Mathematica Slovaca

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Mathematica Slovaca, Vol. 55 (2005), No. 1, 9--19

Persistent URL: http://dml.cz/dmlcz/130423

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ON SOME IDENTITIES FOR THE FIBONOMIAL COEFFICIENTS

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(*Communicated by Stanislav Jakubec*)

ABSTRACT. The Fibonomial coefficients $\left[{n \atop k}\right]$ are defined for positive integers $n \geq k$ as follows

$$\left[{n \atop k}\right] = \frac{F_n F_{n-1} \cdots F_{n-k+1}}{F_1 F_2 \cdots F_k},$$

with $\left[{n \atop 0}\right] = 1$, where the Fibonacci numbers are given by the recurrence relation $F_{n+2} = F_{n+1} + F_n$, $F_0 = 0$, $F_1 = 1$. In this paper new identities for the Fibonomial coefficients are derived. These identities are related to the generating function of the kth powers of the Fibonacci numbers. Their proofs are based on a reasonable manipulation with these generating functions.

1. Introduction

In 1915 Fontené published a one-page note [2] suggesting a generalization of binomial coefficients, replacing the natural numbers by the terms of an arbitrary sequence $\{A_n\}_{n=0}^{\infty}$ of real or complex numbers.

Jarden considered in [6] the general second order recurrence relation

$$y_{n+2} = g y_{n+1} - h y_n, \tag{1}$$

where $h \neq 0$ and its auxiliary equation had the roots ε, ω. Let $U_n = \frac{\varepsilon^n - \omega^n}{\varepsilon - \omega}$, $\varepsilon \neq \omega$, be the solution of (1), he defined generalized binomial coefficients

$$\left\{{m \atop j}\right\} = \frac{U_m U_{m-1} \cdots U_{m-j+1}}{U_j U_{j-1} \cdots U_1} \qquad \text{with} \quad \left\{{m \atop 0}\right\} = 1.$$

One may also state the generalized factorial $[m]! = U_1 U_2 \cdots U_m$ with $[0]! = 1$, and then

$$\left\{{m \atop j}\right\} = \frac{[m]!}{[j]!\,[m-j]!} \qquad \text{for any nonnegative integers } m \geq j.$$

2000 Mathematics Subject Classification: Primary 11B39, 05A15, 05A19.
Keywords: Fibonacci number, Lucas number, Fibonomial coefficient, generating function.

Jarden showed that for the product z_n of the nth terms of $m-1$ sequences satisfying (1) holds the mth order recurrence relation

$$\sum_{j=0}^{m}(-1)^j\begin{Bmatrix}m\\j\end{Bmatrix}h^{\frac{j}{2}(j+1)}z_{n+m-j}=0\,.$$

Torretto and Fuchs in [9] established the following identity for $\left\{{m \atop j}\right\}$

$$\sum_{j=0}^{m}(-1)^j\begin{Bmatrix}m\\j\end{Bmatrix}h^{\frac{j}{2}(j+1)}U_{a_1+m-j}U_{a_2+m-j}\cdots U_{a_m+m-j}y_{n+m-j}$$
$$=U_1U_2\cdots U_m y_{n+a_1+a_2+\cdots+a_m+\frac{m}{2}(m+1)}\,,$$

where n, $a_1,\dots,a_m$ are any integers and $\{y_n\}_{n=0}^{\infty}$ is an arbitrary sequence satisfying (1).

In [3], Gould rewieved the generalized binomial coefficients and he proved the inversion theorem for $\left\{{m \atop j}\right\}$ and a representation of the bracket function as a linear combination of them.

Since 1964, there has been an accelerated interest in the Fibonomial coefficients, which correspond to the choice $U_n=F_n$, where F_n are the Fibonacci numbers defined by (1) for $g=1$, $h=-1$ and $F_0=0$, $F_1=1$. The Fibonacci numbers can be also expressed by the Binet formula $F_n=\frac{\alpha^n-\beta^n}{\sqrt{5}}$, where $\alpha=\frac{1+\sqrt{5}}{2}$ and $\beta=\frac{1-\sqrt{5}}{2}$. The Lucas numbers L_n satisfy the basic Fibonacci recurrence but $L_0=2$, $L_1=1$ and therefore $L_n=\alpha^n+\beta^n$.

