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REGULARITY OF SEMIGROUP-VALUED SET FUNCTIONS

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Dedicated to Academician Štefan Schwarz on the occasion of his 70th birthday

We present a general version of regularity theorems giving thus a common generalization of several apparently noncompatible cases (see examples (1)—(5) of section 1).

1. Notions. Examples. Results.

The paper is devoted to a study of regularity of a semigroup-valued set function $m: \mathbf{S} \to \mathcal{P}$, where $\mathbf{S}$ is a σ-ring and $\mathcal{P}$ is a semigroup. We moreover assume that $\mathcal{P}$ is a partially ordered commutative semigroup with a binary operation $\oplus$ and a partial ordering $\leqq$ satisfying the following conditions:

(i) There is $\theta \in \mathcal{P}$ such that $\theta \leqq a$ for all $a \in \mathcal{P}$.

(ii) $a \oplus \theta = a$ for all $a \in \mathcal{P}$.

(iii) $a \leqq b$ implies $a \oplus c \leqq b \oplus c$ for all $a, b, c \in \mathcal{P}$.

(iv) $\mathcal{P}$ is conditionally complete (i.e., every bounded subset has the supremum and the infimum in $\mathcal{P}$).

(v) $a_n \to a$, $b_n \to b$ implies $a_n \oplus b_n \to a \oplus b$ for all $a_n, b_n, a, b \oplus \mathcal{P}$ $(n = 1, 2, \ldots)$. (We write $a_n \to a$ if there are $c_n, d_n \in \mathcal{P}$ such that $c_n \leqq a_n \leqq d_n$ and $c_n \uparrow a$, $d_n \downarrow a$.)

(vi) $\mathcal{P}$ is separative, that means, if $\mathcal{P}^< = \{f: \mathcal{P} \to \langle 0, \infty) | f(\theta) = 0;\ a \leqq b$ implies $f(a) \leqq f(b)$ for all $a, b \in \mathcal{P}$; $f(a \oplus b) \leqq f(a) + f(b)$ for all $a, b \in \mathcal{P}$; $a_n \to a$ implies $\lim\limits_{n \to \infty} f(a_n) = f(a)$ for all $a_n, a \in \mathcal{P}$ $(n = 1, 2, \ldots)\}$, then $a, b \in \mathcal{P}$, $a \neq b$ implies that there is an $f \in \mathcal{P}^<$ such that $f(a) \neq f(b)$.

As regards the σ-ring $\mathbf{S}$, we assume that $\mathbf{S}$ is a σ-ring of subsets of a nonempty set X such that there are subsystems $\mathbf{C}$ and $\mathbf{U}$ of $\mathbf{S}$ satisfying axioms (V1)—(V6) and one of the axioms (V7), (V8):

(V1) $\emptyset \in \mathbf{C}$, $\emptyset \in \mathbf{U}$

(V2) If $U_n \in \mathbf{U}$ $(n = 1, 2, \ldots)$, then $\bigcup\limits_{n-1}^{\infty} U_n \in \mathbf{U}$.

(V3) If C_1, $C_2 \in \mathbf{C}$, then $C_1 \cup C_2 \in \mathbf{C}$.
(V4) If $U \in \mathbf{U}$ and $C \in \mathbf{C}$, then $U - C \in \mathbf{U}$ and $C - U \in \mathbf{C}$.
(V5) If $C \in \mathbf{C}$, then there exists $U \in \mathbf{U}$ and $D \in \mathbf{C}$ such that $C \subset U \subset D$
(V6) $\mathbf{S} = \mathbf{S}(\mathbf{C})$ (the σ-ring generated by $\mathbf{C}$) and $\mathbf{U} \subset \mathbf{S}(\mathbf{C})$.
(V7) If $C \in \mathbf{C}$, then there are $U_n \in \mathbf{U}$ $(n = 1, 2, \ldots)$ such that $C = \bigcap_{n=1}^{\infty} U_n$.
(V8) If $U \in \mathbf{U}$ and $U \subset C \in \mathbf{C}$, then there are $C_n \in \mathbf{C}$ $(n = 1, 2, \ldots)$ such that $U = \bigcup_{n=1}^{\infty} C_n$.

Such a σ-ring $\mathbf{S}$ is called $(\mathbf{C}, \mathbf{U})$-regular. On the function $m: \mathbf{S} \to \mathcal{P}$ we assume to fulfil the following requirements:

(vii) $m(A \cup B) \leqq m(A) \oplus m(B)$ for all A, $B \in \mathbf{S}$.
(viii) $A \subset B$ implies $m(A) \leqq m(B)$ for all A, $B \in \mathbf{S}$
(ix) $A_n \downarrow \emptyset$, $A_n \in \mathbf{S}$ $(n = 1, 2, \ldots)$ implies $m(A_n) \downarrow \theta$ (the continuity from above at the empty set).

