Mathematica Slovaca

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Mathematica Slovaca, Vol. 33 (1983), No. 3, 297--302

Persistent URL: http://dml.cz/dmlcz/130569

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A NOTE ON OSCILLATION AND NONOSCILLATION CRITERIA FOR FOURTH ORDER LINEAR DIFFERENTIAL EQUATIONS

JÁN REGENDA

1. Introduction

The present paper is a study of the oscillation and nonoscillation of the differential equation

$$L[y] = y^{(4)} + P(t)y'' + Q(t)y = 0, \tag{L}$$

where $P(t)$, $Q(t)$ are continuous functions on the interval $I = \langle a, \infty)$, $-\infty < a < \infty$. We shall assume throughout that

(A) $P(t) \leqq 0$, $Q(t) \leqq 0$ and $Q(t)$ not identically zero in any subinterval of I.

This note is the continuation of [3] and [4]. So we shall use the notations and results obtained earlier, without explaining them again here. Oscillation and nonoscillation criteria for equation (L) will be obtained by an application of the theory developed in [3] and [4].

It will be proved that (L) is nonoscillatory if

$$u^{(4)} + [P(t) + Q(t)]u = 0 \tag{1}$$

is nonoscillatory. If equation (1) is known to be nonoscillatory, the problem is to find condition on the coefficients in (L) to ensure that (L) also is nonoscillatory. Leighton and Nehari [2], Howard [1] have obtained comparison theorems for the class of self-adjoint linear differential equations of the fourth order. They have based their study of comparison theorems of Sturm's type upon eigenvalue problems of the type

$$[a(t)u'']'' - \lambda c(t)u = 0$$
$$u(\alpha) = u'(\alpha) = u(\beta) = u'(\beta) = 0,$$

where a and c are positive functions of class C^2 and C, respectively.

The results of the above authors will be used in the present.

2. Preliminary results

The following Lemma and Theorems 1, 2, 3 and 4 will be basic in our investigation.

Lemma [4]. *Let* $f(t) \in C^2\langle c, \infty)$ *and* $f(t) > 0$, $f'(t) > 0$, $f''(t) < 0$ *in* $\langle c, \infty)$, $c \geqq a$. *Then*

$$f(t) - (t-c)f'(t) > 0$$

for $t \in \langle c, \infty)$.

Corollary. *Let there be a function* $w(t) \in C^3\langle c, \infty)$ *and let* $w(t) > 0$, $w'(t) > 0$, $w''(t) > 0$ *and* $w'''(t) < 0$ *in* $\langle c, \infty)$, $c \geqq a$. *It follows from the Lemma that*

$$w(t) > \frac{(t-c)^2}{2} w''(t) \quad \text{for} \quad t \in \langle c, \infty).$$

Theorem 1 [3]. *Suppose that* (A) *holds. Then* (L) *is nonoscillatory on* I *if and only if there exists a number* $t_0 \in I$ *and a solution* $y(t)$ *of* (L) *such that either*

$$y(t) > 0, \quad y'(t) > 0, \quad y''(t) < 0$$

or

$$y(t) > 0, \quad y'(t) > 0, \quad y''(t) > 0, \quad y'''(t) < 0$$

for all $t \geqq t_0$.

Theorem 2 [3]. *Suppose that* (A) *holds. Then equation* (L) *is nonoscillatory on* I *if and only if there exists a function* $w(t) \in C^4\langle t_0, \infty)$, $t_0 \in I$, *such that either*

$$w(t) > 0, \quad w'(t) > 0, \quad w''(t) < 0, \quad L[w] \geqq 0$$

or

$$w(t) > 0, \quad w'(t) > 0, \quad w''(t) > 0, \quad w'''(t) < 0, \quad L[w] \geqq 0.$$

Theorem 3 [4]. *Suppose that* (A) *holds and let* $-2t^{-2} \leqq P(t)$ *for* $t > t_0 \geqq \max\{a, 0\}$. *Then there does not exist a solution* $y(t)$ *of* (L) *such that* $y(t) > 0$, $y'(t) > 0$, $y''(t) < 0$ *for* $t > t_1 \geqq t_0$.

Theorem 4 [4]. *Suppose that* (A) *holds and let*

$$\int_{t_0}^{\infty} sP(s)\,\mathrm{d}s > -\infty, \quad t_0 > \max\{a, 0\}.$$

Then there does not exist a solution $y(t)$ *of* (L) *with the properties* $y(t) > 0$, $y'(t) > 0$, $y''(t) < 0$ *for* $t \geqq t_0$.

3. Oscillation and nonoscillation theorems

We assume that the coefficients of (L) satisfy (A). If, in addition,

$$-\frac{2}{t^2} \leqq P(t) \tag{1}$$

or

$$\int_T^\infty tP(t)\,\mathrm{d}t > -\infty \tag{2}$$

for $t \geqq T > \max\{a, 0\}$, then by Theorem 3 and 4, respectively, the equation (L) has no solution $y(t)$ with the properties $y(t) > 0$, $y'(t) > 0$, $y''(t) < 0$ in $\langle t_0, \infty)$, $t_0 \geqq T$.

Using this fact, we can prove the following modification of the oscillation Theorem 2.3 [4].

Theorem 5. *Let $\mu(t)$ be a positive and continuous function in (T, ∞), $T > \max\{a, 0\}$ such that*

$$\liminf_{t\to\infty} \frac{t - t_0}{\mu(t)} \geqq 2$$

for arbitrary $t_0 \geqq a$. Suppose that (A) *and* (1) *or* (2) *hold. If the differential equation of the third order*

$$x''' + \Theta\mu(t)Q(t)x = 0$$

for some $\Theta \in (0, 1)$ is oscillatory, then equation (L) *also is oscillatory.*

The proof is very similar to that of Theorem 2.3 [4] and is omitted.

