U of $V(G)$.

In the remainder of this paper we shall characterize randomly complete n-partite graphs. In order to study these graphs, we will use the following notation. We will be using $n \geq 2$ positive integers $p_1, p_2, \ldots, p_n$ and will asume that

$$p_1 \leq p_2 \leq \ldots \leq p_n \qquad \text{and} \qquad p = p_1 + p_2 + \ldots + p_n.$$

When the integers $p_1, p_2, \ldots, p_n$ are given we will use K to denote the complete n-partite graph $K(p_1, p_2, \ldots, p_n)$. We begin with the following observation.

Proposition 1. *If the graph G is randomly K, then*
a) *G contains a subgraph isomorphic to K;*
b) *the order of G is at least p;*
c) *G is connected;*
d) *the minimum degree of G is at least $p - p_n$; and*
e) *the maximum degree of G is at least $p - p_1$.*

We are now ready to prove the first result about randomly complete n-partite graphs.

Proposition 2. *Let G be a randomly K graph with $p_{n-1} \geq 2$. If G contains a subgraph isomorphic to K_{n+1}, then G is complete.*

Proof. Assume that the maximum order of a complete subgraph of G is t. Then $t \geq n + 1$. Let H* be a complete subgraph of G of order t. If H^* has the same order as G, then G is complete. We assume that $|G| > |H^*|$. Since G is randomly K, Proposition 1 implies that G is connected and so there is a vertex v of $V(G) - V(H^*)$ that is adjacent to a vertex v_1 of H^*. If v is adjacent to all of the vertices of H^*, then $V(H^*) \cup \{v\}$ induces a complete subgraph of G of order $t + 1$, which is a contradiction. Thus there is some vertex v_2 of $V(H^*)$ such that v is not adjacent to v_2 in G. Let $v_3, v_4, \ldots, v_{n+1}$ be any other $n - 1$ vertices of H^*. Then the set of vertices $V = \{v_1, v_2, \ldots, v_{n+1}\}$ induces a complete subgraph H of G of order $n + 1$. Also the vertex v is adjacent to v_1, and v is not adjacent to v_2, and $V(H) = V$. Let H' be the n-partite subgraph of G with $V(H') = V(H) \cup \{u\}$, partite sets $V_1 = \{v, v_{n+1}\}$, $V_2 = \{v_1, v_2\}$, and $V_v = \{v_v\}$, $3 \leq v \leq n$, and edge set $E(H') = E(H) \cup \{vv_1\} - \{v_1v_2\}$. Then H' is isomorphic to a subgraph of K (since $p_{n-1} \geq 2$) and can be extended to a complete n-partite subgraph H'' of G isomorphic to K.

We show that v and v_2 are not in the same partite sets of H'' which implies that they are adjacent in G, a contradiction which completes the proof. Each vertex of the set $V - \{v_1, v_2\}$ must be in distinct partite sets of H'' because they are mutually adjacent in H''. This represents $n - 1$ distinct partite sets of H''. Now if v_1 and v_2 are in distinct partite sets of H'' they must be different from the

$n - 1$ partite sets already mentioned because v_1 and v_2 are each adjacent to every vertex of $V - \{v_1, v_2\}$, then H'' would have $n + 1$ distinct partite sets in a complete n-partite graph. Since this is a contradiction, v_1 and v_2 must be in the same partite set of H''. Since v is adjacent to v_1 in H'', it is in a different partite set of H'' than v_1 and so v is also adjacent to v_2. This is the desired contradiction, which completes the proof.

We now characterize those graphs that are randomly complete bipartite. The case $K(1, n)$ was covered in [3].

Theorem 1. *A graph G is randomly $K(p_1, p_2)$, $p_2 \geq p_1 \geq 2$, if and only if G isomorphic to a complete bipartite graph $K(s_1, s_2)$ where $s_1 \geq p_1$ and $s_2 \geq p_2$, or G is isomorphic to a complete graph $K_{p'}$ where $p' \geq p_1 + p_2$.*

Proof. It is easy to see that if G is isomorhic to $K_{p'}$, where $p' \geq p_1 + p_2$ or G is isomorphic to $K(s_1, s_2)$, where $s_1 \geq p_1$ and $s_2 \geq p_2$, then G is randomly $K(p_1, p_2)$.