One checks easily that the latter set function m has also the following properties: $m(A) \geqq 0$ for all $A \in \mathbf{S}$, $m(\emptyset) = 0$ and if $A_n \uparrow A$ $(B_n \downarrow B)$, then $m(A_n) \uparrow m(A)$, $(m(B_n) \downarrow m(B))$ for all A_n, B_n, A, $B \in \mathbf{S}$.

Let us now give some examples of $(\mathbf{C}, \mathbf{U})$-regular σ-rings.

(A) The σ-ring $\mathbf{S}$ of Baire sets on a locally compact Hausdorff topological space is $(\mathbf{C}, \mathbf{U})$-regular for $\mathbf{C}$ — the family of all compact G_δ subsets and $\mathbf{U}$ — the family of all open sets belonging to $\mathbf{S}$.

(B) The σ-ring $\mathbf{S} = \mathbf{S}(\mathbf{C})$ is $(\mathbf{C}, \mathbf{U})$-regular for $\mathbf{C}$ — the family of all closed subsets and $\mathbf{U}$ — the family of all open subsets of a metric space X.

(C) The σ-ring $\mathbf{S} = \mathbf{S}(\mathbf{C})$ is $(\mathbf{C}, \mathbf{U})$-regular for $\mathbf{C}$ — the family of all closed bounded subsets of a metric space X and $\mathbf{U}$ — the family of all open subsets of X.

Some examples of semigroups $\mathcal{P}$ satisfying axioms (i)—(vi) and set functions m satisfying axioms (vii)—(ix) are listed as follows:

(1) Let $\mathcal{P}$ be the interval $\langle 0, \infty)$ with the usual ordering and let the operation $\oplus$ be the usual addition. Then any σ-additive measure $m: \mathbf{S} \to \langle 0, \infty)$ is an example of the set function satisfying axioms (vii)—(ix). More generally, such an example is any set function $m: \mathbf{S} \to \langle 0, \infty)$ which is monotone, subadditive, continuous from above at the empty set and satisfying the condition $m(\emptyset) = 0$.

(2) Let $\mathcal{P} = \langle 0, \infty \rangle$ be extended real numbers (i.e., $a \leqq \infty$, and $a \oplus \infty = \infty$ for all $a \in \langle 0, \infty \rangle$). Then the example of the set function $m: \mathbf{S} \to \langle 0, \infty \rangle$ satisfying axioms (vii)—(ix) is any set function which is countably additive and continuous from above at the empty set unconditionally (i.e., $E_n \downarrow \emptyset$ implies that $\lim_{n \to \infty} m(E_n) = 0$ for all $E_n \in \mathbf{S}$, $n = 1, 2, \ldots$).

(3) Let $\mathcal{P}$ be the interval $\langle 0, \infty)$ with the usual ordering and with the operation

$\oplus$ defined by the formula $a \oplus b = \max\{a, b\}$ for all $a, b \in \langle 0, \infty)$. Let m: $\mathbf{S} \to \langle 0, \infty)$ be a set function such that

(a) m is finitely maxitive, that is, if $E_1, E_2, \ldots, E_n$ are mutually disjoint sets in $\mathbf{S}$, then $M\left(\bigcup_{i=1}^{n} E_i\right) = \max\{m(E_1), m(E_2), \ldots, m(E_n)\}$.

(b) m is continuous from above at the empty set, that is, if $E_n \downarrow \emptyset$ then $\lim_{n\to\infty} m(E_n) = 0$, for all $E_n \in \mathbf{S}$ $(n = 1, 2, \ldots)$.

Such a set function m satisfies the axioms (vii)—(ix).

(4) Let $\mathscr{P} = \langle 0, \infty \rangle$ as in example 2 and define $\max\{a, \infty\} = \infty$ for all $a \in \langle 0, \infty \rangle)$. Then any function m: $\mathbf{S} \to \langle 0, \infty \rangle$ which is finitely maxitive and continuous from above at the empty set satisfies the axioms (vii)—(ix). The continuity from above at the empty set is asummed unconditional (i.e., $E_n \downarrow \emptyset$ implies $\lim_{n\to\infty} m(E_n) = 0$).

