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STATISTICAL MAPS I. BASIC PROPERTIES

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ABSTRACT. A statistical map is defined to be an affine map between two simplexes of probability measures satisfying a condition of measurability. Relations between statistical maps and some other concepts like: measurable functions, Markov kernels, dual statistical maps, statistical functions and effect valued measures are discussed. Continuity properties of statistical maps and their duals are considered. A product of statistical maps is defined and proved to be a statistical map.

1. Introduction

1.1. *Statistical map* is an affine (i.e. convex combinations preserving) map $M_1^+(\Omega) \to M_1^+(\Xi)$ of the convex set $M_1^+(\Omega)$ of probability measures on a measurable space Ω into such a set $M_1^+(\Xi)$ on another measurable space Ξ, satisfying a measurability condition (formula (1) below). Maps of that kind belong to the formal apparatus of standard probability theory (SPT for brevity): every standard random variable defines a statistical map and can be equivalently described by that map, while every Markov kernel is fully characterized by an associated statistical map (see [14] or Section 2 below). Statistical maps appear also in various branches of science originating from standard probability theory; for instance, the theory of information transmission employs them to describe noisy channels ([7]), in the statistical decision theory they represent random strategies ([3], [18]), statistical maps describe general stochastic processes ([10], [11], [12]), etc.

All that would make the concept of statistical map worth of noticing. What makes it more important is the fact that statistical maps occupy the central position in *operational probability theory* (OPT for brevity, originally called fuzzy

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probability theory), a recent generalization of standard probability theory ([9], [14], [10], [16]). The main idea of OPT is to extend the standard theory by extending the traditional concept of random variable. Instead of measurable functions on a space of elementary events, OPT admits generalized random variables (called *operational random variables*, originally — fuzzy random variables) formally represented by statistical maps. The interpretation of statistical maps as operational random variables is discussed in the second part of this paper [13].

In the present paper we define statistical maps (Section 2) and collect some basic formal properties of them and of their duals (Section 3). Then we prove that an arbitrary family of statistical maps possesses a product which is a statistical map (Section 5). We discuss also several closely related formal concepts like these of Markov kernels (Subsection 2.3), dual statistical maps (Subsection 3.4), statistical functions, and effect valued measures (Section 4).

In our considerations we will refer to notions and facts of measure theory (as presented, for instance, in [4], [18], [8]) and these of the theory of ordered linear spaces [22], [1], [2]. We will also need some fundamental theorems of functional analysis which can be found in many textbooks (see for instance [21], [19]).

1.2. Let $(\Omega, \mathcal{B}(\Omega))$ be a measurable space, we assume that all singletons $\{\omega\}$, $\omega \in \Omega$, are measurable. $M_1^+(\Omega)$ denotes then the convex set of all σ-additive probability measures on $(\Omega, \mathcal{B}(\Omega))$, while $\delta\Omega$ is the set of all Dirac measures concentrated at points of Ω, $\delta\Omega := \{\delta_\omega : \omega \in \Omega\}$. The set of all extreme points of the convex set $M_1^+(\Omega)$ will be denoted $\partial M_1^+(\Omega)$; in a genuine case $\partial M_1^+(\Omega)$ is larger than $\delta\Omega$ (an example can be found in [14]).

Lemma. *$\mu \in \partial M_1^+(\Omega)$ if and only if $\mu(X) \in \{0,1\}$ for every $X \in \mathcal{B}(\Omega)$.*

Proof. Let $\mu \in \partial M_1^+(\Omega)$ and assume that there is $X_0 \in \mathcal{B}(\Omega)$ such that $\mu(X_0) = \lambda \neq 0, 1$. Take $\mu_1, \mu_2 \in M_1^+(\Omega)$ defined by: $\mu_1(X) := \frac{1}{\lambda}\mu(X \cap X_0)$, and $\mu_2(X) := \frac{1}{1-\lambda}\mu(X \cap (\Omega \setminus X_0))$ for every $X \in \mathcal{B}(\Omega)$. It is evident that $\mu = \lambda\mu_1 + (1-\lambda)\mu_2$, so $\mu \neq \partial M_1^+(\Omega)$. Thus $\mu \in \partial M_1^+(\Omega)$ implies $\mu(X) \in \{0,1\}$ for every $X \in \mathcal{B}(\Omega)$.

Assume that for some $\mu \in M_1^+(\Omega)$ one has $\mu(X) \in \{0,1\}$ for every $X \in \mathcal{B}(\Omega)$. If $\mu = \lambda\mu_1 + (1-\lambda)\mu_2$ for some $\mu_1, \mu_2 \in M_1^+(\Omega)$ and $0 < \lambda < 1$, then it is easy to see that $\mu_1(X) = \mu_2(X) = \mu(X)$ for every $X \in \mathcal{B}(\Omega)$. That means that $\mu \in \partial M_1^+(\Omega)$. □

The convex set $M_1^+(\Omega)$ will be considered occasionally as the base of the base normed Banach space $M(\Omega)$ of all bounded σ-additive signed measures on $(\Omega, \mathcal{B}(\Omega))$. It explains the symbol we adopted for the set of all probability measures on $(\Omega, \mathcal{B}(\Omega))$, namely $M_1^+(\Omega) = M^+(\Omega) \cap M_1(\Omega)$, where $M^+(\Omega)$ is the positive cone of $M(\Omega)$, and $M_1(\Omega)$ is the hyperplane of normalized measures (i.e. $\mu \in M_1(\Omega)$ if and only if $\mu(\Omega) = 1$). $M_1^+(\Omega)$ is known to be closed in the base-norm topology of $M(\Omega)$ what is equivalent to its σ-convexity.

The set of all measurable functions on Ω taking values in the real interval $[0,1]$ will be denoted $\mathcal{E}(\Omega)$; elements of $\mathcal{B}(\Omega)$, the measurable subsets of Ω, will be identified with their characteristic functions, so $\mathcal{B}(\Omega) \subset \mathcal{E}(\Omega)$. The set $\mathcal{E}(\Omega)$ will be considered as the order interval of the vector lattice $\mathcal{F}(\Omega)$ of all real bounded measurable functions on Ω. The real linear space $\mathcal{F}(\Omega)$ carries the natural structure of an order-unit Banach space with $\mathcal{E}(\Omega)$ as its (norm) closed order-unit interval.

1.3. Every function $f \in \mathcal{E}(\Omega)$ can be integrated with any measure $\mu \in M_1^+(\Omega)$ to get the number $\int_\Omega f(\omega)\,\mu(\mathrm{d}\omega) \in [0,1]$. The integral extends over all bounded measurable functions on Ω and all bounded measures (including signed ones) on Ω, hence defines the natural duality between Banach spaces $M(\Omega)$ and $\mathcal{F}(\Omega)$. The functional on $M(\Omega)$ corresponding to $f \in \mathcal{F}(\Omega)$ will be denoted $\overline{a_f}$, while the restriction of $\overline{a_f}$ to the base $M_1^+(\Omega)$ of $M(\Omega)$ — by a_f. Clearly,

$$a_f(\mu) = \int_\Omega f(\omega)\,\mu(\mathrm{d}\omega)$$

for every $\mu \in M_1^+(\Omega)$. The affine function on $M_1^+(\Omega)$ generated by χ_X, the characteristic function of a measurable subset $X \in \mathcal{B}(\Omega)$, will be denoted a_X.

The duality between $M(\Omega)$ and $\mathcal{F}(\Omega)$ defines a natural embedding of $\mathcal{F}(\Omega)$ into $M(\Omega)^*$, the Banach dual of $M(\Omega)$, which identifies $\mathcal{F}(\Omega)$ with a weak*-dense subspace of $M(\Omega)^*$. Some essential properties of the dual pair $\langle M(\Omega), \mathcal{F}(\Omega)\rangle$ are collected in [23]. The dual pair $\langle M(\Omega), \mathcal{F}(\Omega)\rangle$ of Banach spaces is a particular case of a general concept of a statistical duality [26], [24].

1.4. Let us finally mention that $M(\Omega)^*$ is isomorphic as an order-unit Banach space with $A^b(M_1^+(\Omega))$, the space of all real bounded affine functions on the convex set $M_1^+(\Omega)$ endowed with the sup-norm (a general proof of that fact can be found, for instance, in [6]). The relations $\mathcal{F}(\Omega) \subseteq M(\Omega)^* \simeq A^b(M_1^+(\Omega))$ introduce the natural ambiguity: an element f of $\mathcal{F}(\Omega)$ can be understood, depending on a context, simply as a bounded real measurable function on Ω, or as the corresponding functional on $M(\Omega)$ (denoted $\overline{a_f}$ above), or as the the bounded real affine function on $M_1^+(\Omega)$ (denoted a_f above). In particular, the set $\mathcal{E}(\Omega)$ of *effects* on Ω will be identified with the corresponding subsets of $\mathcal{F}(\Omega)$, $M(\Omega)^*$, and $A^b(M_1^+(\Omega))$.

2. Statistical maps

2.1. Let $(\Omega, \mathcal{B}(\Omega))$, $(\Xi, \mathcal{B}(\Xi))$ be measurable spaces.

DEFINITION. A map $A\colon M_1^+(\Omega) \to M_1^+(\Xi)$ is called *affine* if
$$A\big(\lambda\mu_1 + (1-\lambda)\mu_2\big) = \lambda A\mu_1 + (1-\lambda)A\mu_2$$
for every $\mu_1, \mu_2 \in M_1^+(\Omega)$ and $\lambda \in [0,1]$.

