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ON THE UNIQUE SOLVABILITY OF SEMI-LINEAR ELLIPTIC SYSTEMS

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ABSTRACT. In this paper, we study the unique solvability of semi-linear elliptic systems of partial differential equations. Our method of the proof is based on the Banach fixed point theorem.

1. Introduction

In this paper, we study the unique solvability of semi-linear elliptic systems of partial differential equations of the form

$$\begin{aligned} -\Delta u &= f(x,u,\nabla u) && \text{in} \quad \Omega\,, \\ u &= 0 && \text{on} \quad \partial\Omega\,, \end{aligned} \tag{b}$$

where Ω is a smooth bounded domain in $\mathbb{R}^n$, $u\colon \overline{\Omega} \to \mathbb{R}^M$, $f\colon \Omega \times \mathbb{R}^M \times \mathbb{R}^{Mn} \to \mathbb{R}^M$.

It is known that the system (b) possesses multiple solutions if the nonlinearity f interacts suitably with the spectrum of the operator $-\Delta_D$ (i.e., the operator $-\Delta$ with a homogeneous Dirichlet boundary condition) (see, e.g., [AZ] or [H]). In this paper, we are concerned with the complementary case, where f does not interact with this spectrum.

In his paper [A], H. Amann has given unique solvability results for semi-linear systems in the case, where f does not interact with the spectrum of $-\Delta_D$, and f does not depend on the gradient:

$$\begin{aligned} -\Delta u &= f(x,u) && \text{in} \quad \Omega\,, \\ u &= 0 && \text{on} \quad \partial\Omega\,. \end{aligned} \tag{a}$$

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The gradient-dependent case (b) is studied in [QŽ] for a single equation $(M=1)$. In this paper, f satisfies

i) $a \le \frac{f(x,u_1,p)-f(x,u_2,p)}{u_1-u_2} \le b, \quad (a,b)\cap\sigma(-\Delta_D)=\emptyset$,

ii) $|f(x,u,p_1)-f(x,u,p_2)| \le c|p_1-p_2|$, where $c>0$ is sufficiently small.

The method of proof is based on the Banach fixed point theorem: it is shown that the mapping $w \mapsto u$, where u is the weak solution of the problem

$$\begin{aligned} -\Delta u &= f(x,u,\nabla w) && \text{in } \Omega\,, \\ u &= 0 && \text{on } \partial\Omega \end{aligned}$$

is a contraction.

Our method of proof of unique solvability of (b) is analogous to that in [QŽ], but the estimates are carried out in a more precise way. Moreover, using the uniform contraction theorem, we show the continuous dependence of solutions of systems of the form

$$\begin{aligned} -\Delta u &= f(x,u,\nabla u,\lambda) && \text{in } \Omega\,, \\ u &= 0 && \text{on } \partial\Omega \end{aligned} \qquad (b_\lambda)$$

on the parameter λ. Our main result is formulated in Theorem 4.4.

2. Preliminaries

Let us first introduce some notation. By $\mathbb{R}$ we shall denote the set of all reals. $\Omega\subset\mathbb{R}^n$, for some $n\ge 1$, is a bounded domain with smooth boundary $\partial\Omega$ (C^2). For $p_j=(p_j^1,\dots,p_j^s)\in\mathbb{R}^s$, $j=1,2$, we define the scalar product $(p_1,p_2)_{\mathbb{R}^s}=\sum\limits_{i=1}^{s}p_1^ip_2^i$. The Lebesgue space $L_2(\Omega,\mathbb{R}^M)$ will be equipped with the usual scalar product $(u,v)_0=\int\limits_\Omega\big(u(x),v(x)\big)_{\mathbb{R}^M}\,\mathrm{d}x$, while the Sobolev space $H_0^1(\Omega,\mathbb{R}^M)$ will be equipped with the scalar product

$$\begin{aligned} (u,v)_1 &= \sum_{i=1}^{n}\left(\frac{\partial}{\partial x_i}u,\frac{\partial}{\partial x_i}v\right)_0 \\ &= \sum_{i=1}^{n}\left(\left(\frac{\partial u_1}{\partial x_i},\dots,\frac{\partial u_M}{\partial x_i}\right),\left(\frac{\partial v_1}{\partial x_i},\dots,\frac{\partial v_M}{\partial x_i}\right)\right)_0 . \end{aligned}$$