(5) Let $\mathscr{P}$ be a conditionally complete upper semi-lattice with the least element θ and the semi lattice operation $\oplus$ (i.e. $a \oplus b = a \vee b$ for all $a, b \in \mathscr{P}$). Let m: $\mathbf{S} \to \mathscr{P}$ have the following properties:

(a) $A \subset B$ implies $m(A) \leqq m(B)$ for all $A, B \in \mathbf{S}$

(b) $m(A \cup B) \leqq m(A) \vee m(B)$ for all $A, B \in \mathbf{S}$

(c) $A_n \downarrow \emptyset$ implies $m(A_n) \downarrow \theta$ for all $A_n \in \mathbf{S}$, $n = 1, 2, \ldots$.

Then m satisfies the axioms (vii)—(ix).

Let us now state our first result.

Theorem 1.1. *Let* $\mathbf{S}$ *be a* $(\mathbf{C}, \mathbf{U})$*-regular* σ*-ring of subsets of a set* X. *Let* m: $\mathbf{S} \to \mathscr{P}$ *be a set function satisfying the axioms* (vii)—(ix). *Then* m *is* $(\mathbf{C}, \mathbf{U})$*-regular. That is, if* $E \in \mathbf{S}$, *then*

$$m(E) = \sup\{m(C) | E \supset C \in \mathbf{C}\} = \inf\{m(U) | E \subset U \in \mathbf{U}\}.$$

Proof. Let $f \in \mathscr{P}^{<}$. Define a set function $f \circ m$: $S \to \langle 0, \infty)$ by the formula $f \circ m(A) = f(m(A))$ for all $A \in \mathbf{S}$. Then:

(1) $f \circ m(\emptyset) = 0$.

(2) $f \circ m(A) \leqq f \circ m(B)$ for all $A, B \in \mathbf{S}$ such that $A \subset B$.

(3) $f \circ m(A \cup B) \leqq f \circ m(A) + f \circ m(B)$ for all $A, B \in \mathbf{S}$.

(4) $A_n \downarrow \emptyset$ implies $\lim_{n\to\infty} f \circ m(A) = 0$ for all $A_n \in \mathbf{S}$ $(n = 1, 2, \ldots)$.

Put $\mathscr{N}_n = \left\{E \in \mathbf{S} | f \circ m(E) < \frac{1}{n}\right\}$ for $n = 1, 2, \ldots$ and $\mathscr{N}_0 = \mathbf{S}$. Then $\{\mathscr{N}_n\}_{n=0}^{\infty}$ is a sequence of subsystems of $\mathbf{S}$ with the properties (i)—(v) of [4], page 117. Recall the properties:

(i) $\emptyset \in \mathscr{N}_n$ for $n = 0, 1, 2, \ldots$

(ii) To any positive integer n there exists a sequence $\{k_i\}_{i=1}^{\infty}$ of positive integers such that $\bigcup_{i=1}^{\infty} E_i \in \mathcal{N}_n$, whenever $E_i \in \mathcal{N}_{k_i}$ $(i=1, 2, \ldots)$.

(iii) If $\{E_i\}_{i=1}^{\infty}$ is a sequence of sets of $\mathbf{S}$, $E_{i+1} \subset E_i$ $(i=1, 2, \ldots)$, $\bigcap_{i=1}^{\infty} E_i = \emptyset$, then to any positive integer n there is a positive integer m such that $E_m \in \mathcal{N}_n$.

(iv) If $E \in \mathcal{N}_n$, $F \subset E$, $F \in \mathbf{S}$, then $F \in \mathcal{N}_n$ $(n=0, 1, 2, \ldots)$

(v) $C \in \mathcal{N}_0$ for every $C \in \mathbf{C}$.

Put

$\mathbf{R}_1 = \{E \in \mathbf{S}|$ to any positive integer n there is a set $U \in \mathbf{U}$ such that $E \subset U$, $U - E \in \mathcal{N}_n\}$.

$\mathbf{R}_2 = \{E \in \mathbf{S}|$ to any positive integer n there is a set $C \in \mathbf{C}$ such that $C \subset E$, $E - C \in \mathcal{N}_n\}$ and a set $\mathcal{P} = \mathcal{R}_1 \cap \mathcal{R}_2$.

One can prove that $\mathbf{P} = \mathbf{S}$ (see also [4], Theorems 3, 4). Hence, if $E \in \mathbf{S}$, then to any positive integer n there are $U \in \mathbf{U}$ and $C \in \mathbf{C}$ such that $C \subset E \subset U$ and $f \circ m(U-E) < \frac{1}{n}$, $f \circ m(E-C) < \frac{1}{n}$. According to the property (3) of $f \circ m$, it follows that

$$f \circ m(E) = \sup \{f \circ m(C) | E \supset C \in \mathbf{C}\} = \inf \{f \circ m(U) | E \subset U \in \mathbf{U}\}$$

for all $f \in \mathcal{P}^{<}$.