Thus, the Fibonomial coefficients can be expressed for integers $n\ge k\ge 1$ as

$$\begin{bmatrix}n\\k\end{bmatrix}=\prod_{i=0}^{k-1}\frac{F_{n-i}}{F_{k-i}}=\frac{F_nF_{n-1}\cdots F_{n-k+1}}{F_kF_{k-1}\cdots F_1}\,,$$

with $\left[{n \atop 0}\right]=1$ and $\left[{n \atop k}\right]=0$ for $n<k$. It is easy to find the important recurrence formula for the Fibonomial coefficients in the form

$$\begin{bmatrix}n\\k\end{bmatrix}=F_{k+1}\begin{bmatrix}n-1\\k\end{bmatrix}+F_{n-k-1}\begin{bmatrix}n-1\\k-1\end{bmatrix}\tag{2}$$

using the well-known identity $F_n=F_{n-k}F_{k+1}+F_{n-k-1}F_k$ (see e.g. [5]).

In the past much attention has been focused on the generating function $f_k(x)=\sum\limits_{n=0}^{\infty}F_n^kx^n$ for the kth powers of F_n. In [7] Riordan found the general recurrence for $f_k(x)$, and Carlitz in [1] and Horadam in [4] generalized his result and found similar recurrences for the generating functions of different types of generalized Fibonacci numbers. They found closed form for the polynomial $N_k(x)$ in the numerator and the polynomial $D_k(x)$ in the denominator of the generating function $f_k(x)$. As a special case of Horadam's result in [4] it is possible to get the following relation for the generating function of an integer powers of the Fibonacci numbers

$$f_k(x) = \frac{\sum\limits_{i=0}^{k}\sum\limits_{j=0}^{i}(-1)^{\frac{j(j+1)}{2}}\begin{bmatrix}k+1\\ j\end{bmatrix}F_{i-j}^k x^i}{\sum\limits_{i=0}^{k+1}(-1)^{\frac{i(i+1)}{2}}\begin{bmatrix}k+1\\ i\end{bmatrix}x^i}\,. \tag{3}$$

Shannon found in [8] some special results for the numerator and the denominator in the expression of the generating function $f_k(x)$.

It is easy to obtain for any odd integer k that

$$f_k(x) = 5^{-\frac{k-1}{2}}\sum_{j=0}^{\frac{k-1}{2}}\binom{k}{j}\frac{F_{k-2j}\,x}{1-(-1)^j L_{k-2j}\,x - x^2} \tag{4}$$

and for any even integer k that

$$f_k(x) = 5^{-\frac{k}{2}}\sum_{j=0}^{\frac{k-2}{2}}(-1)^j\binom{k}{j}\frac{2-(-1)^j L_{k-2j}\,x}{1-(-1)^j L_{k-2j}\,x + x^2} + \binom{k}{\frac{k}{2}}\frac{(-5)^{-\frac{k}{2}}}{1-(-1)^{\frac{k}{2}}x} \tag{5}$$

after simplification of one of Shannon's results.

The integers $d_i = (-1)^{\frac{i}{2}(i+1)}\begin{bmatrix}k+1\\ i\end{bmatrix}$ are terms of the sequence which was named as "signed Fibonomial triangle" in the on-line encyclopedia of integer sequences (maintained by N. J. A. Sloane) with ID Number A055870. The encyclopedia gives only the following identity in the connection with this sequence (see [10])

$$\sum_{j=0}^{k+1}(-1)^{\frac{j}{2}(j+1)}\begin{bmatrix}k+1\\ j\end{bmatrix}F_{n-j}^k = 0\,,$$

where n, k are any positive integers such that $n \geq k+1$. It is clear that this identity correspond to the sum in the numerator of the generating function (3) for $i \geq k+1$.

From (3), (4) and (5) we get the following generating functions of d_i

$$D^{(o)}(x) = \prod_{j=0}^{\frac{k-1}{2}}\left(1-(-1)^j L_{k-2j}\,x - x^2\right) = \sum_{i=0}^{k+1} d_i x^i \tag{6}$$

for any odd positive integer k and

$$D^{(e)}(x) = \left(1-(-1)^{\frac{k}{2}}x\right)\prod_{j=0}^{\frac{k}{2}-1}\left(1-(-1)^j L_{k-2j}\,x + x^2\right) = \sum_{i=0}^{k+1} d_i x^i \tag{7}$$

for any even positive integer k.