By combining Theorem 5 with the known oscillation criteria for the third-order equation $x''' + r(t)x = 0$ we obtain oscillation criteria for (L).

The next theorem gives sufficient conditions for equation (L) to be nonoscillatory.

Theorem 6. *Suppose that* (A) *holds. Then equation* (L) *is nonoscillatory if*

$$(\mathscr{L}) \qquad u^{(4)} + [P(t) + Q(t)]u = 0$$

is nonoscillatory.

Proof. Suppose that $(\mathscr{L})$ is nonoscillatory. Since the coefficient of u'' in $(\mathscr{L})$ has vanished, it follows from Theorem 3 that $(\mathscr{L})$ has no solution $u(t)$ such that $u(t) > 0$, $u'(t) > 0$ and $u''(t) < 0$ for $t \geqq t_1$, $t_1 \in I$. By Theorem 1 there exists a number $t_2 \in I$ and a solution $u(t)$ of $(\mathscr{L})$ such that $u(t) > 0$, $u'(t) > 0$, $u''(t) > 0$ and $u'''(t) < 0$ for all $t \geqq t_2$. Applying Lemma and Corollary to the solution $u(t)$ we obtain

$$u(t) > \frac{(t-t_2)^2}{2} u''(t) \tag{3}$$

for $t \geqq t_2$. It follows from (3) and the assumption $P(t) \leqq 0$ that $P(t)u(t) \leqq P(t)u''(t)$ for $t \geqq t_2 + \sqrt{2} = t_0$. From the last inequality and from ($\mathscr{L}$) we obtain that for all $t \geqq t_0$.

$$u^{(4)} + P(t)u'' + Q(t)u \geqq 0.$$

Since $u(t) > 0$, $u'(t) > 0$, $u''(t) > 0$, $u'''(t) < 0$ and $L[u] \geqq 0$, it follows from Theorem 2 that equation

$$y^{(4)} + P(t)y'' + Q(t)y = 0$$

is nonoscillatory on I. Theorem 6 is proved.

4. Nonoscillation criteria

The following nonoscillation criteria for equation (L) will now be obtained by combining Theorem 6 with the known nonoscillation theorems for equation ($\mathscr{L}$).

Theorem 7' (Leighton and Nehari). *The equation $u^{(4)} + c(t)u = 0$, $c(t) < 0$ is nonoscillatory if*

$$\limsup_{t \to \infty} t^4 |c(t)| < \frac{9}{16}.$$

Theorem 7. *Suppose that the coefficients of* (L) *satisfy the assumptions* (A) *and let $P(t) + Q(t) < 0$ in I. If*

$$\limsup_{t \to \infty} t^4 |P(t) + Q(t)| < \frac{9}{16},$$

then (L) *is nonoscillatory.*

Theorem 8' (Leighton and Nehari). *If the equation*

$$u'' + [4t^2 c(t) - 2t^{-2}]u = 0$$

$(c(t) > 0)$ is nonoscillatory in (α, ∞), the same is true of $u^{(4)} = c(t)u (\alpha > 0)$.

Theorem 8. *Suppose that the coefficients of* (L) *satisfy the assumptions* (A) *and let $P(t) + Q(t) < 0$ in I. If the equation*

$$u'' - [4t^2(P(t) + Q(t)) + 2t^{-2}]u = 0$$

is nonoscillatory in (α, ∞), the same is true of (L) *$(\alpha > \max\{a, 0\})$.*

The next theorem will be based on the results of Howard H. [1].

Theorem 9'. *Let f be a positive nonincreasing function of class C^1 in $\langle \alpha, \infty)$ $\alpha \geqq 0)$ such that*

$$\int_\alpha^\infty t^{-4} f(t)\, dt < \infty. \tag{4}$$

The sufficient conditions for the equation $u^{(4)} = c(t)u$ $(c(t) > 0)$ to be nonoscillatory is

$$\limsup_{t\to\infty} \left(\int_t^\infty c(s) f(s)\, ds\right)\left(\int_t^\infty s^{-4} f(s)\, ds\right)^{-1} < \frac{9}{16}.$$

Theorem 9. *Let f be a positive nonincreasing function of class C^1 in $\langle \alpha, \infty)$ $(\alpha \geqq \max\{a, 0\})$ with the property* (4). *Suppose that* (A) *holds and let $P(t) + Q(t) < 0$ in I. The sufficient condition for equation* (L) *to be nonoscillatory is*

$$\limsup_{t\to\infty} \left(\int_t^\infty |P(s) + Q(s)| f(s)\, ds\right)\left(\int_t^\infty s^{-4} f(s)\, ds\right)^{-1} < \frac{9}{16}. \tag{5}$$

The condition (5) can be replaced by condition

$$\limsup_{t\to\infty} \left(\int_t^\infty (s-\alpha)^p |P(s) + Q(s)|\, ds\right)\left(\int_t^\infty s^{-4} (s-\alpha)^p\, ds\right)^{-1} < \frac{9}{16},$$

where p is a number satisfying $p \leqq 2$, according to Howard's Theorem 3.86 [5].

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Received November 20, 1981

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ЗАМЕЧАНИЕ ОБ ОСЦИЛЛЯЦИОННЫХ И НЕОСЦИЛЛЯЦИОННЫХ ПРИЗНАКАХ ДЛЯ ЛИНЕИНЫХ ДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИИ ЧЕТВЕРТОГО ПОРЯДКА

Jan Regenda

Резюме

В работе даны некоторые признаки осцилляции и неосциляции дифференциальноı о уравнения четвертого порядка

(L) $$y^{(4)} + P(t)y'' + Q(t)y - 0$$

с непрерывными коеффиц нтами в интер ле $I = \langle a, \infty)$

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