Let $E \in \mathbf{S}$. By the property (iv) of $\mathcal{P}$ there exists $\sup \{m(C) | E \supset C \in \mathbf{C}\} = a \in \mathcal{P}$. Evidently, $a \leqq m(E)$. Suppose that $a \neq m(E)$. Since $a \leqq m(E)$, we have $f(a) \leqq f \circ m(E)$ for all $f \in \mathcal{P}^{<}$. Since $m(C) \leqq a$ for all $C \subset E$, $C \in \mathbf{C}$, we have $f \circ m(C) \leqq f(a)$. Further, we have $f \circ m(E) = \sup \{f \circ m(C) | E \supset C \in \mathbf{C}\} \leqq f(a)$ for all $f \in \mathcal{P}^{<}$. Hence $f(a) = f \circ m(E)$ for all $f \in \mathcal{P}^{<}$, which violates the property (vi) of $\mathcal{P}$. Therefore $a = m(E)$.

Similarly, according to the property (iv) of $\mathcal{P}$, there exists $\inf \{m(U) | E \subset U \in \mathbf{U}\} = b \in \mathcal{P}$. Assume that $b \neq m(E)$. Since $m(E) \leqq b$, we have $f \circ m(E) \leqq f(b)$ for all $f \in \mathcal{P}^{<}$. Since $b \leqq m(U)$ for all $E \subset U \in \mathbf{U}$ and therefore $f(b) \leqq f \circ m(U)$, we have $f \circ m(E) = \inf \{f \circ m(U) | E \subset U \in \mathbf{U}\} \leqq f(b)$ for all $f \in \mathcal{P}^{<}$. Hence $f \circ m(E) = f(b)$ for all $f \in \mathcal{P}^{<}$, which is a contradiction with the property (vi) of $\mathcal{P}$. Thus $b = m(E)$, which completes the proof.

2. A theorem on σ-maxitive measures

Throughout this section, let X denote a locally compact Hausdorff topological space, $\mathbf{S}$ the σ-ring generated by the class $\mathbf{C}$ of all compact G_δ's in X (Baire sets) and $\mathbf{T}$ the σ-ring generated by the class $\mathbf{D}$ of all compact sets in X (Borel sets). The class of all open Baire sets is denoted by $\mathbf{U}$ and the class of all open Borel sets is denoted by $\mathbf{V}$.

A σ-maxitive measure is a set function m: $\mathbf{S} \to \langle 0, \infty \rangle$ such that $m(\emptyset)=0$ and $m\left(\bigcup_{n=1}^{\infty} E_n\right) = \sup_n m(E_n)$ for all aequences $\{E_n\}_{n=1}^{\infty}$ in $\mathbf{S}$ (see [2]). The σ-maxitive measure need not be $(\mathbf{C}, \mathbf{U})$-regular as the following example shows.

Example 2.1. Let $m(E)=\sup_{x \in E} f(x)$ for all $E \in \mathbf{S}$, where $\mathbf{S}$ is the class of Baire subsets of $(-\infty, \infty)$ and $f(x)=1$ if $0 \leqq x \leqq 1$ and $f(x)=2$ if $x<0$ or $x>1$. Then $1=m(\langle 0, 1 \rangle) \neq \inf \{m(U) | \langle 0, 1 \rangle \subset U \in \mathbf{U}\} = 2$.

If the σ-maxitive measure m: $\mathcal{S} \to \langle 0, \infty)$ is continuous from above at the empty set then, by Theorem 1.1., m is $(\mathbf{C}, \mathbf{U})$-regular (see also Example (3), section 1).

If a σ-maxitive measure m: $\mathbf{S} \to \langle 0, \infty)$ is $(\mathbf{C}, \mathbf{U})$-regular, then m need not be continuous from above at the empty set (for example, let $m(E)=0$ or 1 according to $E=\emptyset$ or not).

If m is the σ-maxitive measure on Borel sets, then the following theorem holds:

Theorem 2.2. *Let* m: $\mathbf{T} \to \langle 0, \infty)$ *be a* $(\mathbf{D}, \mathbf{V})$*-regular σ-maxitive measure. Then the following propositions hold:*

(α) If $D \in \mathbf{D}$, then for any $\varepsilon>0$ there exists a point $x \in X$ such that $m(D)< m(\{x\})+\varepsilon$.

(β) If $m(\{x\})=0$ for all $x \in X$, then $m(E)=0$ for all $E \in \mathbf{T}$.

(γ) If m is continuous from above at the empty set, then there exists an at most countable set $A \in \mathbf{T}$ such that $m(E-A)=0$ for all $E \in \mathbf{T}$.