2. The main results

One of important features of the generating function of a sequence is the possibility to find a family of relations for its terms by suitable manipulation with it. Concretely, proofs of Theorems in this paper are based on divisibility of the polynomial $D^{(o)}(x)$ by factors $x+(-1)^j\alpha^{k-2j}$ and $x+(-1)^j\beta^{k-2j}$ or on divisibility of the polynomial $D^{(e)}(x)$ by factors $x+(-1)^{j+1}\alpha^{k-2j}$, $x+(-1)^{j+1}\beta^{k-2j}$ and $1-(-1)^{\frac{k}{2}}x$.

The main results are given in the following theorems:

THEOREM 1. *Let m be any odd positive integer. Then*

$$\sum_{i=0}^{m}(-1)^{\frac{i}{2}(m+i)}\begin{bmatrix} m\\ i\end{bmatrix}=0\,.$$

THEOREM 2. *Let k be any positive integer and $l\le\frac{k-1}{2}$, $m>k$ be any nonnegative integers. Then*

$$\sum_{i=0}^{m}(-1)^{\frac{i}{2}(2l+i+1)}\frac{F_{(k-i)(k-2l)}}{F_{k-2l}}\begin{bmatrix} k+1\\ i\end{bmatrix}=0\,.$$

THEOREM 3. *Let k be any positive integer, $l\le\frac{k-1}{2}$, n and $m>k$ be any nonnegative integers. Then*

$$\sum_{i=0}^{m}(-1)^{\frac{i}{2}(2l+i+(-1)^k)}L_{(k-2l)(i+n)}\begin{bmatrix} k+1\\ i\end{bmatrix}=0\,. \tag{8}$$

3. The preliminary results

Let k be an arbitrary nonnegative integer. Suppose $\{x_n\}_{n=0}^{\infty}$ is any sequence of real numbers satisfying the recurrence relation

$$x_{n+2}-\lambda x_{n+1}+(-1)^k x_n=0\,,\qquad x_0=0\,,\quad x_1=1\,, \tag{9}$$

where λ is a real number. As (9) is a special case of (1) it is evident that

$$x_n=\frac{1}{\sqrt{\lambda^2-4(-1)^k}}\left(\left(\frac{\lambda+\sqrt{\lambda^2-4(-1)^k}}{2}\right)^n-\left(\frac{\lambda-\sqrt{\lambda^2-4(-1)^k}}{2}\right)^n\right) \tag{10}$$

for any nonnegative integer n.

Lemma 1. *Let $l \le \frac{k-1}{2}$ be any nonnegative integer. Let $\{x_n\}_{n=0}^{\infty}$ be any sequence of real numbers defined by the recurrence $x_{n+2} = (-1)^l L_{k-2l}\, x_{n+1} - (-1)^k x_n$ for $n \ge 0$, with $x_0 = 0$, $x_1 = 1$. Then*

$$x_n = (-1)^{l(n+1)} \frac{F_{n(k-2l)}}{F_{k-2l}} .$$

P r o o f. The assertion follows from (10) using the well-known formula $L_n^2 - 4(-1)^n = 5F_n^2$ and the Binet formulas for F_n and L_n. □

Lemma 2. *Let $\{a_n\}_{n=0}^{\infty}$, $\{b_n\}_{n=0}^{\infty}$ be any sequences of real numbers, k be any nonnegative integer and $\{x_n\}_{n=0}^{\infty}$ be any sequence (9). Then for $n > 1$*

$$b_n = a_n - \lambda a_{n-1} + (-1)^k a_{n-2} \tag{11}$$

if and only if

$$a_n = \sum_{i=0}^{n-2} x_{i+1} b_{n-i} + x_n a_1 - (-1)^k x_{n-1} a_0 . \tag{12}$$

P r o o f. Let us show that the identity (11) implies the identity (12). We have

$$\begin{aligned}
&\sum_{i=0}^{n-2} x_{i+1} b_{n-i} + x_n a_1 - (-1)^k x_{n-1} a_0 \\
&= \sum_{i=0}^{n-2} x_{i+1} \big(a_{n-i} - \lambda a_{n-1-i} + (-1)^k a_{n-2-i}\big) + x_n a_1 - (-1)^k x_{n-1} a_0 \\
&= \sum_{i=0}^{n-2} x_{i+1} a_{n-i} - \lambda \sum_{i=0}^{n-3} x_{i+1} a_{n-1-i} + (-1)^k \sum_{i=0}^{n-3} x_{i+1} a_{n-2-i} + a_1 \big(x_n - \lambda x_{n-1}\big) \\
&= x_1 a_n + x_2 a_{n-1} + \sum_{i=2}^{n-2} x_{i+1} a_{n-i} - \lambda x_1 a_{n-1} - \lambda \sum_{i=1}^{n-3} x_{i+1} a_{n-1-i} \\
&\qquad\qquad + (-1)^k \sum_{i=0}^{n-4} x_{i+1} a_{n-2-i} \\
&= a_n + a_{n-1}\big(x_2 - \lambda x_1\big) + \sum_{i=2}^{n-2} a_{n-i} \big(x_{i+1} - \lambda x_i + (-1)^k x_{i-1}\big) = a_n .
\end{aligned}$$