Assume that G is randomly $K(p_1, p_2)$ and that G is not complete. Then G contains a subgraph H isomorphic to $K(p_1, p_2)$. By Proposition 2, G contains no subgraph isomorphic to K_3, and so the subgraph induced by each partite set of H is empty, implying that H is an induced subgraph of G. Let G' be a complete bipartite subgrph of G containing a subgraph isomorphic to $K(p_1, p_2)$ of maximum order, say t. Then $t \geq p_1 + p_2$. Let V_1 and V_2 be the partite sets of G'. Proposition 2 implies that G' is an induced subgraph of G.

If $V(G) = V(G')$, then, since G' is an induced subgraph of G, $G = G'$. Suppose that G' is a proper subgraph of G. Since G is connected, there exists a vertex v of $V(G) - V(G')$ that is adjacent to a vertex of $V(G')$. Since G contains no subgraphs isomorphic to K_3, there exists a partite set V_i, $i = 1$ or 2, such that v is adjacent to none of the vertices of V_i. Select v_1 of V_1 and v_2 of V_2. Without loss of generality we may assume that vv_1 is not an edge of G and that vv_2 is an edge of G. Let w be any other vertex of V_2. Then the subgraph of G induced by $\{v, v_1, v_2, w\}$ is the path vv_2v_1w which is isomorphic to a subgraph of $K(p_1, p_2)$. Since G is randomly $K(p_1, p_2)$, $\langle\{v, v_1, v_2, w\}\rangle$ can be extended to a complete bipartite subgraph G'' of G that is isomorphic to $K(p_1, p_2)$. Since the path vv_2v_1w is a subgraph of G'', vw must be an edge of G'' (and of G). Thus v is adjacent to each vertex of V_2. The graph T induced by $V_1 \cup V_2 \cup \{v\}$ is a complete bipartite subgraph of G containing G'' of order $t + 1$. Since this is a contradiction, $G = G'$, which completes the proof.

We now look at some special cases of the complete n-partite graphs where $n \geq 3$. We begin by considering the case where the graph G is a complete n-partite graph.

Proposition 3. *Let n, $p_1, p_2, \ldots, p_n$, $s_1, s_2, \ldots, s_n$ be positive integers with $n \geq 3$, $p_1 \leq p_2 \leq \ldots \leq p_n$ and $s_1 \leq s_2 \leq \ldots \leq s_n$. If $G = K(s_1, s_2, \ldots, s_n)$ is randomly K, then $s_1 \geq p_1$ and $s_\nu = p_\nu$, $2 \leq \nu \leq n$.*

Proof. Since $G = K(p_1, p_2, \ldots, p_n)$ is randomly K, G contains a subgraph isomorphic to K. Thus $p_\nu \leq s_\nu$, $1 \leq \nu \leq n$.

Let $V_1, V_2, \ldots, V_n$ be the partite sets of G with $|V_\nu| = s_\nu$, $1 \leq \nu \leq n$. We begin by showing that $p_n = s_n$. Suppose that $s_n > p_n$. Let v be a vertex of V_1 and let V_n' be a subset of $p_n + 1$ vertices of V_n. Let $K(1, p_n + 1)$ be a star with center at v and endvertices in V_n'. Since $n \geq 3$, $K(1, p_n + 1)$ is isomorphic to a subgraph of K. The graph G is randomly K and so $K(1, p_n + 1)$ can be extended to a complete n-partite subgraph H of G that is isomorphic to K. There exist two vertices x and y in V_n' such that x and y belong to different partite sets of H, so that x and y are adjacent in H. But x and y belong to the same partite set of G and so x and y are non-adjacent in G. This is a contridiction and so $s_n = p_n$.