An affine map $A\colon M_1^+(\Omega) \to M_1^+(\Xi)$ is called a *statistical map* if for every $X \in \mathcal{B}(\Xi)$ there is $f_X \in \mathcal{E}(\Omega)$ such that
$$\int_X (A\mu)(\mathrm{d}\xi) = \int_\Omega f_X(\omega)\, \mu(\mathrm{d}\omega) \tag{1}$$
for every $\mu \in M_1^+(\Omega)$.

The statistical map is *pure* if $A\big(\partial M_1^+(\Omega)\big) \subseteq \partial M_1^+(\Xi)$, *sharp* if $A(\delta\Omega) \subseteq \partial M_1^+(\Xi)$, and *strict* if $A(\delta\Omega) \subseteq \delta\Xi$.

Of course, if A is pure, then A is sharp. The reverse implication is proved in Subsection 3.1, hence a statistical map is pure if and only if it is sharp. In 2.2 it is proved that strict statistical maps $M_1^+(\Omega) \to M_1^+(\Xi)$ correspond exactly to measurable functions $\Omega \to \Xi$ *via* the concept of the distribution functional. The equivalence "sharp $\iff$ strict" holds if and only if $\partial M_1^+(\Xi) = \delta\Xi$ (see [14]).

Statistical maps are distinguished by the property that any such a map is uniquely defined by its restriction to the set of Dirac measures:

PROPOSITION.

(a) *An affine map $A\colon M_1^+(\Omega) \to M_1^+(\Xi)$ is a statistical map if and only if*
$$(A\mu)(X) = \int_\Omega (A\delta_\omega)(X)\, \mu(\mathrm{d}\omega) \tag{2}$$
for every $\mu \in M_1^+(\Omega)$ and $X \in \mathcal{B}(\Xi)$.

(b) *A statistical map A is pure if and only if $(A\mu)(X) \in \{0,1\}$ for every $X \in \mathcal{B}(\Xi)$ and $\mu \in \partial M_1^+(\Omega)$.*

(c) *A statistical map A is sharp if and only if $\big(A\delta_\omega\big)(X) \in \{0,1\}$ for every $X \in \mathcal{B}(\Xi)$ and $\omega \in \Omega$.*

Proof.

(a) Clearly, $\int_X (A\mu)(\mathrm{d}\xi) = (A\mu)(X)$. Take $\mu = \delta_{\omega_0}$, then condition (1) reads $(A\delta_{\omega_0})(X) = \int_\Omega f_X(\omega)\, \delta_{\omega_0}(\mathrm{d}\omega) = f_X(\omega_0)$. Hence, $f_X(\omega) = (A\delta_\omega)(X)$ for all $\omega \in \Omega$, what gives (2). The reverse implication is evident.

(b) and (c) are easy corollaries of Lemma of 1.2. □

COROLLARY. *A sharp statistical map $A\colon M_1^+(\Omega) \to M_1^+(\Xi)$ defines the unique function $\varphi_A\colon \Omega \to \partial M_1^+(\Xi)$ such that $\varphi_A(\omega) := A\delta_\omega$ for every $\omega \in \Omega$; the function $\omega \to \varphi_A(\omega)(X)$ for an arbitrary fixed $X \in \mathcal{B}(\Xi)$ is a characteristic function of a measurable subset of Ω.*

2.2. Let $(\Omega, \mathcal{B}(\Omega))$ and $(\Xi, \mathcal{B}(\Xi))$ be two measurable spaces such that all their one-point subsets are measurable. A measurable function $F\colon \Omega \to \Xi$ associates with any $\mu \in M_1^+(\Omega)$ a probability measure μ_F on Ξ, the *distribution* of F at μ, defined by

$$\mu_F(X) := \mu\bigl(F^{-1}(X)\bigr)$$

for every $X \in \mathcal{B}(\Xi)$. The map $M_1^+(\Omega) \ni \mu \to \mu_F \in M_1^+(\Xi)$, denoted D_F, is affine; moreover, it is a statistical map. Indeed, the last equality translates into

$$\int_X (D_F\mu)(\mathrm{d}\xi) = \int_\Omega \chi_{F^{-1}(X)}(\omega)\, \mu(\mathrm{d}\omega),$$

what is a special case of the condition (1). It is evident that D_F is strict, i.e. transforms Dirac measures into Dirac measures:

$$D_F\delta_\omega = \delta_{F(\omega)}. \tag{3}$$

Thus every measurable function $F\colon \Omega \to \Xi$ defines (and is uniquely determined by) the strict statistical map $D_F\colon M_1^+(\Omega) \to M_1^+(\Xi)$ called the *distribution functional* of F. Also the reverse statement holds (comp. [14]):

THEOREM. *There is a natural one-to-one correspondence between measurable functions $\Omega \to \Xi$ and strict statistical maps $M_1^+(\Omega) \to M_1^+(\Xi)$, such that:*

(a) *any measurable function $F\colon \Omega \to \Xi$ defines the unique strict statistical map $D_F\colon M_1^+(\Omega) \to M_1^+(\Xi)$ such that $(D_F\mu)(X) := \mu\bigl(F^{-1}(X)\bigr)$ for every $\mu \in M_1^+(\Omega)$ and every $X \in \mathcal{B}(\Xi)$;*

(b) *any strict statistical map $A\colon M_1^+(\Omega) \to M_1^+(\Xi)$ defines a unique measurable function $F_A\colon \Omega \to \Xi$ such that $A\delta_\omega = \delta_{F_A(\omega)}$ for every $\omega \in \Omega$;*

(c) *$F_{D_F} = F$ for every measurable function F, $D_{F_A} = A$ for every strict statistical map A.*

Proof.

(a) is proved above.

(b) Clearly, $A\delta_\omega = \delta_\xi$ for some $\xi \in \Xi$. Thus the statistical map $A\colon M_1^+(\Omega) \to M_1^+(\Xi)$ defines a function $F_A\colon \Omega \to \Xi$ such that $A\delta_\omega = \delta_{F_A(\omega)}$. We have to show that F_A is measurable. Let $X \in \mathcal{B}(\Xi)$. Then, $(A\delta_\omega)(X) = \delta_{F_A(\omega)}(X) = \chi_{F_A^{-1}(X)}(\omega)$, what implies that $F_A^{-1}(X)$ is measurable. Thus F_A must be measurable.

(c) The two equalities: $D_F\delta_\omega = \delta_{F(\omega)}$ (see formula (3)), and $D_F\delta_\omega = \delta_{F_{D_F}(\omega)}$ (see (b)) imply $F_{D_F} = F$. It is easy to find that $\big(D_{F_A}\delta_\omega\big)(X) := \delta_\omega\big(F_A^{-1}(X)\big) = \chi_{F_A^{-1}(X)}(\omega) = \delta_{F_A(\omega)}(X) = (A\delta_\omega)(X)$ for every $\omega \in \Omega$ and $X \in \mathcal{B}(\Xi)$. Thus, the two statistical maps D_{F_A} and A are identical on $\delta\Omega$ what implies (see the proposition of 2.1) that $D_{F_A} = A$. □

If $\partial M_1^+(\Xi) = \delta\Xi$, then any sharp statistical map $A\colon M_1^+(\Omega) \to M_1^+(\Xi)$ is strict, hence defines the unique measurable function $F_A\colon \Omega \to \Xi$ such that $A\delta_\omega = \delta_{F_A(\omega)}$ for every $\omega \in \Omega$. The condition $\partial M_1^+(\Xi) = \delta\Xi$ holds for all measurable spaces appearing in practical applications of probability theory, see [14].

2.3. Let us remind that a *Markov kernel* from $\big(\Omega, \mathcal{B}(\Omega)\big)$ to $\big(\Xi, \mathcal{B}(\Xi)\big)$ is a map

$$K\colon \Omega \times \mathcal{B}(\Xi) \to [0,1]$$

such that for any fixed $\omega \in \Omega$ the map $\mathcal{B}(\Xi) \to [0,1]$ defined by $X \to K(\omega, X)$ is a probability measure on Ξ, and for any fixed $X \in \mathcal{B}(\Xi)$ the map $\Omega \to [0,1]$ defined by $\omega \to K(\omega, X)$ is a measurable function on Ω.

A Markov kernel $K\colon \Omega \times \mathcal{B}(\Xi) \to [0,1]$ associates with any $\mu \in M_1^+(\Omega)$ a probability measure μ_K on Ξ defined by

$$\mu_K(X) := \int_\Omega K(\omega, X)\, \mu(\mathrm{d}\omega)$$

for every $X \in \mathcal{B}(\Xi)$, see for instance [4; p. 326]. In particular, if $\mu = \delta_{\omega_0}$, then the corresponding measure $(\delta_{\omega_0})_K$ on Ξ is given by $\big(\delta_{\omega_0}\big)_K(X) = K(\omega, X)$ according to the definition of Markov kernel. As the definition of Markov kernel ensures that $K(\omega, X)$ for a fixed X is an element of $\mathcal{E}(\Omega)$, the map $D_K\colon M_1^+(\Omega) \to M_1^+(\Xi)$, $D_K\mu := \mu_K$, is a statistical map.