The corresponding norms will be denoted by $|\cdot|_{\mathbb{R}^s}$, $\|\cdot\|_0$ and $\|\cdot\|_1$, respectively. We define now a linear operator $A_0\colon \operatorname{dom}(A_0)\subset L_2(\Omega,\mathbb{R})\to L_2(\Omega,\mathbb{R})$ by

$$\operatorname{dom}(A_0)=H_0^1(\Omega,\mathbb{R}^M)\cap H^2(\Omega,\mathbb{R}^M) \qquad \text{and}$$

$$A_0u := -\sum_{i=1}^{n}\frac{\partial^2 u}{\partial x_i^2} \qquad \text{for} \quad u\in\operatorname{dom}(A_0)\,.$$

It is well known that A_0 is self-adjoint, that it has a compact resolvent and that the spectrum of A_0, $\sigma(A_0) = \{\lambda_i\}_{i=1}^{\infty}$, where $0 < \lambda_1 \le \lambda_2 \le \dots$ and $\lambda_i \to \infty$. Let $\{\varphi_i\}_{i=1}^{\infty}$ be the corresponding sequence of eigenfunctions, which we assume to be normalized in $L_2(\Omega, \mathbb{R})$, i.e.,

$$\|\varphi_i\|^2_{L_2(\Omega,\mathbb{R})} = \int_\Omega \varphi_i^2 \,\mathrm{d}x = 1\,, \qquad i = 1, 2, \dots .$$

Thus it forms an orthonormal base in $L_2(\Omega, \mathbb{R})$ and it is also a complete orthogonal set in $H_0^1(\Omega, \mathbb{R})$ such that

$$\|\varphi_i\|^2_{H_0^1(\Omega,\mathbb{R})} = \int_\Omega (\nabla\varphi_i, \nabla\varphi_i)_{\mathbb{R}^n} \,\mathrm{d}x = \lambda_i\,, \qquad i = 1, 2, \dots .$$

Finally we define a self-adjoint linear operator with compact resolvent:

$$A\colon\ \mathrm{dom}(A) \subset L_2(\Omega, \mathbb{R}^M) \to L_2(\Omega, \mathbb{R}^M)$$

for some $M \ge 1$ by:

$$\mathrm{dom}(A) := \big[\mathrm{dom}(A_0)\big]^M \qquad \text{and} \qquad A := \mathrm{diag}(A_0, \dots, A_0)\,.$$

Now we suppose that the function $f\colon \Omega \times \mathbb{R}^M \times \mathbb{R}^{Mn} \to \mathbb{R}^M$ satisfies the hypothesis (H):

$$(\mathrm{H}) \begin{cases} \text{i)} & f \textit{ is a Caratheodory function.} \\ \text{ii)} & \textit{There exist } c_1, c_2 > 0 \textit{ such that} \\ & |f(x,u_1,p_1) - f(x,u_2,p_2)|_{\mathbb{R}^M} \le c_1|u_1 - u_2|_{\mathbb{R}^M} + c_2|p_1 - p_2|_{\mathbb{R}^{Mn}} \\ & \textit{for a.a. } x \in \Omega \textit{ and all } u_1, u_2 \in \mathbb{R}^M \textit{ and } p_1, p_2 \in \mathbb{R}^{Mn}. \\ \text{iii)} & f(\cdot\,, 0, 0) \in L_2(\Omega, \mathbb{R}^M)\,. \end{cases}$$

Then, by a weak solution of the semi-linear elliptic system

$$\begin{aligned} -\Delta u &= f(x, u, \nabla u) && \text{in} \quad \Omega\,, \\ u &= 0 && \text{on} \quad \partial\Omega \end{aligned}$$

we mean a function $u \in H_0^1(\Omega, \mathbb{R}^M)$ such that

$$\sum_{i=1}^{n} \int_\Omega \left(\frac{\partial u}{\partial x_i}, \frac{\partial v}{\partial x_i}\right)_{\mathbb{R}^M} \mathrm{d}x = \int_\Omega \big(f(x,u,\nabla u), v\big)_{\mathbb{R}^M} \,\mathrm{d}x$$

$$\text{for all} \quad v \in H_0^1(\Omega, \mathbb{R}^M)\,.$$

We get from standard regularity theory that $u \in H_0^1(\Omega, \mathbb{R}^M) \cap H^2(\Omega, \mathbb{R}^M)$ and u is a solution of the operator equation

$$Au = F(u)\,,$$

where F is the Nemytskii operator corresponding to f.