Let ν be the largest integer such that $s_\nu > p_\nu$, $2 \leq \nu < n$. Since $2 \leq \nu < n$, $K = K(p_1, p_2, \ldots, p_n)$ contains a subgraph isomorphic to the graph $T = K(p_\nu + 1, p_{\nu+1}, \ldots, p_n)$. Let V_ν' be a $p_\nu + 1$ subset of V_ν. Let H be the subgraph of G induced by the set $V_\nu' \cup V_{\nu+1} \cup V_{\nu+2} \cup \ldots \cup V_n$. Then H is isomorphic to $T = K(p_\nu + 1, s_{\nu+1}, \ldots, s_n)$ which is isomorphic to a subgraph of K and can be extended to a complete n-partite subgraph H' of G which is isomorphic to K. There exist two verties x and y of V_ν such that x and y belong to different paritie sets of H' and so are adjacent in H' (and in G), while they belong to the same partite set V_ν of G and so are non-adjacent in G. This is a contradiction, and so $p_\nu = s_\nu$. Thus $p_\mu = s_\mu$ for $2 \leq \mu \leq n$.

Proposition 4. *Let n, s_1, $p_1, p_2, \ldots, p_n$ be positive integers with $n \geq 3$ and $p_1 < s_1 \leq p_2 \leq \ldots \leq p_n$. If $G = K(s_1, p_2, \ldots, p_n)$ is randomly K, then $s_1 = p_2 = \ldots$ $\ldots = p_n$ and $p_1 = s_1 - 1$.*

Proof. Assume that not all of the p's are equal to s_1. Let ν be the smallest subscript such that $s_1 < p_\nu$, $2 \leq \nu \leq n$. That is, $s_1 = p_2 = \ldots = p_{\nu-1}$. Let V_1, $V_2, \ldots, V_n$ be the partite sets of G such that $|V_1| = s_1$ and $|V_\mu| = p_\mu$, $2 \leq \mu \leq n$. Let $V_1' = \{v_1, . v_2, \ldots, v_t\}$ be a set of $t = p_1 + 1$ vertices of V_1, let u be a vertex of V_ν and let $V_\nu' = V_\nu - \{u\}$. Let T be the complete n-parite subgraph of G with partite sets $V_1', V_2, \ldots, V_{\nu-1}, V_\nu', V_{\nu+1}, \ldots, V_n$. Here $|V_1'| = p_1 + 1$, $|V_\nu'| = p_\nu - 1$, and $|V_\mu| = p_\mu$, $2 \leq \mu \leq n$, ν not equal to μ. Let S be the subgraph of T (and of G) with vertex set $V(S) = V(T)$ and edge set

$$E(S) = E(T) - \{vv_1 : v \text{ is a vertex of } V_\nu'\}.$$

Then S is isomorphic to a subgraph of K and can be extended to a complete n-partite subgraph S' of G that is isomorphic to K. Because of the construction, each of the sets V_μ, $2 \leq \mu \leq n$, μ not equal to ν, is a partite set of S'. The vertices of $V_1' \cup V_\nu'$ cannot be contained in any of these $n-2$ partite sets because they are adjacent to vertices of these sets. Since each vertex of V_ν' is adjacent to each vertex of $V_1' - \{v_1\}$, all the vertices of $V_1' - \{v_1\}$ must be contained in one of the partite sets of S', say U_1, while all of the vertices of V_ν' must be contained in a different partite set U_ν of S'. Since $|V_1' - \{v_1\}| = p_1$, $|V_\nu'| = p_\nu - 1$, and $p_\nu > p_1 + 1$, we have that v_1 must be in the partite set U_ν of S'. Then in S' (and G), v_1 is adjacent to all of the vertices of U_1, and in particular, v_1 is adjacent to v_2 of U_1. Thus in S', v_1 and v_2 are adjacent, which contradicts the fact that v_1 and v_2 are non-adjacent in G. Thus, $s_1 = p_2 = p_3 = \ldots = p_n$. We assume next that $s_1 > p_1 + 1$. Let $V_1, V_2, \ldots, V_n$ be the partite sets of $G = K(s_1, p_2, \ldots, p_n)$, where $|V_1| = s_1$ and $|V_\nu| = p_\nu$, $2 \leq \nu \leq n$. Let $t = p_1 + 1$ and let $V_1' = \{v_1, v_2, \ldots, v_t\}$ be a set of $t = p_1 + 1$ vertices of V_1 and let v be a vertex of V_2. Let $V_2' = V_2 - \{u\}$. Let T be the complete n-partite subgraph of G with partite sets $V_1', V_2', V_3, \ldots, V_n$. Here $|V_1'| = t = p_1 + 1$ and $|V_2'| = p_2 - 1$. Let S be the subgraph of T with vertex set $V(S) = V(T)$ and edge set