Thus, this part of the statement is true and similarly we can prove that the reversed implication holds too. □

Lemma 3. *Let* $\{a_n\}_{n=0}^{\infty}$, $\{b_n\}_{n=0}^{\infty}$ *be any sequences of real numbers and* $\lambda \neq 0$ *be any real number. Then for an arbitrary positive integer* n

$$a_n = b_{n-1} + \lambda b_n \tag{13}$$

if and only if

$$b_n = \lambda^{-n}\left(\sum_{i=1}^{n} \lambda^{i-1}(-1)^{i+n}a_i + (-1)^n b_0\right). \tag{14}$$

P r o o f . Let us show that identity (13) implies identity (14). Hence we have to prove that

$$\lambda^n b_n = \sum_{i=1}^{n} \lambda^{i-1}(-1)^{i+n}\big(b_{i-1} + \lambda b_i\big) + (-1)^n b_0 \tag{15}$$

for any positive integer n. We use induction on n. It is evident that for $n = 1$ identity (15) holds. If we suppose that (15) holds for any n its validity for $n+1$ is implied by

$$\begin{aligned}
&\sum_{i=1}^{n+1}\lambda^{i-1}(-1)^{i+n+1}\big(b_{i-1} + \lambda b_i\big) + (-1)^{n+1}b_0\\
&\quad= \sum_{i=1}^{n}\lambda^{i-1}(-1)^{i+n+1}\big(b_{i-1} + \lambda b_i\big) + \lambda^n\big(b_n + \lambda b_{n+1}\big) + (-1)^{n+1}b_0\\
&\quad= -\lambda^n b_n + \lambda^n\big(b_n + \lambda b_{n+1}\big) = \lambda^{n+1}b_{n+1}\,.
\end{aligned}$$

Hence this part of the assertion is true and similarly we can prove the reversed implication. □

4. The proofs of the main theorems

P r o o f o f T h e o r e m 1. We define a polynomial $P_k(x) = \sum\limits_{n=0}^{k} p_n(k)x^n$ by

$$P_k(x) = \frac{D^{(\mathrm{e})}(x)}{1-(-1)^{\frac{k}{2}}x} = \prod_{j=0}^{\frac{k-2}{2}}\big(1-(-1)^j L_{k-2j}\,x + x^2\big) \tag{16}$$

for any even nonnegative integer k.

The following relations are implied by (7) and (16)

$$\begin{aligned}
d_0 &= p_0(k) = 1\,,\\
d_i &= p_i(k) + (-1)^{\frac{k}{2}+1}p_{i-1}(k)\,, \qquad i = 1,2,\dots,k\,,\\
d_{k+1} &= (-1)^{\frac{k}{2}+1}p_k(k) = (-1)^{\frac{k}{2}+1}\,.
\end{aligned}$$

Putting $p_i(k) = 0$ for $i < 0$ or $i > k$ we obtain the general recurrence

$$p_i(k) + (-1)^{\frac{k}{2}+1} p_{i-1}(k) = d_i\,, \tag{17}$$

which holds for any integer i.

We will prove the relation

$$p_n(k) = \sum_{i=0}^{n} (-1)^{\frac{k}{2}(n+i)} d_i\,, \tag{18}$$

where n is any nonnegative integer.

(i) Let $\frac{k}{2} \equiv 0 \pmod 2$.

From (17) we get $d_i = p_i(k) - p_{i-1}(k)$ for any integer i. Hence

$$\sum_{i=0}^{n} d_i = \sum_{i=0}^{n} \big(p_i(k) - p_{i-1}(k)\big) = p_n(k) - p_{-1}(k)$$

and

$$p_n(k) = \sum_{i=0}^{n} d_i\,.$$

(ii) Let $\frac{k}{2} \equiv 1 \pmod 2$.