3. Apriori estimates

We introduce the following hypothesis:

(H1)
- i) *$h\colon \Omega\times\mathbb{R}^M \to \mathbb{R}^M$ is a Caratheodory function such that $h(x,\cdot)\in C^1(\mathbb{R}^M,\mathbb{R}^M)$ with a symmetric derivative $\mathrm{D}_2\, h(x,u) := \frac{\partial}{\partial u}h(x,u)$ for a.a. $x\in\Omega$ and all $u\in\mathbb{R}^M$.*
- ii) *There exist two symmetric $M\times M$-matrices B^+, B^- such that*
$$B^- \le \mathrm{D}_2\, h(x,u) \le B^+ \quad \textit{and} \quad \bigcup_{i=1}^{M}[\mu_i^-,\mu_i^+]\cap\sigma(A)=\emptyset\,,$$
where $\mu_1^\pm \le \mu_2^\pm \le \dots \le \mu_M^\pm$ are the eigenvalues of the matrices $B^\pm$, respectively.
- iii) $h(\cdot\,,0)\in L_2(\Omega,\mathbb{R}^M)$.

The inequalities in ii) are to be understood in the sense that $B^+\ge B^-$ means that B^+-B^- is positive semi-definite. From [VK; p. 109, Courant-Fischer theorem], we have the equalities $\mu_i^\pm = \min\limits_{\substack{\dim(V)=i\\ V\subset\mathbb{R}^M}} \max\limits_{\substack{x\ne 0\\ x\in V}} \dfrac{(B^\pm x,x)}{(x,x)}$ for $i=1,\dots,M$, which implies the inequalities $\mu_i^-\le\mu_i^+$ for $i=1,\dots,M$.

Now we put

$$\nu := \min\{\lambda_i \mid \lambda_i\in\sigma(A)\,,\ \lambda_i>\mu_M^+\}\,,$$
$$\varepsilon := \begin{cases} \operatorname{dist}\Big(\bigcup\limits_{i=1}^{M}[\mu_i^-,\mu_i^+],\sigma(A)\Big) & \text{for } \mu_M^+\ge 0\,,\\ \lambda_1 & \text{for } \mu_M^+<0\,. \end{cases} \tag{3.1}$$

Recall that (H1) ii) implies $\varepsilon>0$.

Let us formulate the apriori estimates lemma:

Lemma 3.1. *Let $B\colon\Omega\times\mathbb{R}^M\to\mathbb{R}^M$ satisfy* (H1) *with h replaced by B and $B(x,0)=0$ for a.a. $x\in\Omega$. Let $f\colon\Omega\times\mathbb{R}^M\to\mathbb{R}^M$ be a Caratheodory function, and let there exist $0\le\gamma<\frac{\varepsilon}{2}$ (ε as in* (3.1)*) and $\rho\in L_2(\Omega,\mathbb{R})$ such that:*

$$|f(x,u)|_{\mathbb{R}^M} \le \gamma|u|_{\mathbb{R}^M} + \rho(x) \tag{3.2}$$

for a.a. $x\in\Omega$ and all $u\in\mathbb{R}^M$. Let $z\in H_0^1(\Omega,\mathbb{R}^M)\cap H^2(\Omega,\mathbb{R}^M)$ be a solution of the equation

$$Az = G(z)\,, \tag{3.3}$$

where G denotes the Nemytskii operator corresponding to $B+f$. Then

$$\|z\|_0 \le \frac{\|\rho\|_0}{\frac{\varepsilon}{2}-\gamma} \tag{3.4}$$

and

$$\|z\|_1 \le \frac{\sqrt{\nu}}{\frac{\varepsilon}{2}-\gamma}\|\rho\|_0 \,. \tag{3.5}$$

P r o o f . It is obvious that $\sigma(A)=\sigma(A_0)$. We put $H:=L_2(\Omega,\mathbb{R}^M)$ and

$$\sigma_{H^-} := \{\lambda_i \in \sigma(A):\ \lambda_i < \mu_1^-\}, \qquad \sigma_{H^+} := \{\lambda_i \in \sigma(A):\ \mu_M^+ < \lambda_i\},$$
$$\sigma_Z := \{\lambda_i \in \sigma(A):\ \mu_1^- < \lambda_i < \mu_M^+\}.$$

Let $H = H^- \oplus Z \oplus H^+$ be the orthogonal decomposition of H corresponding to the decomposition of the spectrum $\sigma(A) = \sigma_{H^-} \cup \sigma_Z \cup \sigma_{H^+}$. Let $e_i^\pm$ (for $i=1,\dots,M$) be the eigenvectors of $B^\pm$ corresponding to $\mu_i^\pm$, respectively. We now put

$$\begin{aligned} Z^- &= \operatorname{span}\{\varphi_i e_j^- :\ \mu_1^- < \lambda_i < \mu_j^-,\ i\in\mathbb{N},\ j=1,2,\dots,M\},\\ Z^+ &= \operatorname{span}\{\varphi_i e_j^+ :\ \mu_j^+ < \lambda_i < \mu_M^+,\ i\in\mathbb{N},\ j=1,2,\dots,M\}. \end{aligned} \tag{3.6}$$