$$E(S) = E(T) - \{vv_1 : v \text{ is a vertex of } V_2'\}.$$

Then S is isomorphic to a subgraph of K and can be extended to a complete n-partite subgraph S' of G that is isomorphic to K. The sets $V_3, V_4, \ldots, V_n$ must be partite sets of S' and the verices of $V_1' \cup V_2'$ cannot be contained in any of these $n-2$ partite sets. Thus there are two partite sets of S' that can contain the vertices of $V_1' \cup V_2'$. Since each vertex of V_2' is adjacent to each vertex of $V_1' - \{v\}$, all of the vertices of $V_1' - \{v_1\}$ must be contained in one partite set U_1 of S' and all of the vertices of V_2' must be contined in another partite set U_2 of S_1'. If v_1 is in U_1, then S' has two partite set with different sizes than the partite sets of K, since $s_1 = p_2 > p_1 + 1$; which is a contradiction. Thus v_1 must be a vertex of U_2. But then v_1 and v_2 are adjacent in S' (a subgraph of G), but non-adjancent in G. This is a contradiction, and so $s_1 = p_2 \leq p_1 + 1$. Since $p_1 < s_1$, $p_1 = s_1 - 1$.

We now look at the complete n-partite graph $K(p_1, p_2, \ldots, p_n)$ where

$$p_1 = p_2 = \ldots = p_{n-1} = 1 \qquad \text{and} \qquad p_n \leq 2\,.$$

We can rewrite such a complete n-partite graph as the join of a complete graph with $n-1$ vertices and a totally disconnected graph on p_n vertices. We shall use the notation $K_m + \bar{K}_n$ for such a complete n-partite graph. The complete graph K_p we can denote by $K_p + \bar{K}_0$.

Theorem 2. *Let $m \geq 2$ and $n \geq 3$ be integers. The graph G is randomly $K_{n-1} + \bar{K}_m$ if and only if G is isomorphic to $K_r + \bar{K}_s$ where*

$$r \geq n-1, m \geq s \geq 0, \quad \text{and} \quad r + s \geq m + n - 1\,.$$

Proof. Assume that G is randomly $K_{n-1} + \bar{K}_m$. Then G contains a subgraph isomorphic to $K_{n-1} + \bar{K}_m$ and the order of G is greater than $m + n - 1$. Let H be a complete subgraph of G of maximum order t. Then the order of H is at least n, that is, $t \geq n$. If $|H| = |G|$, then G is isomorphic to K_t (we can write K_t as $K_t + \bar{K}_0$) and we have finished. We assume that $V(G) = V(G)$ is not empty. Since G is connected, let v be a vertex of $V(G) - V(H)$ that is adjacent to a vertex u of H. We claim that $k = |N(v) \cap V(H)| = |V(H)| - 1$. If v is adjacent to all the vertices of H, then $V(H) \cup \{v\}$ induces a complete graph of order $t + 1$; a contradiction. So $k \leq |V(H)| - 1$. Let x, y be vertices of $V(H) - \{u\}$ such that v is not adjacent to either x or y. Let S be a subset of $V(H)$ of order n containing x, y, and u. Then the subgraph $\langle S \rangle$ of G induced by S is isomorphic to K_n. Let H' be the subgraph of G whose vertex set is $V(H') = V(\langle S \rangle) \cup \{v\}$ and whose edge set is $E(H') = E(\langle S \rangle) \cup \{uv\}$. Then, since $m \geq 2$, H' is isomorphic to a subgraph of $K_{n-1} + \bar{K}_m$. Since G is randomly $K_{n-1} + \bar{K}_m$, H' can be extended to a complete n-partite graph H'' of G that is isomorphic to $K_{n-1} + \bar{K}_m$. Thus v is adjacent to exactly one of the vertices x and y, which is a contradiction. Thus $k = |V(H)| - 1$.