Proof. (α) Let $\varepsilon>0$ be given. For any $x \in X$ there exists $V_x \in \mathbf{V}$ such that $x \in V_x$ and $m(\{x\})+\varepsilon>m(V_x)$. Let $D \in \mathbf{D}$. Since $D \subset \bigcup_{x \in D} V_x$ and D compact, we can choose $x_1, x_2, \ldots, x_n \in D$ such that $D \subset \bigcup_{i=1}^{n} V_{x_i}$. Hence $m(D) \leqq \max \{m(V_{x_1}), m(V_{x_2}), \ldots, m(V_{x_n})\} = m(V_{x_k})$ for some $k (1 \leqq k \leqq n)$. Thus $m(D)<m(\{x_k\})+\varepsilon$.

(β) Let $m(\{x\})=0$ for all $x \in X$. Then according to (α) of Theorem 2.2, $m(D)=0$ for all $D \in \mathbf{D}$ and thus $m(E)=0$ for all $E \in \mathbf{T}$.

(γ) Let $A_n = \left\{x \in X | m(\{x\}) > \frac{1}{n}\right\}$. Suppose that A_n is an infinite set. Let $x_k \in A_n$ $(k=1, 2, \ldots)$, and $E_k = \{x_k, x_{k+1}, \ldots\}$ for $k=1, 2, \ldots$. Since $E_k \downarrow \emptyset$, we conclude that $\lim_{k \to \infty} m(E_k)=0$ and thus $\lim_{k \to \infty} m(\{x_k\})=0$ is a contradiction with the definition of A_n.

Hence $A = \bigcup_{n=1}^{\infty} A_n$ is at most countable.

Let $E \in \mathbf{T}$. Choose any $\varepsilon>0$. If $x \notin A$, then $m(\{x\})=0$. By the regularity of m there exists $D \in \mathbf{D}$ such that $D \subset E-A$ and $m(E-A)-\varepsilon<m(D)$. By (α) of Theorem 2.2., there exists $x \in D$ such that $m(D)<m(\{x\})+\varepsilon$. Thus $m(E-A)<\varepsilon$ for any $\varepsilon>0$ and hence $m(E-A)=0$.

Corollary 2.3. In a locally compact Hausdorff topological space X every

σ-maxitive measure m on Borel sets which is finite, continuous from above at the empty set and (**D**, **V**)-regular must be one of the following types:

(1) There exist $\alpha_i \in (0, \infty)$ and $x_i \in X$ $(i = 1, 2, \ldots, n)$ such that

$$m(E) = \max_{1 \le i \le n} \alpha_i \chi_{\{x_i\}},$$

for all Borel sets E.

(2) There exist $\alpha_i \in (0, \infty)$ $(i = 1, 2, \ldots)$ such that $\lim_{i \to \infty} \alpha_i = 0$ and $x_i \in X$ $(i = 1, 2, \ldots)$, such that

$$m(E) = \sup_{i = 1, 2, \ldots} \alpha_i \chi_{\{x_i\}}$$

for all Borel sets E.

Note 2.4. Let $X = (-\infty, \infty)$. Let $m\colon 2^X \to \langle 0, \infty)$ be σ-maxitive and continuous from above at the empty set. If $m(\{x\}) = 0$ for all $x \in (-\infty, \infty)$, then $m(E) = 0$ for all $E \subset (-\infty, \infty)$. (Since $\mathbf{S} = \mathbf{T}$, $m/\mathbf{T}$ is (**C**, **U**)-regular and hence (**D**, **V**)-regular. By Theorem 2.2., $m(E) = 0$ for all $E \in \mathbf{T}$ and therefore $m(E) = 0$ for all $E \subset (-\infty, \infty)$). This is an analogue of a Banach—Kuratowski theorem (see [5]).

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О РЕГУЛЯРНОСТИ ФУНКЦИЙ МНОЖЕСТВА СО ЗНАЗЕНИЯМИ В ЧАСТИЧНО УПОРЯДОЧЕННОЙ ПОЛУГРУППЕ

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Резюме

В статье доказывается теорема о (**C**, **U**) – регулярности для функций множества, принимающих значения в частично упорядоченной полугруппе. Если областью определения этих функции является система подмножеств в абстрактном пространстве, то системы множеств **C** и **U** должны обладать свойствами (V1)—(V7). Примеры некоторых пространств и в них систем **C** и **U** приводятся в параграфе 3. В параграфе 4 показывается необходимое и достаточное условие для регулярности непрерывной макситивной меры в локально компактном хаусдорфовом пространстве.