Thus any Markov kernel $K\colon \Omega \times \mathcal{B}(\Xi) \to [0,1]$ defines the corresponding statistical map $D_K\colon M_1^+(\Omega) \to M_1^+(\Xi)$, its distribution functional. On the other hand, it is evident that any statistical map $A\colon M_1^+(\Omega) \to M_1^+(\Xi)$ defines a Markov kernel $K_A\colon \Omega \times \mathcal{B}(\Xi) \to [0,1]$ by

$$K_A(\omega, X) := \big(A\delta_\omega\big)(X)$$

for every $(\omega, X) \in \Omega \times \mathcal{B}(\Xi)$. Moreover, the statistical map defined by K_A is again A. It is equally evident that, for an arbitrary Markov kernel $K\colon \Omega \times \mathcal{B}(\Xi) \to [0,1]$, the Markov kernel defined by the statistical map $D_K\colon M_1^+(\Omega) \to M_1^+(\Xi)$ is again K.

We summarize above remarks (comp. [14]).

THEOREM. *There is a natural one-to-one correspondence between statistical maps $M_1^+(\Omega) \to M_1^+(\Xi)$ and Markov kernels $\Omega \times \mathcal{B}(\Xi) \to [0,1]$ such that:*

(a) *a Markov kernel $K\colon \Omega \times \mathcal{B}(\Xi) \to [0,1]$ defines the unique statistical map $D_K\colon M_1^+(\Omega) \to M_1^+(\Xi)$ such that $(D_K\mu)(X) := \int_\Omega K(\omega, X)\, \mu(\mathrm{d}\omega)$ for every $\mu \in M_1^+(\Omega)$ and $X \in \mathcal{B}(\Xi)$,*

(b) *a statistical map $A\colon M_1^+(\Omega) \to M_1^+(\Xi)$ defines the unique Markov kernel $K_A\colon \Omega \times \mathcal{B}(\Xi) \to [0,1]$ such that $K_A(\omega, X) := (A\delta_\omega)(X)$ for every $(\omega, X) \in \Omega \times \mathcal{B}(\Xi)$,*

(c) $K_{D_K} = K$, $D_{K_A} = A$.

3. Continuity properties of statistical maps and their duals

3.1. The convex set $M_1^+(\Omega)$ linearly generates the base normed Banach space $M(\Omega)$ (the Hahn-Jordan decomposition theorem), hence any affine map $A\colon M_1^+(\Omega) \to M_1^+(\Xi)$ extends to a unique linear transformation (an operator) $\overline{A}\colon M(\Omega) \to M(\Xi)$. Notice by the way that $\overline{A}$ is an instance of the general concept of a stochastic operator [15].

Elementary properties of $\overline{A}$ and of its Banach dual $\overline{A}^*$ are collected below. Recall that the Banach dual space $M(\Omega)^*$ is isomorphic as an order-unit Banach space with $A^b(M_1^+(\Omega))$ (see 1.4), hence $\overline{A}^*\colon A^b(M_1^+(\Xi)) \to A^b(M_1^+(\Omega))$.

PROPOSITION. *For any affine map $A\colon M_1^+(\Omega) \to M_1^+(\Xi)$:*

(a) *the extension $\overline{A}\colon M(\Omega) \to M(\Xi)$ is a bounded positive operator between base norm Banach spaces, with the operator norm $\|\overline{A}\| = 1$,*

(b) *the Banach dual $\overline{A}^*\colon A^b(M_1^+(\Xi)) \to A^b(M_1^+(\Omega))$ is a bounded positive unital operator between order unit Banach spaces with the operator norm $\|\overline{A}^*\| = 1$.*

Proof.

(a) It is evident that $\overline{A}$ is positive, i.e. maps the positive cone $M^+(\Omega)$ into the positive cone $M^+(\Xi)$. The base norm of $M(\Omega)$ is known to be equal to the total variation norm, and $M_1^+(\Omega)$ is closed in the norm topology, hence the closed unit ball of $M(\Omega)$ equals $\mathrm{Conv}\big(M_1^+(\Omega) \cup \big(-M_1^+(\Omega)\big)\big)$. Now we see that $\overline{A}$ maps the closed unit ball of $M(\Omega)$ into the closed unit ball of $M(\Xi)$, hence must be bounded and its operator norm $\|\overline{A}\|$ is not bigger than 1. In fact $\|\overline{A}\| = 1$ as it maps $M_1^+(\Omega)$ into $M_1^+(\Xi)$.

(b) Any bounded operator possesses the bounded Banach dual of the same operator norm, hence $\overline{A}^*$ does exist and $\|\overline{A}^*\| = 1$. The Banach dual of any positive operator has to be positive. Finally, it is evident that $\overline{A}^* a_\Xi = a_\Omega$, where a_Ξ is the order unit of $A^b(M_1^+(\Xi))$ and a_Ω is the order unit of $A^b(M_1^+(\Omega))$. □

COROLLARY 1. *For any affine map $A\colon M_1^+(\Omega) \to M_1^+(\Xi)$, $\overline{A}$ and $\overline{A}^*$ are norm-to-norm continuous.*

P r o o f. The norm boundedness of an operator between normed spaces is equivalent to its norm-to-norm continuity. □

COROLLARY 2. *Let $A\colon M_1^+(\Omega) \to M_1^+(\Xi)$ be an affine map. Every one of the following conditions is necessary and sufficient for A to be a statistical map:*

(a) $\overline{A}^*(\mathcal{B}(\Xi)) \subset \mathcal{E}(\Omega)$;

(b) $\overline{A}^*(\mathcal{F}(\Xi)) \subseteq \mathcal{F}(\Omega)$;

(c) $\overline{A}^*(\mathcal{E}(\Xi)) \subseteq \mathcal{E}(\Omega)$.

P r o o f.

(a) Condition (1) easily translates into

$$\overline{A}^*(\mathcal{B}(\Xi)) \subset \mathcal{E}(\Omega). \tag{4}$$

(b) The positivity of $\overline{A}^*$ (see the last Proposition) implies: if $\overline{A}^*(\mathcal{F}(\Xi)) \subseteq \mathcal{F}(\Omega)$, then $\overline{A}^*(\mathcal{B}(\Xi)) \subset \mathcal{E}(\Omega)$, so A is a statistical map.

On the other hand, it is known that the linear hull of $\mathcal{B}(\Xi)$ is norm-dense in $\mathcal{F}(\Xi)$, hence the norm continuity of $\overline{A}^*$ implies: if $\overline{A}^*(\mathcal{B}(\Xi)) \subset \mathcal{E}(\Omega)$, then $\overline{A}^*(\mathcal{F}(\Xi)) \subseteq \mathcal{F}(\Omega)$.

(c) is an immediate implication of the above. □

COROLLARY 3. *A statistical map $A\colon M_1^+(\Omega) \to M_1^+(\Xi)$ is sharp if and only if $\overline{A}^*(\mathcal{B}(\Xi)) \subseteq \mathcal{B}(\Omega)$.*

Now we can prove the mentioned equivalence of the concepts of pure and sharp statistical maps.

COROLLARY 4. *A statistical map is sharp if and only if it is pure.*

P r o o f. We have to show that if a statistical map $A\colon M_1^+(\Omega) \to M_1^+(\Xi)$ has the property $A(\delta\Omega) \subseteq \partial M_1^+(\Xi)$, then $A(\partial M_1^+(\Omega)) \subseteq \partial M_1^+(\Xi)$. Let $\mu \in \partial M_1^+(\Omega)$. The sharpness of A implies that $\overline{A}^*(\mathcal{B}(\Xi)) \subseteq \mathcal{B}(\Omega)$, hence $(\overline{A}^* a_X)(\mu) \in \{0,1\}$ for every $X \in \mathcal{B}(\Xi)$, see Lemma of 1.2. The definition of the dual $\overline{A}^*$ implies that $a_X(A\mu) = (\overline{A}^* a_X)(\mu)$, hence $a_X(A\mu) \in \{0,1\}$ for every $X \in \mathcal{B}(\Xi)$. Thus $A\mu \in \partial M_1^+(\Xi)$ by Lemma of 1.2. □

3.2. Taking into account that $\langle M(\Omega), A^b(M_1^+(\Omega))\rangle$ and $\langle M(\Omega), \mathcal{F}(\Omega)\rangle$ are dual pairs of ordered linear spaces (see Section 1. above) we can apply the basic theorem on adjoint maps (see [22; Chapter IV, §2, Subsection 2.1]) to show that $\overline{A}$ and $\overline{A}^*$ are weakly continuous.