We next show that if u and v are vertices of $V(G) - V(H)$ such that

$$|N(u) \cap V(H)| = |N(v) \cap V(H)| = |V(H)| - 1,$$

then $N(u) \cap V(H) = N(v) \cap V(H)$. Suppose to the contrary that these two sets are not equal. Let $\{x\} = V(H) - N(u)$ and $\{y\} = V(H) - N(v)$. Then x and y are distinct vertices of H, u is not adjacent to x but to all of the other vertices of H, and v is not adjacent to y but to all of the other vertices of H. Let T be a subset of $V(H)$ of order $n - 1$ containing x and y. The subgraph $\langle T \rangle$ of G induced by T is isomorphic to K_{n-1}. Let H' be the subgraph of G whose vertex set is $V(H') = V(\langle T \rangle) \cup \{u, v\}$ and whose edge set is $E(H') = E(\langle T \rangle) \cup \cup \{uy, vx\}$. Then H' is a subgraph of G that is isomorphic to a subgraph of $K_{n-1} + \bar{K}_m$ and so can be extended to a complete n-partite subgraph of G isomorphic to $K_{n-1} + \bar{K}_m$. Thus ux and vy are edges of G, a contradiction, and we have that $N(u) \cup V(H) = N(v) \cup V(H)$.

We now show that $\langle V(G) - V(H) \rangle$ is a totally disconnected subgraph of G, that is, if u and v are vertices of $\langle V(G) - V(H) \rangle$, then u and v are non-adjacent. Suppose to the contrary that $\langle V(G) - V(H) \rangle$ is not totally disconnected. Since G is connected, there are vertices u and v of $\langle V(G) - V(H) \rangle$ that are adjacent and such that u is adjacent to a vertex w of $V(H)$. Using the result from above, u is adjacent to $|V(H)| - 1$ vertices of H. Suppose that x is the vertex of H such that x is not adjacent to u. Let S be a subset of H of order $n - 1$ that contains x and w. Then the subgraph $\langle S \rangle$ of G induced by S is isomorphic to K_{n-1}. Let H' be the subgraph of G with vertex set $V(H') = V(\langle S \rangle) \cup \{u, v\}$ and edge set $E(H) = E(\langle S \rangle) \cup \{uw, uv\}$. The subgraph H' of G is isomorphic to a subgraph of $K_{n-1} + \bar{K}_m$ and can be extended to a complete n-partite subgraph H'' of G

that is isomorphic to $K_{n-1} + \bar{K}_m$. Let T be the set of vertices of H'' that have degree $m+n-2$ and let T' be the set of vertices of degree $n-1$. Then $|T| = n-1$ and $|T'| = m$. Since x and u are not adjacent G (and in H''),

$$\deg_{H''} x < m+n-2 \quad \text{and} \quad \deg_{H''} u < m+n-2 .$$

So x and u are vertices of T'. Since u and v are adjacent, and u is in T', v must be in T. So in H'', x and v must be adjacent. By the above results,

$$N(u) \cap V(H) \quad \text{and} \quad N(v) \cap V(H)$$

are the same set, say $V(H) - \{y\}$. Then the graph induced by $V(H) \cap \{u, v\} - \{y\}$ is a complete subgraph of G of order $|V(H)| + 1$, which has one more vertex than H. Since this is a contradiction, the graph induced by $V(G) - V(H)$ is totally disconnected.