Analogously from (17) we get $d_i = p_i(k) + p_{i-1}(k)$ and

$$\sum_{i\ 0}^{n} (-1)^{i+1} d_i = \sum_{i=0}^{n} (-1)^{i+1} \big(p_i(k) + p_{i-1}(k)\big) = (-1)^{n+1} p_n(k) - p_{-1}(k)\,.$$

Thus, the relation

$$p_n(k) = (-1)^{n+1} \sum_{i=0}^{n} (-1)^{i+1} d_i$$

is true.

Setting $d_i = (-1)^{\frac{i(i+1)}{2}} \left[{k+1 \atop i}\right]$ in (18) and replacing $k+1$ by m, the proof is finished. □

P r o o f o f T h e o r e m 2. We need consider two cases.

(i) Let k be any odd positive integer.

Define polynomials $D_l^{(o)}(x) = \sum\limits_{i=0}^{k-1} p_i(k,l) x^i$ by

$$D_l^{(o)}(x) = \prod_{\substack{j=0 \\ j\neq l}}^{\frac{k-1}{2}} \big(1 - (-1)^j L_{k-2j}\, x - x^2\big) = \frac{D^{(o)}(x)}{1 - (-1)^l L_{k-2l}\, x - x^2}\,, \tag{19}$$

where $l \le \frac{k-1}{2}$ is any nonnegative integer.

Multiplying $D_l^{(o)}(x)$ by $1-(-1)^l L_{k-2l}x - x^2$ and comparing with $D^{(o)}(x)$ we have

$$\begin{aligned} p_0(k,l) &= d_0 = 1\,, \\ p_1(k,l) - (-1)^l L_{k-2l}\, p_0(k,l) &= d_1 = -F_{k+1}\,, \\ p_i(k,l) - (-1)^l L_{k-2l}\, p_{i-1}(k,l) - p_{i-2}(k,l) &= d_i\,, \qquad i = 2,3,\dots,k-1\,, \\ -(-1)^l L_{k-2l}\, p_{k-1}(k,l) - p_{k-2}(k,l) &= d_k = (-1)^{\frac{k+1}{2}} F_{k+1}\,, \\ p_{k-1}(k,l) &= -d_{k+1} = (-1)^{\frac{k-1}{2}}\,. \end{aligned}$$

Putting $p_i(k,l) = 0$ for $i < 0$ or $i > k-1$ we can rewrite the previous relations into the recurrence

$$p_i(k,l) - (-1)^l L_{k-2l}\, p_{i-1}(k,l) - p_{i-2}(k,l) = d_i\,,$$

which holds for any integer i. Hence using Lemma 2 we get

$$p_0(k,l) = 1\,, \qquad p_1(k,l) = (-1)^l L_{k-2l} - F_{k+1}\,,$$

$$p_n(k,l) = \sum_{i=0}^{n-2} x_{i+1} d_{n-i} + \big((-1)^l L_{k-2l} - F_{k+1}\big) x_n + x_{n-1}\,, \qquad n > 1\,.$$

Further, Lemma 1 and the formula $L_p F_{pq} = F_{p(q+1)} + (-1)^p F_{p(q-1)}$ imply the relation

$$p_n(k,l) = \sum_{i=0}^{n} (-1)^{l(n-i)} \frac{F_{(n-i+1)(k-2l)}}{F_{k-2l}}\, d_i\,. \tag{20}$$

Setting $n = k-1$ in (20) we obtain

$$\sum_{i=0}^{k-1} (-1)^{l(k-1-i)} \frac{F_{(k-i)(k-2l)}}{F_{k-2l}}\, d_i = -d_{k+1}\,.$$

As the summand $(-1)^{l(k-1-i)} \frac{F_{(k-i)(k-2l)}}{F_{k-2l}} d_i$ in the previous sum is equal to zero for $i = k$ and it is equal to d_{k+1} for $i = k+1$ the following relation

$$\sum_{i=0}^{k+1} (-1)^{l(k-1-i)} \frac{F_{(k-i)(k-2l)}}{F_{k-2l}}\, d_i = 0$$

is true. As for $i > k+1$ the integers d_i are equal to zero we get

$$\sum_{i=0}^{m} (-1)^{l(k-1-i)} \frac{F_{(k-i)(k-2l)}}{F_{k-2l}}\, d_i = 0$$

for any integer $m > k$. Putting $d_i = (-1)^{\frac{i(i+1)}{2}} \left[{k+1 \atop i}\right]$ in the previous identity we obtain the assertion.