PROPOSITION 1. *Let $A\colon M_1^+(\Omega) \to M_1^+(\Xi)$ be an affine map, let $\overline{A}^*\colon A^b(M_1^+(\Xi)) \to A^b(M_1^+(\Omega))$ be the Banach dual of its extension $\overline{A}$. Then*

(a) *$\overline{A}^*$ is continuous for the weak topologies of $A^b(M_1^+(\Xi))$ and $A^b(M_1^+(\Omega))$ defined by the Banach dual pairs*
$$\langle M(\Xi), A^b(M_1^+(\Xi))\rangle \text{ and } \langle M(\Omega), A^b(M_1^+(\Omega))\rangle$$
(i.e. $\overline{A}^*$ is weak-to-weak* continuous in the standard sense),*

(b) *$\overline{A}$ is continuous for the weak topologies of $M(\Omega)$ and $M(\Xi)$ defined by the Banach dual pairs*
$$\langle M(\Xi), A^b(M_1^+(\Xi))\rangle \text{ and } \langle M(\Omega), A^b(M_1^+(\Omega))\rangle$$
(i.e. $\overline{A}$ is weak-to-weak continuous in the standard sense).

PROPOSITION 2. *Let $A\colon M_1^+(\Omega) \to M_1^+(\Xi)$ be a statistical map, let now $\overline{A}^*\colon \mathcal{F}(\Xi) \to \mathcal{F}(\Omega)$ denote the restriction of the Banach dual of $\overline{A}$ to the subspace $\mathcal{F}(\Xi) \subseteq A^b(M_1^+(\Xi))$. Then*

(a) *$\overline{A}^*$ is continuous for the weak topologies of $\mathcal{F}(\Xi)$ and $\mathcal{F}(\Omega)$ defined by the dual pairs $\langle M(\Xi), \mathcal{F}(\Xi)\rangle$ and $\langle M(\Omega), \mathcal{F}(\Omega)\rangle$,*

(b) *$\overline{A}$ is continuous for the weak topologies of $M(\Omega)$ and $M(\Xi)$ defined by the dual pairs $\langle M(\Xi), \mathcal{F}(\Xi)\rangle$ and $\langle M(\Omega), \mathcal{F}(\Omega)\rangle$.*

P r o o f . Both propositions are direct consequences of the mentioned basic theorem on adjoint maps. □

An important consequence of the above propositions is:

COROLLARY 1. *Let $A\colon M_1^+(\Omega) \to M_1^+(\Xi)$ be a statistical map. If $\{f_n : n \in \mathbb{N}\} \subset \mathcal{F}(\Xi)$ is a sequence of real measurable functions on Ξ such that the pointwise limit $f = \lim_{n\to\infty} f_n$ does exist and belongs to $\mathcal{F}(\Xi)$, then*
$$\overline{A}^* a_f = \lim_{n\to\infty} \overline{A}^* a_{f_n}\,.$$

P r o o f . For any sequence of real measurable bounded functions on a measurable space, the pointwise convergence is equivalent to the weak* convergence, see for instance [23; Proposition A.3]. □

Let us remind that a linear map $B\colon W_1 \to W_2$, where W_1 and W_2 are two normed ordered monotone σ-complete spaces, is called *σ-normal* if for every norm bounded monotone increasing sequence $\{w_n : n \in \mathbb{N}\}$ in W,
$$\text{l. u. b.}\{Bw_n : n \in \mathbb{N}\} = B\big(\text{l. u. b.}\{w_n : n \in \mathbb{N}\}\big)\,.$$

It is known that both $M(\Omega)$, $\mathcal{F}(\Omega)$ are monotone σ-complete; the monotone σ-completeness of $M(\Omega)$ is a direct consequence of the σ-convexity of $M_1^+(\Omega)$, while the monotone σ-completeness of $\mathcal{F}(\Omega)$ is implied by the mentioned fact that the linear hull of $\mathcal{B}(\Omega)$ is norm dense in $\mathcal{F}(\Omega)$. Now it is easy to show the following corollary.

COROLLARY 2. *If $A\colon M_1^+(\Omega) \to M_1^+(\Xi)$ is a statistical map, then $\overline{A}$ and $\overline{A}^*$ are σ-normal.*

Proof. If $\{\mu_n : n \in \mathbb{N}\}$ is a norm bounded monotone increasing sequence in $M^+(\Omega)$, then $\text{l.u.b.}\{\mu_n : n \in \mathbb{N}\} = \|\cdot\| - \lim_{n\to\infty} \mu_n$. Hence the norm-to-norm continuity of $\overline{A}$ (Corollary 1 of 3.1) implies that $\overline{A}$ is σ-normal. If $\{f_n : n \in \mathbb{N}\}$ is a norm bounded monotone increasing sequence in $\mathcal{F}(\Omega)$, then $\text{l.u.b.}\{f_n : n \in \mathbb{N}\} = \text{weak*-}\lim_{n\to\infty} f_n$, see [23]. Hence the weak* continuity of $\overline{A}^*$ (see above) implies that $\overline{A}^*$ is σ-normal. □

3.3. The statistical map $A\colon M_1^+(\Omega) \to M_1^+(\Xi)$ inherits the continuity properties of its extension $\overline{A}$. The same concerns the restriction of $\overline{A}^*\colon A^b(M_1^+(\Xi)) \to A^b(M_1^+(\Omega))$ to the subset $\mathcal{E}(\Xi)$ of $A^b(M_1^+(\Xi))$. Let us recall that, according to Corollary 3 of 3.1, the restriction of $\overline{A}^*$ to $\mathcal{E}(\Xi)$ maps $\mathcal{E}(\Xi)$ into $\mathcal{E}(\Omega)$. The resulting map $\mathcal{E}(\Xi) \to \mathcal{E}(\Omega)$ will be denoted A^* and called the *dual of the statistical map* $A\colon M_1^+(\Omega) \to M_1^+(\Xi)$. The corollary below refers to the topologies induced on $M_1^+(\Omega)$ and $M_1^+(\Xi)$ by the topologies of $M(\Omega)$ and $M(\Xi)$ mentioned in Corollary 1 of 3.1 and in Propositions 1 and 2 of 3.2; the same concerns the topologies of $\mathcal{E}(\Xi)$ and $\mathcal{E}(\Omega)$.

COROLLARY. *Let $A\colon M_1^+(\Omega) \to M_1^+(\Xi)$ be a statistical map. Then:*

(a) *A is norm, Banach weakly, and $\mathcal{F}(\Omega)$-weak-to-$\mathcal{F}(\Xi)$-weak continuous;*

(b) *A^* is norm, and Banach weakly* continuous;*

(c) *if $\{\mu_n\colon n \in \mathbb{N}\}$ is a sequence of measures $\mu_n \in M_1^+(\Omega)$ and $\{\lambda_n\colon n \in \mathbb{N}\}$ is a sequence of numbers such that $\lambda_n \in [0,1]$ and $\sum_n \lambda_n = 1$, then*

$$A\Big(\sum_n \lambda_n \mu_n\Big) = \sum_n \lambda_n A\mu_n;$$

(d) *if $\{f_n : n \in \mathbb{N}\} \subset \mathcal{E}(\Xi)$ is a sequence of real measurable functions on Ξ such that the pointwise limit $f = \lim_{n\to\infty} f_n$ does exist and belongs to $\mathcal{E}(\Xi)$, then*

$$A^* a_f = \lim_{n\to\infty} A^* a_{f_n}.$$

3.4. Let $A\colon M_1^+(\Omega) \to M_1^+(\Xi)$ be a statistical map. It is clear that $A^*\colon \mathcal{E}(\Xi) \to \mathcal{E}(\Omega)$, the dual of A, has the following properties:

(a) $A^* a_\Xi = a_\Omega$ (Proposition of 3.1),

(b) if $\{f_n : n \in \mathbb{N}\} \subset \mathcal{E}(\Xi)$ is such that the pointwise limit $f = \lim_{n\to\infty} f_n$ does exist and belongs to $\mathcal{E}(\Xi)$, then $A^* a_f = \lim_{n\to\infty} A^* a_{f_n}$ (Corollary of 3.3).

Definition. An affine map $B\colon \mathcal{E}(\Xi) \to \mathcal{E}(\Omega)$, where Ξ and Ω are measurable spaces, will be called a *dual statistical map* if it satisfies the following conditions:

(a) $Ba_\Xi = a_\Omega$,

(b) if $\{f_n : n \in \mathbb{N}\} \subset \mathcal{E}(\Xi)$ is such that the pointwise limit $f = \lim_{n\to\infty} f_n$ does exist and belongs to $\mathcal{E}(\Xi)$, then $Ba_f = \lim_{n\to\infty} Ba_{f_n}$.

Any dual statistical map $B\colon \mathcal{E}(\Xi) \to \mathcal{E}(\Omega)$ is the dual of a unique statistical map $D_B\colon M_1^+(\Omega) \to M_1^+(\Xi)$ defined by:

$$(D_B\mu)(X) := (Ba_X)(\mu)$$

for any $\mu \in M_1^+(\Omega)$ and $X \in \mathcal{B}(\Xi)$; on the other hand, any statistical map defines the corresponding dual statistical map, see Subsection 3.3.

4. Statistical functions and effect valued measures

4.1. In Corollary of 2.1 we have met a function $\Omega \to M_1^+(\Xi)$ defined by a sharp statistical map; that function was in fact a sharp statistical function according to the following definition.

