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Mathematica Slovaca, Vol. 44 (1994), No. 2, 139--141

Persistent URL: http://dml.cz/dmlcz/130425

Math. Slovaca, **44** (1994), No. 2, 139–141

Mathematica Slovaca

Dedicated to Academician Štefan Schwarz on the occasion of his 80th birthday

MODULAR ORDERED SETS

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(*Communicated by Tibor Katriňák*)

ABSTRACT. A notion of modular ordered set is introduced. There is shown that a modular ordered set fulfils the Jordan-Dedekind condition for chains.

By an ordered set (= o. set) we mean a partially ordered set. In [2], notions of modularity and distributivity of o. sets are introduced. There was shown that modularity and distributivity of o. sets can be characterized by the non-existence of specific subsets of these o. sets. In the present paper another notion of modular o. set is given.

We first introduce some notations. Let $(P;\leq)$ be an o. set. Given $a\in P$, we write $[a)=\{x\in P;\ a\leq x\}$ and $(a]=\{x\in P;\ x\leq a\}$. For $a,b\in P$ such that $a\leq b$, set $[a,b]=\{x\in P;\ a\leq x\leq b\}$. If $a\leq b$ and $[a,b]=\{a,b\}$, we say that b covers a (in symbols $a\prec b$). For $a,b\in P$ we set $l(a,b)=(a]\cap(b]$ and $u(a,b)=[a)\cap[b)$. If $c\in l(a,b)$, we write $l(a,b;c)=[c)\cap l(a,b)$, and when $c\in u(a,b)$, $u(a,b;c)=(c]\cap u(a,b)$.

An o. set is said to have locally finite length if all its maximal bounded chains are finite. In what follows we will suppose that o. sets under consideration have locally finite length.

DEFINITION. *An o. set P is said to be modular if it fulfils the following condition:*

Let $a,b,c\in P$ and $b\leq c$. If there are $p\in l(a,b)\cap l(a,c)$, $q\in u(a,b)\cap u(a,c)$ with $l(a,b;p)=l(a,c;p)$ and $u(a,b;q)=u(a,c;q)$, then $b=c$.

In [2], there was shown that the o. set which consists of elements a, b, c, p such that $p<a$ and $p<b<c$, is not modular according to the notion

AMS Subject Classification (1991): Primary 06A99.
Key words: Ordered set, Modular.

of modularity in [2]. But it is modular in our sense of modularity. Hence these notions of modularity of o. sets are different.

LEMMA. *Let P be a modular o. set. Assume that $a, b, c \in P$ and $a \prec c \prec b$. Then any maximal chain in the interval $[a, b]$ has length 2.*

P r o o f. Consider a maximal chain

$$a = a_0 \prec a_1 \prec \ldots \prec a_n = b.$$

Either $a_1 = c$, and therefore $a_2 = b$, or $a_1 \neq c$. In the last event, $a_2 \neq b$ implies $l(c, a_1; a) = l(c, a_2; a)$ and $u(c, a_1; b) = u(c, a_2; b)$. By the modularity of P, we have $a_1 = a_2$, a contradiction. Thus $a_2 = b$ and $n = 2$.

THEOREM. *A modular o. set P of locally finite length fulfils the Jordan-Dedekind condition for chains, that means, all maximal chains between the same endpoints in P have the same length.*

P r o o f. We will proceed by induction on the length of maximal chains in the intervals of P. More precisely, let (I_n) denote the following property of P:

If $[a, b]$ in P has a maximal chain of length at most n, then all maximal chains of $[a, b]$ have the same length.

Lemma shows that (I_2) is true in P. Assume that (I_{n-1}) is true in P for $n \geq 3$. Let $a = a_0 \prec a_1 \prec \ldots \prec a_n = b$ be a maximal chain of length n in P. Moreover, assume that $a = b_0 \prec b_1 \prec \ldots \prec b_m = b$ is another maximal chain of length $m \geq n$.

Two cases can occur: (i) $a_1 = b_1$ or (ii) $a_1 \neq b_1$.

In the first event, $n = m$ by the induction hypothesis. Suppose that $a_1 \neq b_1$. Evidently, a_1 and b_1 are incomparable, and $l(a_1, b_1; a) = l(a_1, b_2; a)$. $u(a_1, b_1; a) = u(a_1, b_2; a)$ is impossible as $m \geq 3$. Therefore $u(a_1, b_1; b) \neq u(a_1, b_2; b)$. Take a minimal element c from the set $u(a_1, b_1; b) - u(a_1, b_2; b)$. It is easy to verify that c and b_2 are incomparable. Now we claim that $b_1 \prec c$. Really, assume to the contrary that there exists $b_1 \prec c_1 < c$. Clearly, $l(a_1, b_1; a) = l(a_1, c_1; a)$. $u(a_1, b_1; c) = u(a_1, c_1; c)$ follows from the minimality of c in $u(a_1, b_1; b) - u(a_1, b_2; b)$. Therefore, by modularity of P, $b_1 = c_1$, which is a contradiction. Hence $b_1 \prec c$ as claimed. Now $a_1 \prec c$ by Lemma. Using the induction hypothesis we see that the length of all maximal chains in intervals $[a_1, b]$ and $[b_1, b]$ is $n - 1$. Thus $n = m$ and the proof is complete.

R e m a r k. One can try to generalize the Jordan-Dedekind condition for modular multilattices and lines (see [3]). Let us recall that an o. set P is called a multilattice if

(i) for any $a, b, c \in P$ with $l(a, b; c)$ there exists a maximal element u of $(a] \cap (b]$ such that $c \leq u$;
(ii) the dual of (i).

Then $a \vee b$ means the set of minimal elements of $[a) \cap [b)$. Similarly, $a \wedge b$ is the set of all maximal elements of $(a] \cap (b]$.

We can now define the betweenness relation $a\,x\,b$ on P as follows: $a\,x\,b$ if and only if

$$\big[(a \wedge x) \vee (b \wedge x)\big] \cap (x] = \{x\} = \big[(a \vee x) \wedge (b \vee x)\big] \cap (x] \,.$$

A subset of a multilattice P is called a line if for $a, b, c \in P$ one of the relations $a\,b\,c$, $b\,a\,c$, or $a\,c\,b$ is true. Obviously, any chain is a line.

The following example shows that the Jordan-Dedekind condition for lines is not true for modular multilattices.

Take $P = \{a, b, p, r, s, t, v\}$ and define the partial order as follows: $t < p < a < r$, $p < b$, $a < v$ and $s < b$. It can easily be shown that this is a modular multilattice, and that the sets $\{a, p, b\}$ and $\{a, r, s, b\}$ are lines in P. They do not have the same length.

The author is grateful to the referee for critical comments.

REFERENCES

[1] BIRKHOFF, G.: *Lattice Theory. Amer. Math. Soc. Colloq. Publ. Vol. 25*, Amer. Math. Soc., Providence, R.I., 1967.

[2] CHAJDA, I.—RACHŮNEK, J.: *Forbidden configurations for distributive and modular ordered sets*, Order **5** (1989), 407–423.

[3] KOLIBIAR, M.: *Linien in Verbänden*, An. Ştiinţ. Univ. "Al. I. Cuza" Iaşi Secţ. I a Mat. (N.S.) **XIB** (1965), 89–98.

Received July 30, 1993

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