Combining the results from above, we get that G must be isomorphic to the graph $K_{|H|-1} + \bar{K}_{|G|-|H|+1}$. Since $|H| \geq n$, we have $|H| - 1 \geq n-1$. Suppose that $|G| - |H| + 1 > m$. Then G contains an induced subgraph R that is isomorphic to $K(1, m+1)$. Also $K_{n-1} + \bar{K}_m$ contains a subgraph isomorphic to $K(1, m+1)$. Because R is an induced subgraph of G it is not possible to extend R to asubgraph of G isomorphic to $K_{n-1} + \bar{K}_m$. Since this is a contradiction, $|G| - |H| + 1 \leq m$. Letting $r = |H| - 1$ and $s = |G| - |H| + 1$, we have the desired result.

Conversely, we need to show that if $r \leq n-1$, $m \geq s \geq 0$, and $r + s \geq m + n - 1$, then $G = K_r + \bar{K}_s$ is randomly $K_{n-1} + \bar{K}_m$. Suppose that V_1 and V_2 are subsets of $V(G)$ such that $\langle V_1 \rangle$ is isomorphic to K_r and $\langle V_2 \rangle$ is isomorphic to $\bar{K}_s$. Let H be a subgraph of $K_r + \bar{K}_s$ that is isomorphic to a subgraph of $K_{n-1} + \bar{K}_m$. If $|V(H) \cap V_1| \leq n-1$, let S be any $(n-1)$-subset of V_1 containing $V(H) \cap V_1$. Then H is a subgraph of $\langle S \cup V_2 \rangle$ and $\langle S \cup V_2 \rangle$ is isomorphic to $K_{n-1} + \bar{K}_m$. If $|V(H) \cap V_1| = n - 1 + t$, $0 < t \leq m$, then $|V(H)| = n - 1 + t + |V(H) \cap V_2| \leq n - 1 + m$, so that $|V(H) \cap V_2| \leq m - t$. Not that H is a subgraph of $K_{n-1} + \bar{K}_m$ and $|V(H) \cap V_1| = n - 1 + t$, $V(H) \cap V_1$ contains an independent set (with respect to H) of t vertices, say S_1. Let S_2 be a $(m-t)$-subset of V_2 that contains $V(H) \cap V_2$. Then H is a subgraph of $\langle (V(H) \cap V_1) \cap S_2 \rangle - - E(\langle S_1 \rangle)$ which is isomorphic to $K_{n-1} + \bar{K}_m$. Hence, in each case, H can be extended to a subgraph of $K_r + \bar{K}_s$ that is isomorphic to $K_{n-1} + \bar{K}_m$. This completes the proof.

We are now ready to consider the case where $p_{n-1} \geq 2$.

Theorem 3. *Let $n \geq 3$, $p_1 \leq p_2 \leq \ldots \leq p_n$ be positive integers such that $p_{n-1} \geq 2$. A graph G is randomly K if and only if G is isomorphic to one of the following*

(i) K_p, *where* $p = p_1 + p_2 + \dots + p_n$,
(ii) K, *or*
(iii) $K(s_1, s_2, \dots, s_n$ *where* $s_i = p_1 + 1 = p_j$, $1 \le i \le n$, and $2 \le j \le n$.

Proof. Assume that G is randomly K and that G is not complete. Then G contains a subgraph H isomorphic to K. By Proposition 2, G contains no subgraph isomorphic to K_{n+1}, and so the subgraph induced by each partite set of H is empty, implying that H is an induced subgraph of G. Let G' be an induced complete n-partite subgraph of G containing a subgraph isomorphic to K of maximum order. If the order of G' is t, then $t \ge p$. Let V_v, $1 \le n \le n$, be the partite sets of G'. Again, Proposition 2 implies that G' is an induced subgraph of G.