Thus we have

$$\begin{aligned} ((A-B^-)u,u)_0 &< 0 \qquad \text{for all} \quad u\in Z^-\setminus\{0\},\\ ((A-B^+)u,u)_0 &> 0 \qquad \text{for all} \quad u\in Z^+\setminus\{0\}, \end{aligned}$$

hence $Z^+\cap Z^- = \{0\}$. It also follows from (3.6) that $\dim Z = \dim Z^+ + \dim Z^-$. Consequently,

$$Z = Z^+ \oplus Z^-.$$

Putting $X^- := H^-\oplus Z^-$ and $X^+ := H^+\oplus Z^+$ we get the decomposition of H

$$H = X^+ \oplus X^- .$$

In X^+, X^-, we can choose the orthonormal bases

$$\{\psi_{ij}^- = \varphi_i e_j^- :\ \lambda_i < \mu_j^-,\ i\in\mathbb{N},\ j=1,2,\dots,M\},$$
$$\{\psi_{ij}^+ = \varphi_i e_j^+ :\ \lambda_i > \mu_j^+,\ i\in\mathbb{N},\ j=1,2,\dots,M\},$$

respectively. Let $z\in H_0^1(\Omega,\mathbb{R}^M)\cap H^2(\Omega,\mathbb{R}^M)$ be a solution of (3.3). We have $z = z^- + z^+$, where $z^\pm \in X^\pm$ respectively. Define also $\tilde z = -z^- + z^+$. For a.a. $x\in\Omega$, the mean value theorem implies the existence of $\xi(x)\in\mathbb{R}^M$ such that

$$Az(x) = B\big(x,z(x)\big) + f\big(x,z(x)\big) = \mathrm{D}_2 B\big(x,\xi(x)\big)z(x) + f\big(x,z(x)\big). \tag{3.7}$$

Since by (H1) i) $S(x) := \mathrm{D}_2 B\big(x,\xi(x)\big)$ is a symmetric $M\times M$-matrix, we get from (H1) ii) that

$$\begin{aligned} \big(S(x)z(x),\tilde z(x)\big)_{\mathbb{R}^M} &= \big(S(x)z^+(x),z^+(x)\big)_{\mathbb{R}^M} - \big(S(x)z^-(x),z^-(x)\big)_{\mathbb{R}^M}\\ &\le \big(B^+z^+(x),z^+(x)\big)_{\mathbb{R}^M} - \big(B^-z^-(x),z^-(x)\big)_{\mathbb{R}^M} \end{aligned} \tag{3.8}$$

for a.a. $x \in \Omega$. Testing the equation (3.7) by $\tilde{z}$ and using (3.2), (3.8) we get

$$\begin{aligned}(Az,\tilde{z})_0 &= (Az^+,z^+)_0 - (Az^-,z^-)_0 \\ &\le (B^+z^+,z^+)_0 - (B^-z^-,z^-)_0 + \gamma\|z\|_0\|\tilde{z}\|_0 + \|\rho\|_0\|\tilde{z}\|_0 .\end{aligned} \tag{3.9}$$

Let $z = \overset{-}{\sum} z_{ij}^- \psi_{ij}^- + \overset{+}{\sum} z_{ij}^+ \psi_{ij}^+$, $z_{ij}^\pm \in \mathbb{R}$, be the Fourier series of z with respect to the base $\{\psi_{ij}^-\} \cup \{\psi_{ij}^+\}$ in $L_2(\Omega, \mathbb{R}^M)$, where $\overset{-}{\sum} = \sum\limits_{\substack{i\in\mathbb{N}\\ j=1,\dots,M\\ \lambda_i<\mu_j^-}}$ and $\overset{+}{\sum} = \sum\limits_{\substack{i\in\mathbb{N}\\ j=1,\dots,M\\ \lambda_i>\mu_j^+}}$.