Definition. A *statistical function* on a measurable space $(\Omega, \mathcal{B}(\Omega))$ with the outcome space $(\Xi, \mathcal{B}(\Xi))$ is a measure-valued function $\varphi\colon \Omega \to M_1^+(\Xi)$ such that for every $X \in \mathcal{B}(\Xi)$ the function $\varphi_X\colon \Omega \to [0,1]$, defined by

$$\varphi_X(\omega) := \varphi(\omega)(X),$$

belongs to $\mathcal{E}(\Omega)$;

$$\varphi_X \in \mathcal{E}(\Omega), \qquad X \in \mathcal{B}(\Xi). \tag{5}$$

A statistical function $\varphi\colon \Omega \to M_1^+(\Xi)$ is called *sharp* if $\varphi(\Omega) \subseteq \partial M_1^+(\Xi)$, and *strict* if $\varphi(\Omega) \subseteq \delta\Xi$.

A motive, and an interpretation of the concept of statistical function can be found in [5], [9], [14]. Statistical functions are called *transition probabilities* in [18]. We will show below that statistical functions are closely related to statistical maps.

4.2. Any statistical map $A\colon M_1^+(\Omega) \to M_1^+(\Xi)$ defines a function $\varphi_A\colon \Omega \to M_1^+(\Xi)$ by

$$\varphi_A(\omega) := A\delta_\omega .$$

Proposition of 2.1 shows that φ_A is a statistical function.

On the other hand, if $\varphi\colon \Omega \to M_1^+(\Xi)$ is a statistical function, then for any $X \in \mathcal{B}(\Xi)$ the measurable function $\varphi_X \in \mathcal{E}(\Omega)$ can be integrated over Ω, hence any $\mu \in M_1^+(\Omega)$ defines the set function

$$\mathcal{B}(\Xi) \ni X \to \int_\Omega \varphi_X(\omega)\, \mu(\mathrm{d}\omega)$$

which actually is a measure. Indeed, for any sequence $\{X_n : n \in \mathbb{N}\}$ of pairwise disjoint measurable subsets of Ξ we get the increasing and bounded sequence of measurable functions $\{f_n : n \in \mathbb{N}\}$, $f_n(\omega) := \sum_{i=1}^{n} \varphi(\omega)(X_i)$, which converges pointwise to f, $f(\omega) := \varphi(\omega)\left(\bigcup_{i=1}^{\infty} X_i\right)$, hence $\int_\Omega f_n(\omega)\, \mu(\mathrm{d}\omega)$ converges (because of the monotone convergence theorem) to $\int_\Omega f(\omega)\, \mu(\mathrm{d}\omega)$ for any $\mu \in M_1^+(\Omega)$. Thus, any statistical function $\varphi\colon \Omega \to M_1^+(\Xi)$ determines the affine map $D_\varphi\colon M_1^+(\Omega) \to M_1^+(\Xi)$ by

$$(D_\varphi \mu)(X) := \int_\Omega \varphi_X(\omega)\, \mu(\mathrm{d}\omega), \qquad X \in \mathcal{B}(\Xi). \tag{6}$$

It is evident that D_φ is a statistical map.

Finally, if $A\colon M_1^+(\Omega) \to M_1^+(\Xi)$ is a statistical map and $\varphi_A\colon \Omega \to M_1^+(\Xi)$ — the corresponding statistical function, the statistical map D_{φ_A} returns A, and *vice versa*: the statistical function defined by D_φ is φ. Thus we obtain:

THEOREM. *There is a natural one-to-one correspondence between statistical maps $M_1^+(\Omega) \to M_1^+(\Xi)$ and statistical functions $\Omega \to M_1^+(\Xi)$ such that:*

(a) *a statistical function $\varphi\colon \Omega \to M_1^+(\Xi)$ defines the unique statistical map $D_\varphi\colon M_1^+(\Omega) \to M_1^+(\Xi)$ such that $(D_\varphi\mu)(X) := \int_\Omega \varphi_X(\omega)\, \mu(\mathrm{d}\omega)$ for every $\mu \in M_1^+(\Omega)$ and $X \in \mathcal{B}(\Xi)$,*

(b) *a statistical map $A\colon M_1^+(\Omega) \to M_1^+(\Xi)$ defines the statistical function $\varphi_A\colon \Omega \to M_1^+(\Xi)$ such that $\varphi_A(\omega) := A\delta_\omega$ for every $\omega \in \Omega$,*

(c) $\varphi_{D_\varphi} = \varphi$, $D_{\varphi_A} = A$.

4.3. The one-to-one correspondence between statistical functions and statistical maps proved above implies the correspondence between strict statistical functions $\Omega \to M_1^+(\Xi)$ and measurable functions $\Omega \to \Xi$, parallel to the one stated in Theorem of 2.2. One can easily find an independent proof of that correspondence.

THEOREM. *There is a natural correspondence between measurable functions $\Omega \to \Xi$ and strict statistical functions $\Omega \to M_1^+(\Xi)$ such that:*

(a) *any measurable function $F\colon \Omega \to \Xi$ defines the unique strict statistical function $\varphi_F\colon \Omega \to M_1^+(\Xi)$ such that $\varphi_F(\omega) = \delta_{F(\omega)}$ for every $\omega \in \Omega$,*

(b) *any strict statistical function $\varphi\colon \Omega \to M_1^+(\Xi)$ defines the unique measurable function $F_\varphi\colon \Omega \to \Xi$ such that $\varphi(\omega) = \delta_{F_\varphi(\omega)}$ for every $\omega \in \Omega$,*

(c) *$F_{\varphi_F} = F$, $\varphi_{F_\varphi} = \varphi$ for every strict φ.*

Proof.

(a) It is evident that the map $\varphi_F\colon \Omega \to M_1^+(\Xi)$, with $\varphi_F(\omega) := \delta_{F(\omega)}$, is a statistical function.

(b) If a statistical function $\varphi\colon \Omega \to M_1^+(\Xi)$ is strict, then there is a function $F_\varphi\colon \Omega \to \Xi$ such that $\varphi(\omega) = \delta_{F_\varphi(\omega)}$ for all $\omega \in \Omega$. It is easy to see that the function F_φ is measurable. Indeed, $\varphi_X(\omega) = \delta_{F_\varphi(\omega)}(X) = \chi_{F_\varphi^{-1}(X)}(\omega)$, and the condition (5) implies now that $F_\varphi^{-1}(X) \in \mathcal{B}(\Omega)$.

(c) $\varphi_F(\omega) = \delta_{F_{\varphi_F}(\omega)} = \delta_{F(\omega)}$, $\varphi_{F_\varphi}(\omega) = \delta_{F_\varphi(\omega)} = \varphi(\omega)$ for all $\omega \in \Omega$. □

4.4. The one-to-one correspondence between statistical functions $\Omega \to M_1^+(\Xi)$ and statistical maps $M_1^+(\Omega) \to M_1^+(\Xi)$ (stated by Theorem of 4.2) together with the one-to-one correspondence between statistical maps $M_1^+(\Omega) \to M_1^+(\Xi)$ and Markov kernels $\Omega \times \mathcal{B}(\Xi) \to [0,1]$ (Theorem of 2.3) implies the (otherwise evident) one-to-one correspondence between statistical functions and Markov kernels.

4.5. Characteristic functions of measurable subsets of Ξ belong to $\mathcal{E}(\Xi)$, actually they form the set $\partial\mathcal{E}(\Xi)$ of extreme elements of the convex set $\mathcal{E}(\Xi)$, see [23; Lemma A.7(a)]. Now, given a statistical map $A\colon M_1^+(\Omega) \to M_1^+(\Xi)$, the restriction of its dual A^* to $\partial\mathcal{E}(\Xi)$ is an $\mathcal{E}(\Omega)$ valued measure on Ξ, denoted E^A,

$$\mathcal{B}(\Xi) \ni X \to E^A(X) := A^* a_X \in \mathcal{E}(\Omega). \tag{7}$$

The concept of an $\mathcal{E}(\Omega)$ valued measure (an *effect valued measure*, or *EV measure*) generalizes that of Mackey's *observable* ([17]).

On the other hand, let E be an $\mathcal{E}(\Omega)$ valued measure on Ξ, i.e. E is a map $\mathcal{B}(\Xi) \to \mathcal{E}(\Omega)$ such that $E(\Xi) = a_\Omega$, and $E\left(\bigcup_{n=1}^{\infty} X_n\right) = \sum_{n=1}^{\infty} E(X_n)$ for any sequence of pairwise disjoint measurable subsets of Ξ, where the series $\sum_{n=1}^{\infty} E(X_n)$ converges pointwise. Obviously, E defines the affine map $D_E\colon M_1^+(\Omega) \to M_1^+(\Xi)$,

$$(D_E\mu)(X) := \mu(E(X)),$$

we will call it the *distribution functional* of the EV measure E.

The map D_E is a statistical map because its dual $(D_E)^*$ coincides on $\mathcal{B}(\Xi)$ with E. Hence we obtain:

THEOREM. *There is a one-to-one correspondence between statistical maps $M_1^+(\Omega) \to M_1^+(\Xi)$, and $\mathcal{E}(\Omega)$ valued measures on Ξ, in particular:*

(a) *any statistical map $A\colon M_1^+(\Omega) \to M_1^+(\Xi)$ defines the unique $\mathcal{E}(\Omega)$ valued measure $E^A\colon \mathcal{B}(\Xi) \to \mathcal{E}(\Omega)$, $E^A(X) := A^* a_X$ for any $X \in \mathcal{B}(\Xi)$,*

(b) *any $\mathcal{E}(\Omega)$ valued measure $E\colon \mathcal{B}(\Xi) \to \mathcal{E}(\Omega)$ defines the unique statistical map $D_E\colon M_1^+(\Omega) \to M_1^+(\Xi)$ by $(D_E\mu)(X) := \mu(E(X))$ for any $X \in \mathcal{B}(\Xi)$, and*

(c) $E^{D_E} = E$, $D_{E^A} = A$.