(ii) Let k be any even positive integer.

Similarly, we now define polynomials $D_l^{(e)}(x) = \sum_{i=0}^{k-1} p_i(k,l)x^i$ by

$$D_l^{(e)}(x) = \left(1-(-1)^{\frac{k}{2}}x\right)\prod_{\substack{j=0\\ j\neq l}}^{\frac{k-2}{2}}\left(1-(-1)^j L_{k-2j}\,x+x^2\right) = \frac{D^{(e)}(x)}{1-(-1)^l L_{k-2l}\,x+x^2}\,,$$

where $l \le \frac{k-2}{2}$ is any nonnegative integer. This fact and Lemma 1 with Lemma 2 lead to the assertion and the proof is over. □

Proof of Theorem 3. The proof falls naturally into two parts.

(i) Let k be any odd positive integer.

We first prove that

$$\sum_{i=0}^{m}(-1)^{\frac{i}{2}(2l+i-1)}\alpha^{(k-2l)(i+n)}\begin{bmatrix}k+1\\ i\end{bmatrix} = 0 \tag{21}$$

for any positive integer $m > k$. The expression $1-(-1)^l L_{k-2l}\,x - x^2$ in $D^{(o)}(x)$ is possible to factorize for an arbitrary integer $l \le \frac{k-1}{2}$ in the form

$$1-(-1)^l L_{k-2l}\,x - x^2 = -\left(x+(-1)^l\alpha^{k-2l}\right)\left(x+(-1)^l\beta^{k-2l}\right).$$

Therefore we can define for any integer $l \le \frac{k-1}{2}$ polynomials

$$Q_l(x) = \sum_{i=0}^{k} q_i(k,l)x^i = \frac{D^{(o)}(x)}{x+(-1)^l\alpha^{k-2l}}\,.$$

Thus, comparing the product $\left(x+(-1)^l\alpha^{k-2l}\right)Q_l(x)$ with $D^{(o)}(x)$ we have

$$\begin{aligned}(-1)^l\alpha^{k-2l}q_0(k,l) &= d_0 = 1\,,\\ q_{i-1}(k,l)+(-1)^l\alpha^{k-2l}q_i(k,l) &= d_i\,, \qquad i=1,2,\dots,k\,,\\ q_k(k,l) &= d_{k+1} = (-1)^{\frac{k+1}{2}}\,.\end{aligned}$$

Putting $q_m(k,l) = 0$ for $m < 0$ or $m > k$ the previous relations can be rewritten into the recurrence

$$q_{m-1}(k,l)+(-1)^l\alpha^{k-2l}q_m(k,l) = d_m$$

for any integer m. With respect to Lemma 3 the equality

$$q_m(k,l) = \sum_{i=0}^{m}(-1)^{l(i-1)+i+l(m+1)}\alpha^{(k-2l)(i-m-1)}d_i$$

holds and hence for $m > k$ we obtain identity (21) putting $d_i = (-1)^{\frac{i(i+1)}{2}} \left[{k+1 \atop i}\right]$ and after a certain modification.

Similarly, we can obtain the identity

$$\sum_{i=0}^{m}(-1)^{\frac{i}{2}(2l+i-1)}\beta^{(k-2l)(i+n)}\begin{bmatrix}k+1\\ i\end{bmatrix}=0 \tag{22}$$

replacing α by β in the previous part of the proof.

The summation of equalities (21) and (22) gives

$$\sum_{i=0}^{m}(-1)^{\frac{i}{2}(2l+i-1)}L_{(k-2l)(i+n)}\begin{bmatrix}k+1\\ i\end{bmatrix}=0\,. \tag{23}$$

(ii) Let k be any even positive integer.

We can prove this case analogously but now we factorize in $D^{(\mathrm{e})}(x)$ the term $1-(-1)^l L_{k-2l}\,x+x^2$ in the form

$$1-(-1)^l L_{k-2l}\,x+x^2=\bigl(x+(-1)^{l+1}\alpha^{k-2l}\bigr)\bigl(x+(-1)^{l+1}\beta^{k-2l}\bigr)\,.$$

It leads to the result

$$\sum_{i=0}^{m}(-1)^{\frac{i}{2}(2l+i+1)}L_{(k-2l)(i+n)}\begin{bmatrix}k+1\\ i\end{bmatrix}=0 \tag{24}$$

and the assertion follows from identities (23) and (24). □

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Received July 2, 2004

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