Then

$$\begin{aligned}\varepsilon\big(\|z^-\|_0^2 + \|z^+\|_0^2\big) &\le \overset{+}{\sum}(\lambda_i - \mu_j^+){z_{ij}^+}^2 + \overset{-}{\sum}(\mu_j^- - \lambda_i){z_{ij}^-}^2 \\ &= \big((A-B^+)z^+,z^+\big)_0 - \big((A-B^-)z^-,z^-\big)_0 \\ &\le \gamma\|z\|_0\|\tilde{z}\|_0 + \|\rho\|_0\|\tilde{z}\|_0 .\end{aligned} \tag{3.10}$$

Now, since

$$\begin{aligned}\|z^+ \pm z^-\|_\alpha^2 &= \|z^+\|_\alpha^2 \pm 2(z^+,z^-)_\alpha + \|z^-\|_\alpha^2 \\ &\le \|z^+\|_\alpha^2 + 2\|z^+\|_\alpha\|z^-\|_\alpha + \|z^-\|_\alpha^2 \\ &\le 2\big(\|z^+\|_\alpha^2 + \|z^-\|_\alpha^2\big)\end{aligned} \tag{3.11}$$

(for $\alpha = 0$ or 1), we get from (3.10)

$$\frac{\varepsilon}{2}\|z\|_0\|\tilde{z}\|_0 \le \gamma\|z\|_0\|\tilde{z}\|_0 + \|\rho\|_0\|\tilde{z}\|_0 ,$$

which implies (3.4). Using the Young inequality $ab \le \frac{a^2}{4s} + sb^2$ and (3.11) we have from (3.10)

$$\overset{+}{\sum}(\lambda_i - \mu_j^+){z_{ij}^+}^2 + \overset{-}{\sum}(\mu_j^- - \lambda_i){z_{ij}^-}^2 \le 2(\gamma+s)\big(\|z^+\|_0^2 + \|z^-\|_0^2\big) + \frac{\|\rho\|_0^2}{4s} \tag{3.12}$$

for arbitrary $s \in \big(0, \frac{\varepsilon}{2} - \gamma\big)$. Hence,

$$\overset{+}{\sum}\left[\frac{\lambda_i - \mu_j^+ - 2\gamma - 2s}{\lambda_i}\right]\lambda_i {z_{ij}^+}^2 + \overset{-}{\sum}\left[\frac{\mu_j^- - \lambda_i - 2\gamma - 2s}{\lambda_i}\right]\lambda_i {z_{ij}^-}^2 \le \frac{\|\rho\|_0^2}{4s} .$$

Since

$$\left[\frac{\lambda_i - \mu_j^+ - 2\gamma - 2s}{\lambda_i}\right] \ge \frac{\varepsilon - 2(\gamma+s)}{\nu} \qquad \text{for} \quad \lambda_i > \mu_j^+ ,$$

$$\left[\frac{\mu_j^- - \lambda_i - 2\gamma - 2s}{\lambda_i}\right] \ge \frac{\varepsilon - 2(\gamma+s)}{\nu} \qquad \text{for} \quad \lambda_i < \mu_j^- ,$$

we get, using (3.11),

$$\frac{\varepsilon - 2(\gamma + s)}{2\nu}\|z\|_1^2 \le \frac{\|\rho\|_0^2}{4s}.$$

Thus we have the inequality

$$\|z\|_1^2 \le \min_{s\in(0,\frac{\varepsilon}{2}-\gamma)} \frac{\nu}{4s\big[(\frac{\varepsilon}{2}-\gamma)-s\big]}\|\rho\|_0^2,$$

which implies (3.5). □

4. Unique solvability results

In the following, we shall use the result of H. Amann [A; p. 166, Theorem 4.2] on the unique solvability of nonlinear elliptic systems.

LEMMA 4.1. *Suppose that the function $g\colon \Omega\times\mathbb{R}^M \to \mathbb{R}^M$ satisfies* (H1) *with h replaced by g. Then the semi-linear elliptic system (a) (with f replaced by g) possesses exactly one weak solution.*

First we prove a more general existence and uniqueness theorem, in which we relax the assumptions on the function g.

THEOREM 4.2. *Let $g\colon \Omega\times\mathbb{R}^M \to \mathbb{R}^M$ be Caratheodory function such that $g(\cdot\,,0) \in L_2(\Omega,\mathbb{R}^M)$. Suppose there exist $g_i\colon \Omega\times\mathbb{R}^M \to \mathbb{R}^M$ for $i=1,2$ such that $g = g_1 + g_2$ and:*

- (G1) *g_1 satisfies* (H1) *with h replaced by g_1,*
- (G2) *g_2 satisfies the Lipschitz condition in the second variable with Lipschitz constant δ:*

$$|g_2(x,u_1) - g_2(x,u_2)|_{\mathbb{R}^M} \le \delta|u_1 - u_2|_{\mathbb{R}^M} \tag{4.1}$$

for a.a. $x\in\Omega$ and all $u_1,u_2\in\mathbb{R}^M$.

If $0\le\delta<\frac{\varepsilon}{2}$, then the system (a) (with f replaced by g) has exactly one weak solution.