The EV measure $E^A\colon \mathcal{B}(\Xi) \to \mathcal{E}(\Omega)$ generated by a statistical map A : $M_1^+(\Omega) \to M_1^+(\Xi)$ is called the *semi-spectral resolution of the statistical map* A.

4.6. Notice that the above theorem establishes also a one-to-one correspondence between EV measures and dual statistical maps (defined in 3.4): it is obvious that every dual statistical map $B\colon \mathcal{E}(\Xi) \to \mathcal{E}(\Omega)$ defines an EV measure by restriction to $\mathcal{B}(\Xi) \subset \mathcal{E}(\Xi)$, while every EV measure $E\colon \mathcal{B}(\Xi) \to \mathcal{E}(\Omega)$ defines the dual statistical map D_E^*.

The one-to-one correspondence between statistical functions $\Omega \to M_1^+(\Xi)$ and statistical maps $M_1^+(\Omega) \to M_1^+(\Xi)$ (see Theorem of 4.2), together with the one-to-one correspondence between statistical maps $M_1^+(\Omega) \to M_1^+(\Xi)$ and $\mathcal{E}(\Omega)$ valued measures on Ξ stated above, imply the one-to-one correspondence between statistical functions and $\mathcal{E}(\Omega)$ valued measures on Ξ.

4.7. Up to now we discussed several formal concepts which are equivalent to the one of statistical map. It would be interesting to mention briefly also some concepts which, being inequivalent, are nevertheless closely related to that of statistical map.

Let $(\Omega, \mathcal{B}(\Omega))$ and $(\Xi, \mathcal{B}(\Xi))$ be two measurable space which (according to our general assumption) have all points measurable. Consider a statistical function $\varphi\colon \Omega \to M_1^+(\Xi)$ with the corresponding statistical map $D_\varphi\colon M_1^+(\Omega) \to M_1^+(\Xi)$, and a measure $\mu_0 \in M_1^+(\Omega)$. The integral $\int_X \varphi_Y(\omega)\, \mu_0(\mathrm{d}\omega)$ converges for every $Y \in \mathcal{B}(\Xi)$, and the obtained set function m_X,

$$\mathcal{B}(\Xi) \ni Y \to m_X(Y) := \int_X \varphi_Y(\omega)\, \mu_0(\mathrm{d}\omega),$$

is a measure which belongs to the cap

$$M_{[0,1]}^+(\Xi) := \{\nu \in M^+(\Xi) : \ \nu(\Xi) \leqslant 1\}$$

of the cone $M^+(\Xi)$ of positive measures on Ξ. Clearly, $m_\Omega = D_\varphi\mu \in M_1^+(\Xi)$; while the measure μ_0 is recovered as $\mu_0(X) = m_X(\Xi)$. The obtained map

$$\mathcal{B}(\Omega) \ni X \to m_X \in M_{[0,1]}^+(\Xi)$$

is an $M_{[0,1]}^+(\Xi)$-valued measure on the measurable space $(\Omega, \mathcal{B}(\Omega))$; we understand it in the standard sense as a vector valued measure ([18; IV.2]) with the following condition of weak convergence: for every $f \in \mathcal{F}(\Xi)$, and every sequence $\{X_i :\ i = 1, 2, \dots\}$ of pairwise disjoint measurable subsets of Ω, the equality

$$\int_\Xi f(\xi)\, m_X(\mathrm{d}\xi) = \sum_{i=1}^{\infty} \int_\Xi f(\xi)\, m_{X_i}(\mathrm{d}\xi)$$

holds. Notice that any $M_{[0,1]}^+(\Xi)$-valued measure $\mathcal{B}(\Omega) \ni X \to m_X \in M_{[0,1]}^+(\Xi)$ defines two probability measures: $\mu_0 \in M_1^+(\Omega)$, and $m_\Omega \in M_1^+(\Xi)$ with: $\mu_0(X) := m_X(\Xi)$, $X \in \mathcal{B}(\Omega)$.

We have demonstrated that any statistical function $\varphi : \Omega \to M_1^+(\Xi)$ associates with every measure $\mu_0 \in M_1^+(\Omega)$ an $M_{[0,1]}^+(\Xi)$-valued measure $\mathcal{B}(\Omega) \ni X \to m_X \in M_{[0,1]}^+(\Xi)$ such that $\mu_0(X) := m_X(\Xi)$, $X \in \mathcal{B}(\Omega)$ and $m_\Omega = D_\varphi\mu$. It is easy to show that the reverse statement does not hold; there are $M_{[0,1]}^+(\Xi)$-valued measures on Ω which are not generated in the way outlined above by any corresponding statistical function. Thus, only a special class of measure-valued measures is generated by statistical maps.

It is evident that any $M_{[0,1]}^+(\Xi)$-valued measure $\mathcal{B}(\Omega) \ni X \to m_X \in M_{[0,1]}^+(\Xi)$ determines the set function $\mathcal{B}(\Omega) \times \mathcal{B}(\Xi) \ni X \times Y \to m_X(Y) \in [0,1]$ which, according to known rules of probability theory, extends to the unique probability measure μ_m on the product measurable space $\langle \Omega \times \Xi, \mathcal{B}(\Omega \times \Xi) \rangle$. Notice that the measures $\mu_0 \in M_1^+(\Omega)$ and $m_\Omega \in M_1^+(\Xi)$ defined by the $M_{[0,1]}^+(\Xi)$-valued measure are just marginal projections of μ_m. In this way, every $M_{[0,1]}^+(\Xi)$-valued measure $\mathcal{B}(\Omega) \ni X \to m_X \in M_{[0,1]}^+(\Xi)$ determines the unique polymorphism of the Lebesgue spaces $\langle \Omega, \mu_0 \rangle$ and $\langle \Xi, m_\Omega \rangle$ (as defined by Veršik [25]).

The reverse statement is also true: every probability measure μ on a product measurable space $\langle \Omega \times \Xi, \mathcal{B}(\Omega \times \Xi) \rangle$ defines the $M_{[0,1]}^+(\Xi)$-valued measure $\mathcal{B}(\Omega) \ni X \to m_X \in M_{[0,1]}^+(\Xi)$, $m_X(Y) := \mu(X \times Y)$ for every $X \in \mathcal{B}(\Omega)$ and $Y \in \mathcal{B}(\Xi)$. It is evident that $\mu_m = \mu$, while $\mu_0 \in M_1^+(\Omega)$ and $m_\Omega \in M_1^+(\Xi)$ are the marginal measures of μ. Hence, every Veršik's polymorphism $\langle \Omega, m_\Omega \rangle \twoheadleftarrow \langle \Omega \times \Xi, \mu \rangle \twoheadrightarrow \langle \Xi, m_\Xi \rangle$ such that μ is a probability measure, defines an $M_{[0,1]}^+(\Xi)$-valued measure on Ω. All that implies that statistical maps generate only a special class of Veršik's polymorphisms.

Let μ_0 be a fixed probability measure on $(\Omega, \mathcal{B}(\Omega))$. The convex set of all probability measures on $(\Omega, \mathcal{B}(\Omega))$ which are absolutely continuous w.r.t. μ_0, denoted $M_1^+(\Omega, \mu_0)$, is a face of $M_1^+(\Omega)$. The Radon-Nikodym theorem provides the known representation for the linear span of $M_1^+(\Omega, \mu_0)$ in $M(\Omega)$ as the (real) space $L^1(\Omega, \mathcal{B}(\Omega), \mu_0)$.

Now, given a statistical map $A\colon M_1^+(\Omega) \to M_1^+(\Xi)$, it is easy to find that $\mu \ll \mu_0$ implies $A\mu \ll A\mu_0$. Thus, A maps $M_1^+(\Omega, \mu_0)$ into $M_1^+(\Xi, A\mu_0)$ and, consequently, defines a linear map

$$U_A\colon L^1(\Omega, \mathcal{B}(\Omega), \mu_0) \to L^1(\Xi, \mathcal{B}(\Xi), A\mu_0).$$

It is clear that U_A is positive, unital, and preserves the L^1-norm, so U_A is a double stochastic transformation. However, in a genuine case a double stochastic transformation $L^1(\Omega, \mathcal{B}(\Omega), \mu_0) \to L^1(\Xi, \mathcal{B}(\Xi), \mu_0')$ does not determine any statistical map $M_1^+(\Omega) \to M_1^+(\Xi)$.

Finally, let us comment a relation between statistical maps and fuzzy-set valued functions (fuzzy random variables, [20]). It is evident that the dual $A^*\colon \mathcal{E}(\Xi) \to \mathcal{E}(\Omega)$ of a statistical map $A\colon M_1^+(\Omega) \to M_1^+(\Xi)$ (see 3.3) defines by restriction to Ξ the $\mathcal{E}(\Omega)$ valued function $\Xi \ni \xi \to A^*a_\xi \in \mathcal{E}(\Omega)$, where a_ξ is the affine function on $M_1^+(\Xi)$ determined by the one-point set $\{\xi\}$, see 1.3. That function, however, does not have to satisfy the conditions defining the fuzzy random variable ([20]). Moreover, the restriction to Ξ does not determine $A^*\colon \mathcal{E}(\Xi) \to \mathcal{E}(\Omega)$.