We claim that G' is the graph G. Suppose to the contrary, that G' is a proper subgraph of G. Since G is connected there exists a vertes v of $V(G) - V(G')$ that is adjacent to a vertex of $V(G')$. Since G contains no subgraph isomorphic to K_{n+1}, there exists a partite set V_ν, for some ν with $1 \le \nu \le n$, such that v is adjacent to none of the vertices of V_ν. Select a vertex v_μ from V_μ for each μ, $1 \le \mu \le n$. Without loss of generality, we may assume that vv_1 is not an edge of G and that vv_2 is an edge of G. The subgraph G'' of G induced by the vertices $v, v_1, v_2, \dots, v_n$ is isomorphic to a subgraph of K. Since G is randomly K, G'' can be extended to a complete n-partite subgraph T of G that is isomorphic to K. The subgraph of G induced by the vertices $v_1, v_2, \dots, v_n$ is isomorphic to K_n, and so the vertices $v_1, v_2, \dots, v_n$ must lie in distinct partite sets of T. The fact that vv_1 is not an edge of G implies that v is adjacent to all of the vertices v_ν, $2 \le \mu \le n$. Since $v_3, v_4, \dots, v_n$ were arbitrary vertices of $V_3, V_4, \dots, V_n$, respectively, for any vertex w of $V^* = V_3 \cup V_4 \cup \dots \cup V_n$, v is adjacent to w. If we now interchange the roles of v_2 and v_3, we see that every vertex of V_2 is adjacent to v. Thus v is adjacent to every vertex of $V_2 \cup V^*$ and so the subgraph T' induced by the set of vertices $V(G') \cup \{v\}$ is isomorphic to a complete n-partite graph which contains a subgraph isomorphic to K and has order $p(G') + 1 = t + 1$. This contradicts the choice of G'. Thus there are no vertices in $V(G) - V(G')$, and so $V(G) = V(G')$. Finally, since G' is an induced subgraph of G, $G' = G$. Since G is a complete n-partite graph, Propositions 3 and 4 imply that G fits part (ii) or (iii) in the statement of the theorem. Conversely, if G is isomorphic to K or $K_{p'}$, $p' \ge p$, it follows that G is randomly K. So assume $c \ge 2$ and $n \ge 3$. We need to show that if $p_1 = c - 1$ and $p_\nu = c$, $2 \le \nu \le n$, then $K' = K(s_1, s_2, \dots, s_n)$, where $s_\mu = c$, $1 \le \mu \le n$, is randomly K. Suppose that H is any subgraph of K' isomorphic to a subgraph of K. Then $p(H) \cap p(K') = p(K) + 1$. Then there is a vertex v in $V(K) - V(K')$. Then H is a subgraph of $\langle V(K') - \{v\} \rangle$ which is isomorphic to K. Since any subgraph of a graph can be extended to the graph itself, H can be extended to a subgraph of K' that is isomorphic to K. Therefore K' is randomly K, which completes the proof.

The general question here is for what classes of graphs H is it possible to characterize all those graphs G that are randomly H. This paper has answered the question for one class of graphs, namely, the complete n-partite graphs.

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СЛУЧАЙНО ПОЛЬНЫЕ n-ДОЛЬНЫЕ ГРАФЫ

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Резюме

Пусть G граф, содержащий субграф H без изолированных вершин. Граф G называется случайно H графом, если произволный субграф графа G без изолированных вершин, который изоморфный субграфу графа H, можно расширить на субграф H_1 графа G который изоморфный графу H. Свойство бить случайно H графом является обобщением таких свойств как «случайно проходящий» или «случайно эулеровые графы», «случайно гамильтоновые графы» или «случайно паросочетания способные грфы». Пусть $K(p_1, p_2, \ldots, p_n)$ означает полный n-дольный граф с классами V_μ порядка p_μ, $1 \leqq \mu \leqq n$. Главный результат — это характеризация тех графов, которые являются случайно $K(p_1, p_2, \ldots, p_n)$ графами.