P r o o f . As a consequence of Lemma 4.1, we obtain that the mapping $T\colon L_2(\Omega,\mathbb{R}^M) \to L_2(\Omega,\mathbb{R}^M)$ defined by $u = Tq$ and

$$\begin{aligned} -\Delta u &= g_1(x,u) + g_2(x,q) &&\text{in } \Omega,\\ u &= 0 &&\text{on } \partial\Omega \end{aligned}$$

is well defined. We show that T is a contraction in the space $L_2(\Omega,\mathbb{R}^M)$, so that Theorem 4.2 will follow from the Banach fixed point theorem. So choose arbitrary $q_1, q_2 \in L_2(\Omega, \mathbb{R}^M)$, and let $u_i = Tq_i$, $i = 1, 2$. Denote $h = q_1 - q_2$, $z = u_1 - u_2$. If we put $B(x, w) = g_1(x, w + u_2(x)) - g_1(x, u_2(x))$ and $f(x) = g_2(x, q_2(x) + h(x)) - g_2(x, q_2(x))$, then z is a solution of the semi-linear system

$$\begin{aligned} -\Delta z &= B(x, z) + f(x) \qquad && \text{in } \Omega\,, \\ z &= 0 && \text{on } \partial\Omega\,, \end{aligned}$$

and the assumptions of Lemma 3.1 are fulfilled with $\gamma = 0$. Hence we get

$$\|z\|_0 \le \frac{2\delta}{\varepsilon} \|h\|_0$$

since

$$|f(x)|_{\mathbb{R}^M} \le \delta |h(x)|_{\mathbb{R}^M}$$

for a.a. $x \in \Omega$ by (4.1). We see that the mapping T is a contraction since $\frac{2\delta}{\varepsilon} < 1$. □

In the sequel, we shall use the following classical generalization of Banach fixed point theorem.

Lemma 4.3. (**Uniform contraction theorem**) *Let (X, d) be a complete metric space, (Λ, d_Λ) a metric space, and let U be an open subset of Λ. We assume that $K\colon X \times U \to X$ is a uniform contraction in the first variable, that is, there exist a constant $0 \le k < 1$ such that*

$$d(K(x, \lambda), K(y, \lambda)) \le k d(x, y)$$

for all $x, y \in X$ and $\lambda \in U$. Assume, moreover, that $K(x, \cdot)\colon U \to X$ is continuous in λ_0 for all $x \in X$. Then for every $\lambda \in U$ there exists a unique x_λ such that $x_\lambda = K(x_\lambda, \lambda)$. Moreover, the mapping $\lambda \mapsto x_\lambda\colon U \to X$ is continuous in λ_0.

After these preparations, we can now prove the main theorem.

Theorem 4.4. *Let Λ be an open subset of a metric space (X, d), and let the function $g\colon \Omega \times \mathbb{R}^M \times \mathbb{R}^{Mn} \times \Lambda \to \mathbb{R}^M$ be such that:*

i) *for every $\lambda \in \Lambda$, $g_\lambda = g(\cdot, \cdot, \cdot, \lambda)\colon \Omega \times \mathbb{R}^M \times \mathbb{R}^{Mn} \to \mathbb{R}^M$ is a Caratheodory function, and* $\mathrm{D}_2\, g(x, u, p, \lambda) := \frac{\partial}{\partial u} g(x, u, p, \lambda)$, $\mathrm{D}_3\, g(x, u, p, \lambda) := \frac{\partial}{\partial p} g(x, u, p, \lambda)$ *exist for all $(x, u, p, \lambda) \in \Omega \times \mathbb{R}^M \times \mathbb{R}^{Mn} \times \Lambda$;*

ii) *$\lambda \to \lambda_0 \in \Lambda$ implies*

$$\sup_{x,u,p} |\,\mathrm{D}_2\, g(x, u, p, \lambda) - \mathrm{D}_2\, g(x, u, p, \lambda_0)|_{\mathbb{R}^{M^2}} \to 0\,,$$

$$\sup_{x,u,p} |\,\mathrm{D}_3\, g(x, u, p, \lambda) - \mathrm{D}_3\, g(x, u, p, \lambda_0)|_{\mathbb{R}^{M^2}{}_n} \to 0\,,$$