In special cases it can happen that the range $A(M_1^+(\Omega))$ of a statistical map $A\colon M_1^+(\Omega) \to M_1^+(\Xi)$ is contained in $M_1^+(\Xi, \mu_0)$ for some $\mu_0 \in M_1^+(\Xi)$. Then to every ω one can associate, via the Radon-Nikodym derivative, a function f_ω which integrated with μ_0 returns $A\delta_\omega$. The obtained map $\Omega \ni \omega \to f_\omega$ again does not have to satisfy the conditions defining the fuzzy random variable ([20]).

5. Products of statistical maps

5.1. We will define a product of an arbitrary family of statistical functions and prove it to be a statistical function. Then as a consequence we obtain the product of statistical maps.

Let $\{\varphi_i : \ i = 1, 2, \dots, n\}$ be a finite family of statistical functions, $\varphi_i\colon \Omega \to M_1^+(\Xi_i)$. With any $\omega \in \Omega$ one can associate the product measure $\varphi_1(\omega) \otimes \varphi_2(\omega) \otimes \cdots \otimes \varphi_n(\omega) \in M_1^+(\Xi_1 \times \Xi_2 \times \cdots \times \Xi_n)$.

Definition. The map $\bigotimes_{i=1}^{n} \varphi_i \colon \Omega \to M_1^+(\Xi_1 \times \Xi_2 \times \cdots \times \Xi_n)$ defined by

$$\Big(\bigotimes_{i=1}^{n} \varphi_i\Big)(\omega) := \varphi_1(\omega) \otimes \varphi_2(\omega) \otimes \cdots \otimes \varphi_n(\omega)$$

for any $\omega \in \Omega$ will be called the *product* of the family $\{\varphi_i : \ i = 1, 2, \ldots, n\}$ of statistical functions.

We want to show that the function $\bigotimes_{i=1}^{n} \varphi_i \colon \Omega \to M_1^+(\Xi_1 \times \Xi_2 \times \cdots \times \Xi_n)$ satisfies the condition (5). To this end we have to consider some properties of general $M_1^+(\Xi)$-valued functions on Ω, and to prove two technical lemmas.

5.2. Let $\varphi \colon \Omega \to M_1^+(\Xi)$ be an arbitrary map. For any $f \in \mathcal{F}(\Xi)$, the composition

$$a_f \circ \varphi \colon \Omega \xrightarrow{\varphi} M_1^+(\Xi) \xrightarrow{a_f} \mathbb{R}$$

is a real valued function on Ω. If $a_f \circ \varphi$ appears to be measurable, i.e. $a_f \circ \varphi \in \mathcal{F}(\Omega)$, we say that f belongs to the *m-space* of the map φ, denoted $m(\varphi)$. Thus

$$m(\varphi) := \{f \in \mathcal{F}(\Xi) : \ a_f \circ \varphi \in \mathcal{F}(\Omega)\}.$$

Obviously $m(\varphi)$ is a linear subspace of $\mathcal{F}(\Xi)$, and $a_\Xi \in m(\varphi)$. It is clear that $m(\varphi) = \mathcal{F}(\Xi)$ if and only if $\varphi \colon \Omega \to M_1^+(\Xi)$ satisfies the condition of formula (5), hence is a statistical function on Ω.

Lemma. *Let $\varphi \colon \Omega \to M_1^+(\Xi)$ be an arbitrary map. If $\{f_n : \ n \in \mathbb{N}\}$ is a norm-bounded increasing sequence of functions belonging to the m-space $m(\varphi)$ of φ, then the pointwise limit of that sequence belongs to $m(\varphi)$.*

Proof. It is clear that the norm-bounded increasing sequence $\{f_n : \ n \in \mathbb{N}\} \subset m(\varphi)$ has the pointwise limit, say f. It can be proved, see [23; Proposition A.3], that f is the weak*-limit of $\{f_n : \ n \in \mathbb{N}\}$ and belongs to $\mathcal{F}(\Xi)$. That means that

$$\lim_{n\to\infty} \int_\Xi f_n(\xi)\, \mu(\mathrm{d}\xi) = \int_\Xi f(\xi)\, \mu(\mathrm{d}\xi)$$

for every $\mu \in M_1^+(\Xi)$. In particular,

$$\lim_{n\to\infty} \int_\Xi f_n(\xi)\, \varphi(\omega)(\mathrm{d}\xi) = \lim_{n\to\infty} (a_{f_n} \circ \varphi)(\omega) = \int_\Xi f(\xi)\, \varphi(\omega)(\mathrm{d}\xi) \tag{8}$$

for every $\omega \in \Omega$.

For any $g_1, g_2 \in \mathcal{F}(\Xi)$, if $g_1 \leqslant g_2$ (pointwise on Ξ), then $\int_\Xi (g_2 - g_1)(\xi)\, \mu(\mathrm{d}\xi) \geqslant 0$ for any $\mu \in M_1^+(\Xi)$. Hence $(a_{g_2} \circ \varphi)(\omega) - (a_{g_1} \circ \varphi)(\omega) = \int_\Xi (g_2 - g_1)(\xi)\, \varphi(\omega)(\mathrm{d}\xi) \geqslant 0$ for any $\omega \in \Omega$, what means that $a_{g_1} \circ \varphi \leqslant a_{g_2} \circ \varphi$ pointwise on Ω. Moreover, for any $g \in \mathcal{F}(\Xi)$ we have

$$|(a_g \circ \varphi)(\omega)| = \left| \int_\Xi g(\xi)\, \varphi(\omega)(\mathrm{d}\xi) \right| \leqslant \sup\{|g(\xi)| :\ \xi \in \Xi\}$$

for all $\omega \in \Omega$. Hence, if the sequence $\{f_n :\ n \in \mathbb{N}\} \subset m(\varphi)$ is norm-bounded and increasing, then the sequence $\{a_{f_n} \circ \varphi :\ n \in \mathbb{N}\} \subset \mathcal{F}(\Omega)$ is norm-bounded and increasing too. As any norm-bounded increasing sequence in $\mathcal{F}(\Omega)$ possesses the pointwise limit belonging to $\mathcal{F}(\Omega)$, the same concerns $\{a_{f_n} \circ \varphi :\ n \in \mathbb{N}\}$.

Let now g denote the pointwise limit of $\{a_{f_n} \circ \varphi :\ n \in \mathbb{N}\}$. Because of the mentioned result of [23], g is the weak*-limit of $\{a_{f_n} \circ \varphi :\ n \in \mathbb{N}\}$ and belongs to $\mathcal{F}(\Omega)$. This ends the proof, because equation (8) means now that $f \in m(\varphi)$. □

The just proved lemma implies that the family $\mathcal{B}(\Xi) \cap m(\varphi)$ of measurable subsets of Ξ (it is not empty, as $a_\Xi \in m(\varphi)$) is a λ-system of Dynkin, see [8; Chapter 1, Section 3].

5.3. Lemma. *Let $\varphi_1 \colon \Omega \to M_1^+(\Xi_1)$, $\varphi_2 \colon \Omega \to M_1^+(\Xi_2)$ be two maps, let $f_1 \in m(\varphi_1) \subseteq \mathcal{F}(\Xi_1)$, $f_2 \in m(\varphi_2) \subseteq \mathcal{F}(\Xi_2)$. Then $f_1 f_2 \colon \Xi_1 \times \Xi_2 \to \mathbb{R}$, defined by $(f_1 f_2)(\xi_1, \xi_2) := f_1(\xi_1) f_2(\xi_2)$, belongs to $m(\varphi_1 \otimes \varphi_2) \subseteq \mathcal{F}(\Xi_1 \times \Xi_2)$.*

P r o o f. The assertion is an easy consequence of Fubini's theorem. Indeed,

$$\begin{aligned}(a_{f_1 f_2} \circ \varphi_1 \otimes \varphi_2)(\omega) &= \int_{\Xi_1 \times \Xi_2} f_1(\xi_1) f_2(\xi_2) \big(\varphi_1(\omega) \otimes \varphi_2(\omega)\big)\big(\mathrm{d}(\xi_1, \xi_2)\big) \\ &= (a_{f_1} \circ \varphi_1)(\omega)(a_{f_2} \circ \varphi_2)(\omega).\end{aligned}$$

As the pointwise product of measurable functions is a measurable function, we conclude that $a_{f_1 f_2} \circ \varphi_1 \otimes \varphi_2 \in \mathcal{F}(\Omega)$. □

Notice the following obvious corollary of the above lemma: if $X_1 \in \mathcal{B}(\Xi_1) \cap m(\varphi_1)$ and $X_2 \in \mathcal{B}(\Xi_2) \cap m(\varphi_2)$, then the measurable rectangle $X_1 \times X_2 \in m(\varphi_1 \otimes \varphi_2)$. Hence, if φ_1 and φ_2 are statistical functions, then $m(\varphi_1 \otimes \varphi_2)$ contains all measurable rectangles of $\mathcal{B}(\Xi_1 \times \Xi_2)$.