$$\|g(\cdot, 0, 0, \lambda) - g(\cdot, 0, 0, \lambda_0)\|_0 \to 0\,;$$

iii) *there exist* $\lambda_0 \in \Lambda$ *such that* $g(\cdot, 0, 0, \lambda_0) \in L_2(\Omega, \mathbb{R}^M)$ *and* $g_{\lambda_0} = g_{1\lambda_0} + g_{2\lambda_0}$, *where* $g_{i\lambda_0} \colon \Omega \times \mathbb{R}^M \times \mathbb{R}^{Mn} \to \mathbb{R}^M$ $(i = 1, 2)$ *and:*
   (G1) $g_{1\lambda_0}(\cdot, \cdot, p)$ *satisfies* (H1) *with* h *replaced by* $g_{1\lambda_0}(\cdot, \cdot, p)$ *for all* $p \in \mathbb{R}^{Mn}$ *with* $B^{\pm}$ *independent of* p,
   (G2) $g_{2\lambda_0}$ *satisfies a Lipschitz condition in the second variable with constant* $0 \le \delta < \frac{\varepsilon}{2}$,
   (G3) g_{λ_0} *satisfies a Lipschitz condition in the third variable with constant* $0 \le c < \frac{\frac{\varepsilon}{2} - \delta}{\sqrt{\nu}}$.

Then there exists an open ball $B := B(\lambda_0, r)$ *in* (X, d) *such that the system*

$$\begin{aligned} -\Delta u &= g(x, u, \nabla u, \lambda) && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega \end{aligned} \tag{b_λ}$$

has exactly one weak solution u_λ *for all* $\lambda \in B$. *Moreover,* $\|u_\lambda - u_{\lambda_0}\|_1 \to 0$ *when* $d(\lambda, \lambda_0) \to 0$.

Proof. It follows from iii) that there are symmetric matrices B^+, B^- and constants ε, δ, c, ν such that

$$0 \le \delta < \frac{\varepsilon}{2} \quad \text{and} \quad 0 \le c < \frac{\frac{\varepsilon}{2} - \delta}{\sqrt{\nu}}, \tag{4.2}$$

$$B^- \le \mathrm{D}_2\, g_{1\lambda_0} \le B^+,$$

$$\begin{aligned} |g_{2\lambda_0}(x, u_1, h) - g_{2\lambda_0}(x, u_2, h)|_{\mathbb{R}^M} &\le \delta |u_1 - u_2|_{\mathbb{R}^M}, \\ |g_{\lambda_0}(x, u, h_1) - g_{\lambda_0}(x, u, h_2)|_{\mathbb{R}^M} &\le c |h_1 - h_2|_{\mathbb{R}^{Mn}} \end{aligned} \tag{4.3}$$

for a.a. $x \in \Omega$ and all $(u_i, h) \in \mathbb{R}^M \times \mathbb{R}^{Mn}$ or $(u, h_i) \in \mathbb{R}^M \times \mathbb{R}^{Mn}$, respectively. Now, it follows from (4.2) that we can choose $L > 0$ such that

$$0 \le \delta + L < \frac{\varepsilon}{2} \quad \text{and} \quad 0 \le c + L < \frac{\frac{\varepsilon}{2} - (\delta + L)}{\sqrt{\nu}}. \tag{4.4}$$

The assumption ii) implies the existence of $r > 0$ such that $B(\lambda_0, r) \subset \Lambda$ and

$$\begin{aligned} |\,\mathrm{D}_2\, g(x, u, p, \lambda) - \mathrm{D}_2\, g(x, u, p, \lambda_0)|_{\mathbb{R}^{M^2}} &< L, \\ |\,\mathrm{D}_3\, g(x, u, p, \lambda) - \mathrm{D}_3\, g(x, u, p, \lambda_0)|_{\mathbb{R}^{M^2 n}} &< L \end{aligned} \tag{4.5}$$

for a.a. $x \in \Omega$ and all $(u, p) \in \mathbb{R}^M \times \mathbb{R}^{Mn}$. For $\lambda \in B(\lambda_0, r)$ we define

$$g_{1\lambda} = g_{1\lambda_0} \quad \text{and} \quad g_{2\lambda} = g_\lambda - g_{1\lambda_0}.$$

From (4.3) and (4.5), we get

$$\begin{aligned} |g_{2\lambda}(x, u_1, h) - g_{2\lambda}(x, u_2, h)|_{\mathbb{R}^M} &\le (\delta + L)|u_1 - u_2|_{\mathbb{R}^M}, \\ |g_\lambda(x, u, h_1) - g_\lambda(x, u, h_2)|_{\mathbb{R}^M} &\le (c + L)|h_1 - h_2|_{\mathbb{R}^{Mn}}. \end{aligned} \tag{4.6}$$

Now we define the operator $T\colon H_0^1(\Omega,\mathbb{R}^M)\times B(\lambda_0,r)\to H_0^1(\Omega,\mathbb{R}^M)$ by $T(h,\lambda)=u$ and

$$\begin{aligned} -\Delta u &= g(x,u,\nabla h,\lambda) && \text{in } \Omega\,,\\ u&=0 && \text{on } \partial\Omega\,. \end{aligned}$$