5.4. Now we can prove that the product $\bigotimes_{i=1}^{n} \varphi_i \colon \Omega \to M_1^+(\Xi_1 \times \Xi_2 \times \cdots \times \Xi_n)$ of statistical functions is a statistical function.

Theorem. *If* $\{\varphi_i : i = 1, 2, \dots, n\}$ *is a finite collection of statistical functions* $\varphi_i \colon \Omega \to M_1^+(\Xi_i)$, *then the map* $\bigotimes_{i=1}^{n} \varphi_i \colon \Omega \to M_1^+(\Xi_1 \times \Xi_2 \times \cdots \times \Xi_n)$ *is a statistical function.*

P r o o f . It suffices to prove the assertion for $n = 2$, so we have to show that the map $\Omega \ni \omega \to \varphi_1(\omega) \otimes \varphi_2(\omega) \in M_1^+(\Xi_1 \times \Xi_2)$ is a statistical function. The lemma of 5.2 implies that the measurable subsets of $\Xi_1 \times \Xi_2$ belonging to $m(\varphi_1 \otimes \varphi_2)$ form a λ-system, while the lemma of 5.3 assures that $m(\varphi_1 \otimes \varphi_2)$ contains the π-system of all measurable rectangles of $\mathcal{B}(\Xi_1 \times \Xi_2)$. Applying the Dynkin's theorem on π-λ-systems ([8]), we find that the σ-field $\mathcal{B}(\Xi_1 \times \Xi_2)$ generated by the measurable rectangles has to belong to $m(\varphi_1 \otimes \varphi_2)$.

Taking into account that any non-negative measurable function on a measurable space $(\Xi, \mathcal{B}(\Xi))$ is a pointwise limit of an increasing sequence of simple functions (see e.g. [4; Theorem 2.3.6]), we infer finally from the lemma of 5.2 that $m(\varphi_1 \otimes \varphi_2) = \mathcal{F}(\Xi_1 \times \Xi_2)$. Thus $\varphi_1 \otimes \varphi_2$ is a statistical function. □

5.5. Taking into account the one-to-one correspondence between statistical maps and statistical functions established in 4.2, we have:

Definition. Let $\{A_i : i = 1, 2, \dots, n\}$ be a finite family of statistical maps, $A_i \colon M_1^+(\Omega) \to M_1^+(\Xi_i)$, let φ_i denote the statistical function corresponding to A_i, $i = 1, 2, \dots, n$. The distribution functional $D_{\bigotimes \varphi_i}$ of the statistical function $\bigotimes_{i=1}^{n} \varphi_i$ will be called the *product* of the family of statistical maps $\{A_i : i = 1, 2, \dots, n\}$ and denoted $\bigotimes_{i=1}^{n} A_i$,

$$\bigotimes_{i=1}^{n} A_i := D_{\bigotimes \varphi_i} .$$

The notion of product of statistical maps has been introduced in [5].

5.6. In the same way one can introduce the concept of product of dual statistical maps and of EV measures. Thus, for instance, if $E_1 \colon \mathcal{B}(\Xi_1) \to \mathcal{E}(\Omega)$, $E_2 \colon \mathcal{B}(\Xi_2) \to \mathcal{E}(\Omega)$ are EV measures, then we define $E_1 \otimes E_2 := E^{\varphi_1 \otimes \varphi_2}$, where φ_i, $i = 1, 2$ is the statistical function corresponding to the EV measure E_i according to 4.6.

Considerations of 5.3. imply the following corollary.

Corollary. *For* $X_1 \in \mathcal{B}(\Xi_1)$, $X_2 \in \mathcal{B}(\Xi_2)$,

$$(E_1 \otimes E_2)(X_1 \times X_2) = E_1(X_1) E_2(X_2) ,$$

where $E_1(X_1)E_2(X_2)$ *is the pointwise product of two measurable functions* $E_1(X_1)$, $E_2(X_2)$ *on* Ω.

The proof of Theorem of 5.4. indicates that an EV measure on the product measurable space $(\Xi_1 \times \Xi_2, \mathcal{B}(\Xi_1) \otimes \mathcal{B}(\Xi_2))$ is uniquely determined by the values it takes on measurable rectangles $X_1 \times X_2$, $X_1 \in \mathcal{B}(\Xi_1)$, $X_2 \in \mathcal{B}(\Xi_2)$, hence one can define $E_1 \otimes E_2$ directly as the unique EV measure which satisfies the assertion of the above corollary ([16]).

5.7. The product of statistical maps reproduces the corresponding concept appearing in the standard probability theory (the product of (standard) random variables ([4]), the random vector ([8]), etc.). Indeed, let $A_1 : M_1^+(\Omega) \to M_1^+(\Xi_1)$, $A_2 : M_1^+(\Omega) \to M_1^+(\Xi_2)$ be two strict statistical maps. Theorem of 2.2 states now that there are two measurable functions $F_1 : \Omega \to \Xi_1$, $F_2 : \Omega \to \Xi_2$ such that $A_1\delta_\omega = \delta_{F_1(\omega)}$, $A_2\delta_\omega = \delta_{F_2(\omega)}$ for all $\omega \in \Omega$. Then, the product statistical map $A_1 \otimes A_2$ satisfies (and is determined by, see Proposition of 2.1)

$$(A_1 \otimes A_2)\delta_\omega = \delta_{F_1(\omega)} \otimes \delta_{F_2(\omega)} .$$

As $\delta_{F_1(\omega)} \otimes \delta_{F_2(\omega)} = \delta_{(F_1(\omega), F_2(\omega))}$, $A_1 \otimes A_2$ is the strict statistical map generated by the measurable function F, $\Omega \ni \omega \to F(\omega) := (F_1(\omega), F_2(\omega)) \in \Xi_1 \times \Xi_2$.

5.8. The concept of product can be extended over arbitrary families of statistical maps. We start with considering the product of an arbitrary family of statistical functions.

Let $\{\varphi_t : t \in \mathbb{T}\}$, with an arbitrary index set $\mathbb{T}$, be a family of statistical functions $\varphi_t : \Omega \to M_1^+(\Xi_t)$. Take $\bigtimes_{t\in\mathbb{T}} \Xi_t$ to denote the set of all mappings $\xi : \mathbb{T} \to \bigcup_{t\in\mathbb{T}} \Xi_t$ such that $\xi(t) \in \Xi_t$. We define the Boolean σ-algebra $\mathcal{B}\left(\bigtimes_{t\in\mathbb{T}} \Xi_t\right)$ of measurable subsets of $\bigtimes_{t\in\mathbb{T}} \Xi_t$ to be the one generated by the π-system $\mathcal{B}_0\left(\bigtimes_{t\in\mathbb{T}} \Xi_t\right)$ of cylinder sets (equivalently: the one generated by all coordinate maps $\pi_t : \bigtimes_{s\in\mathbb{T}} \Xi_s \to \Xi_t$, $t \in \mathbb{T}$).

Now, to every $\omega \in \Omega$ we can associate the infinite product measure ([4; Theorem 5.4.2]) $\bigotimes_{t\in\mathbb{T}} \varphi_t(\omega)$ and, like the case of a finite $\mathbb{T}$, we only need to show that the product map

$$\Omega \ni \omega \to \left(\bigotimes_{t\in\mathbb{T}} \varphi_t\right)(\omega) := \bigotimes_{t\in\mathbb{T}} \varphi_t(\omega) \in M_1^+\left(\bigtimes_{t\in\mathbb{T}} \Xi_t\right)$$

satisfies the condition of equation (5). However, it does satisfy it, as Lemma of 5.2. asserts that the m-space $m\left(\bigotimes_{t\in\mathbb{T}} \varphi_t\right)$ of $\bigotimes_{t\in\mathbb{T}} \varphi_t$ contains a λ-system of measurable subsets of $\bigtimes_{t\in\mathbb{T}} \Xi_t$, while Lemma of 5.3. implies that $\mathcal{B}_0\Big(\bigtimes_{t\in\mathbb{T}} \Xi_t\Big) \subset m\Big(\bigotimes_{t\in\mathbb{T}} \varphi_t\Big)$. Applying again the π-λ-theorem we conclude with the following theorem:

THEOREM. *For any family $\{\varphi_t : t \in \mathbb{T}\}$ of statistical functions $\varphi_t \colon \Omega \to M_1^+(\Xi_t)$, the product $\bigotimes_{t\in\mathbb{T}} \varphi_t$ is a statistical function.*

5.9. Following 5.5, we can define the product of an arbitrary family of statistical map:

DEFINITION. Let $\{A_t : t \in \mathbb{T}\}$ with an arbitrary index set $\mathbb{T}$ be a family of statistical maps $A_t \colon M_1^+(\Omega) \to M_1^+(\Xi_t)$, let φ_t denote the statistical function corresponding to A_t. The distribution functional $D_{\otimes \varphi_t}$ of the statistical function $\bigotimes_{t\in\mathbb{T}} \varphi_t$ will be called the *product* of the family of statistical maps $\{A_t : t \in \mathbb{T}\}$ and denoted $\bigotimes_{t\in\mathbb{T}} A_t$,

$$\bigotimes_{t\in\mathbb{T}} A_t := D_{\otimes \varphi_t}.$$

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