It follows from (4.2) that T is well defined. First we show the uniform contractivity of T. We choose arbitrary $\lambda\in B(\lambda_0,r)$, $q_1,q_2\in H_0^1(\Omega,\mathbb{R}^M)$, and denote $u_i=T(q_i,\lambda)$ for $i=1,2$, $h=q_1-q_2$, and $z=u_1-u_2$. Then $z\in H_0^1(\Omega,\mathbb{R}^M)$ is a weak solution of

$$\begin{aligned} -\Delta z &= B(x,z)+f(x,z) && \text{in } \Omega\,,\\ z&=0 && \text{on } \partial\Omega\,, \end{aligned}$$

where

$$\begin{aligned} B(x,w) &= g_{1\lambda}\big(x,w+u_2(x),\nabla q_1(x)\big)-g_{1\lambda}\big(x,u_2(x),\nabla q_1(x)\big)\\ f(x,w) &= \big[g_{2\lambda}\big(x,w+u_2(x),\nabla q_1(x)\big)-g_{2\lambda}\big(x,u_2(x),\nabla q_1(x)\big)\big]\\ &\quad+\big[g_{\lambda}\big(x,u_2(x),\nabla q_1(x)\big)-g_{\lambda}\big(x,u_2(x),\nabla q_2(x)\big)\big]\,. \end{aligned}$$

The functions B and f satisfy the assumptions of Lemma 3.1, so that we get

$$\|z\|_1\le \frac{(c+L)\sqrt{\nu}}{\frac{\varepsilon}{2}-(\delta+L)}\|h\|_1\,.$$

The inequality (4.4) implies the uniform contractivity of T. Now we put $u_\lambda=T(h,\lambda)$, $u_{\lambda_0}=T(h,\lambda_0)$ for $\lambda\in B(\lambda_0,r)$ and $h\in H_0^1(\Omega,\mathbb{R}^M)$. Then $z:=u_\lambda-u_{\lambda_0}$ is a weak solution of the system

$$\begin{aligned} -\Delta z &= B(x,z)+f(x,z) && \text{in } \Omega\,,\\ z&=0 && \text{on } \partial\Omega \end{aligned}$$

with

$$\begin{aligned} B(x,w) &= g_{1\lambda}\big(x,w+u_{\lambda_0}(x),\nabla h(x)\big)-g_{1\lambda}\big(x,u_{\lambda_0}(x),\nabla h(x)\big)\\ f(x,w) &= \big[g_{2\lambda}\big(x,w+u_{\lambda_0}(x),\nabla h(x)\big)-g_{2\lambda}\big(x,u_{\lambda_0}(x),\nabla h(x)\big)\big]\\ &\quad+\big[(g_\lambda-g_{\lambda_0})\big(x,u_{\lambda_0}(x),\nabla h(x)\big)-(g_\lambda-g_{\lambda_0})(x,0,0)\big]\\ &\quad+\big[(g_\lambda-g_{\lambda_0})(x,0,0)\big]\,. \end{aligned}$$

From (4.5), (4.6), we get

$$\begin{aligned} &|f(x,w)|_{\mathbb{R}^M}\le\\ &\le(\delta+L)|w|_{\mathbb{R}^M}+L_{\lambda,\lambda_0}\big(|u_{\lambda_0}(x)|_{\mathbb{R}^M}+|\nabla h(x)|_{\mathbb{R}^{Mn}}\big)+|(g_\lambda-g_{\lambda_0})(x,0,0)|_{\mathbb{R}^M} \end{aligned}$$

with

$$L_{\lambda,\lambda_0} := \Big[\sup_{x,u,p} |\,\mathrm{D}_2(g_\lambda - g_{\lambda_0})|_{\mathbb{R}^{M^2}} + \sup_{x,u,p} |\,\mathrm{D}_3(g_\lambda - g_{\lambda_0})|_{\mathbb{R}^{M^2 n}} \Big] .$$

We recall that (ii) implies $L_{\lambda,\lambda_0} \to 0$ if $\lambda \to \lambda_0$. Now we have from Lemma 3.1

$$\|z\|_1 \le \frac{\sqrt{\nu}}{\frac{\varepsilon}{2} - (\delta + L)} \big[L_{\lambda,\lambda_0} \big(\|u_{\lambda_0}\|_0 + \|h\|_1 \big) + \|(g_\lambda - g_{\lambda_0})(\cdot\,,0,0)\|_0 \big] .$$

So that, the assumptions of Lemma 4.3 are fulfilled. □

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